King Fahd University of Petroleum and Minerals

Prep-Year Math Program

Math 002 Class Test II
Textbook Sections: 6.3 to 7.5
Term 172
Time Allowed: **90** Minutes

Student's Name: …………… ………………………………………………
ID #:………………………. **Section:** ……. **Serial Number:** …….

Provide neat and complete solutions.
Show all necessary steps for full credit and write the answer in simplest form.

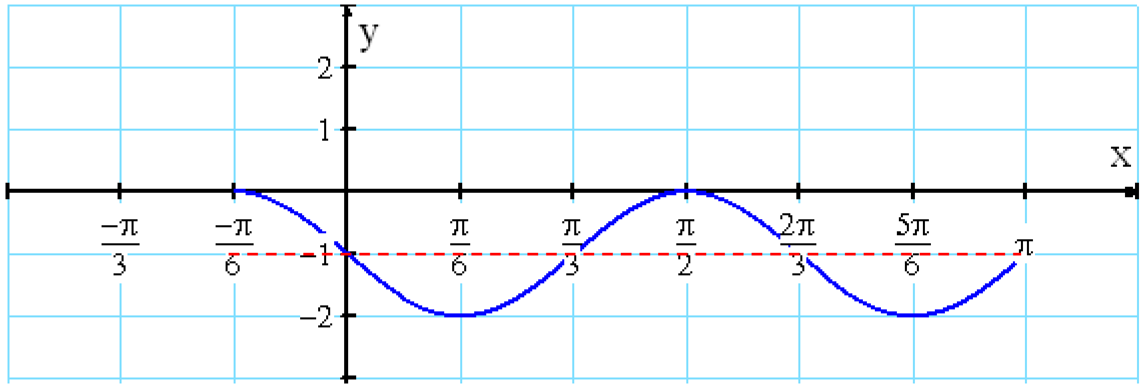
No Calculators, Cameras, or Mobiles are allowed during this exam.

Question	**Points**	Student's Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
Total	**50**	____ / 50
		____ / 100

Q1. (5 points): Given $y = -1 + \cos\left(3x + \dfrac{\pi}{2}\right)$, $0 \le x \le \pi$.

(a): Graph the function over the given interval.
(b): Find the intervals where the function is increasing over the given interval.
(c): Find the intervals where the function is decreasing over the given interval.

Solution (a): $0 \le 3x + \dfrac{\pi}{2} \le 2\pi \iff -\dfrac{\pi}{2} \le 3x \le \dfrac{3\pi}{2} \iff -\dfrac{\pi}{6} \le x \le \dfrac{\pi}{2}$



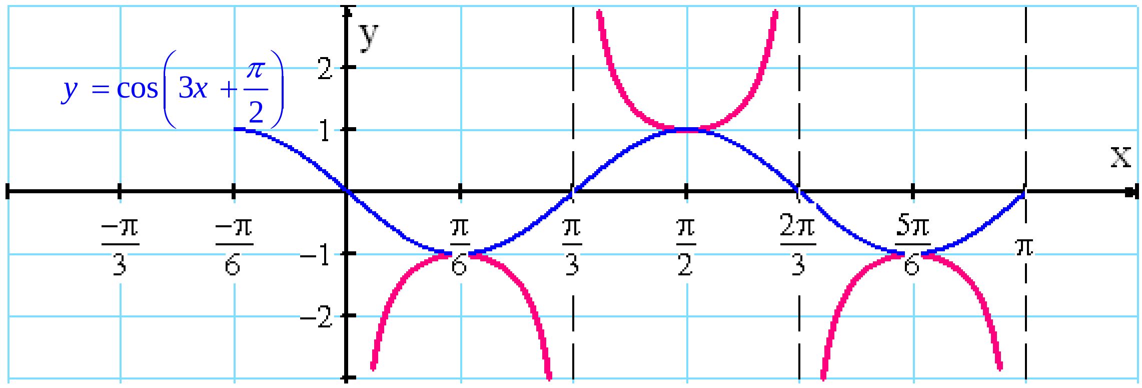
(b): $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right], \left[\dfrac{5\pi}{6}, \pi\right]$

(c): $\left[0, \dfrac{\pi}{6}\right], \left[\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right]$

Q2. (5 points): Given $y = \sec\left(3x + \dfrac{\pi}{2}\right)$, $0 \le x \le \pi$.

