Term 081

CISE302: Linear Control Systems

6. Laplace Transform Properties

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Reading Assignment :

Learning Objective

- To be able to use Laplace transform to solve linear constant coefficient ordinary differential equations

Use of Laplace Transform in solving ODE

Differential Equation

$$\dot{x}(t) + 2x(t) = 0, x(0) = 1$$

→ **Laplace Transform** →

Algebraic Equation

$$sX(s) - 1 + 2X(s) = 0$$

↓

$$X(s) = \frac{1}{s+2}$$

Solution of the Algebraic Equation

← **Inverse Laplace transform** ←

$$x(t) = e^{-2t}$$

Solution of the Differential Equation

Solution Procedure

1. Apply Laplace transform to the differential equation to obtain an algebraic equation
2. Solve the algebraic equation for the unknown function
3. Use Partial fraction expansion to express the unknown function as the sum of simple terms
4. Use inverse Laplace transform to obtain the solution of the original problem

Laplace Transform of Derivative

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - \dot{f}(0)$$

$$L\left\{\frac{d^3 f(t)}{dt^3}\right\} = s^3 F(s) - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$$

Solving ODE

Example 1

Apply Laplace trasform to solve the ODE

$$\dot{x}(t) + 4x(t) = 2, \quad x(0) = 0$$

Step 1: Apply Laplace transform to the ODE

$$L\{\dot{x}(t)\} = sX(s) - x(0)$$

$$L\{x(t)\} = X(s)$$

$$L\{2\} = \frac{2}{s}$$

$$sX(s) - x(0) + 4X(s) = \frac{2}{s}$$

Solving ODE
Example 1

Apply Laplace trasform to solve the ODE

$$\dot{x}(t) + 4x(t) = 2, \quad x(0) = 0$$

Step 2: Solve for the unknown function X(s)

$$sX(s) + 4X(s) = \frac{2}{s}$$

$$(s+4)X(s) = \frac{2}{s}$$

$$X(s) = \frac{2}{s(s+4)}$$

Solving ODE

Example 1

Apply Laplace trasform to solve the ODE

$$\dot{x}(t) + 4x(t) = 2, \quad x(0) = 0$$

Step 3: Partial Fraction Expansion

$$X(s) = \frac{2}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$A = (s)\frac{2}{s(s+4)}\bigg|_{s=0} = 0.5$$

$$B = (s+4)\frac{2}{s(s+4)}\bigg|_{s=-4} = -0.5$$

$$X(s) = \frac{0.5}{s} + \frac{-0.5}{s+4}$$

Solving ODE

Example 1

Apply Laplace trasform to solve the ODE

$$\dot{x}(t) + 4x(t) = 2, \quad x(0) = 0$$

Step 4: Inverse Laplace Transform

$$X(s) = \frac{0.5}{s} + \frac{-0.5}{s+4}$$

$$x(t) = 0.5 \quad - 0.5e^{-4t}$$

Solving ODE
Example 2

Apply Laplace trasform to solve the ODE

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 1, \quad x(0) = 4, \dot{x}(0) = 5$$

$$L\{\ddot{x}(t)\} = s^2 X(s) - sx(0) - \dot{x}(0)$$

$$L\{\dot{x}(t)\} = sX(s) - x(0)$$

$$L\{x(t)\} = X(s)$$

$$L\{1\} = \frac{1}{s}$$

Solving ODE

Example 2 **Step 1: Apply Laplace Transform**

Apply Laplace trasform to solve the ODE

$$\ddot{x}(t)+3\dot{x}(t)+2x(t)=1, \quad x(0)=4, \dot{x}(0)=5$$

$STEP:$

$$L\left\{\ddot{x}(t)+3\dot{x}(t)+2x(t)\right\}=L\left\{u(t)\right\}$$

$$\left[s^2X(s)-sx(0)-\dot{x}(0)\right]+3[sX(s)-x(0)]+2X(s)=\frac{1}{s}$$

$$\left[s^2X(s)-4s-5\right]+3[sX(s)-4]+2X(s)=\frac{1}{s}$$

Solving ODE

Example 2 — Step 2: Solve for X(s)

Apply Laplace trasform to solve the ODE

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 1, \quad x(0) = 4, \dot{x}(0) = 5$$

STEP 2 :

$$\left[s^2 X(s) - 4s - 5\right] + 3[sX(s) - 4] + 2X(s) = \frac{1}{s}$$

$$\left[s^2 + 3s + 2\right]X(s) = -4s - 5 - 12 + \frac{1}{s}$$

$$X(s) = \frac{-4s - 17 + \frac{1}{s}}{s^2 + 3s + 2} = \frac{-4s^2 - 17s + 1}{s(s^2 + 3s + 2)}$$

Solving ODE

Example 2 Step 3: Partial Fraction Expansion

Apply Laplace trasform to solve the ODE

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 1, \quad x(0) = 4, \dot{x}(0) = 5$$

STEP 3:

$$X(s) = \frac{-4s^2 - 17s + 1}{s(s^2 + 3s + 2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = s \left. \frac{-4s^2 - 17s + 1}{s(s^2 + 3s + 2)} \right|_{s=0} = 0.5,$$

$$B = (s+1) \left. \frac{-4s^2 - 17s + 1}{s(s^2 + 3s + 2)} \right|_{s=-1} = -14,$$

$$C = (s+2) \left. \frac{-4s^2 - 17s + 1}{s(s^2 + 3s + 2)} \right|_{s=-2} = 9.5$$

Solving ODE

Example 2 Step 4: Inverse Laplace transform

Apply Laplace trasform to solve the ODE

$$\ddot{x}(t)+3\dot{x}(t)+2x(t)=1, \quad x(0)=4, \dot{x}(0)=5$$

$STEP\,4:$

$$X(s)=\frac{0.5}{s}+\frac{-14}{s+1}+\frac{9.5}{s+2}$$

$$x(t)=0.5-14e^{-t}+9.5e^{-2t} \qquad for\ \ t\geq 0$$

Summary

1. Apply Laplace transform to the differential equation
2. Solve for the unknown function
3. Use Partial fraction expansion to express the unknown function as the sum of simple terms
4. Use inverse Laplace transform to obtain the solution of the original problem