(a): Graph the function over the given interval.
(b): Determine the equations of vertical asymptotes over the given interval.
(c): Find the intervals where the function is decreasing over the given interval.

Solution (a): $0 \le 3x + \dfrac{\pi}{2} \le 2\pi \iff -\dfrac{\pi}{2} \le 3x \le \dfrac{3\pi}{2} \iff -\dfrac{\pi}{6} \le x \le \dfrac{\pi}{2}$



(b): $x = 0$, $x = \dfrac{\pi}{3}$, $x = \dfrac{2\pi}{3}$, $x = \pi$

(c): $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right), \left(\dfrac{\pi}{3}, \dfrac{\pi}{2}\right], \left[\dfrac{5\pi}{6}, \pi\right)$

Q3. (5 points): Given the function $y = 3\tan\left(2x + \frac{\pi}{2}\right)$ where $-\frac{\pi}{2} < x < \pi$

(a) Graph the function over the given interval
(b) Find the x-intercepts over the given interval
(c) Find the equation(s) of vertical asymptote over the given interval.

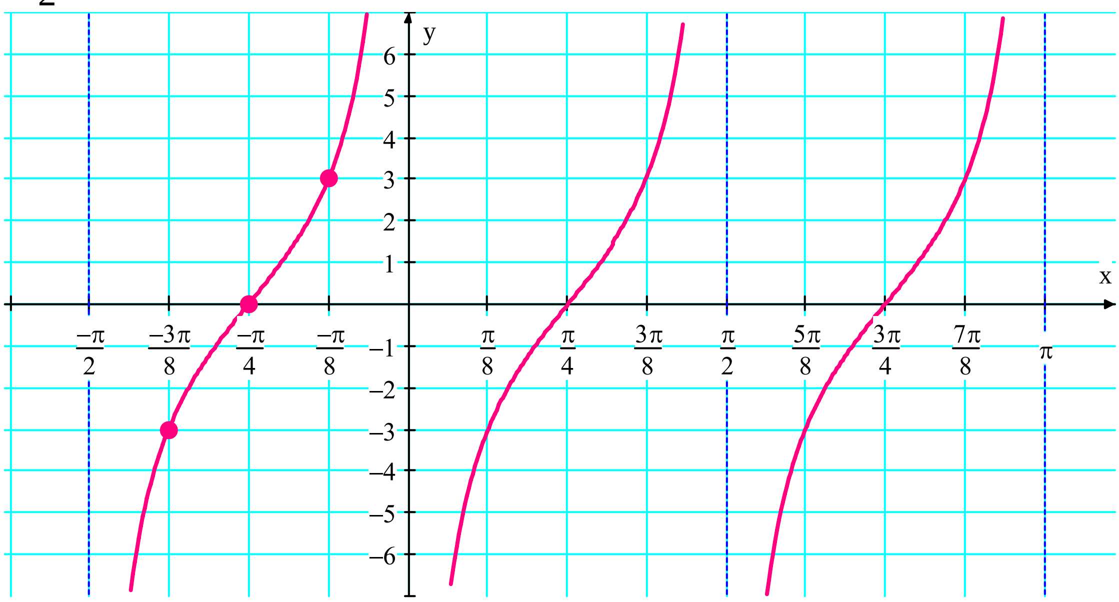
Solution (a):

$$-\frac{\pi}{2} < 2x + \frac{\pi}{2} < \frac{\pi}{2}$$

$$-\pi < 4x + \pi < \pi$$

$$-2\pi < 4x < 0$$

$$-\frac{\pi}{2} < x < 0$$



(b): $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$

(c): $x = 0$, $x = \frac{\pi}{2}$

Q4. (5 points): If $\sin(37°) = t$, then $\sin 863° + \sin 307° =$

Solution:

$$\sin 863° + \sin 307° = \sin(863° - 720°) + [-\sin(360° - 307°)]$$
$$= \sin 143° - \sin(53°)$$
$$= \sin(180° - 143°) - \sin(53°)$$
$$= +\sin 37° - \cos(90° - 53°)$$
$$= t - \cos(37°)$$
$$= t - \sqrt{1 - \sin^2 37}$$
$$= t - \sqrt{1 - t^2}$$

Q5. (5 points): Given: $f(x) = \cos 2x + \sqrt{3}\sin 2x$.

(a): Sketch the graph of f over the interval $\left[-\frac{\pi}{12}, \frac{11\pi}{12}\right]$

(b): Find the interval where the function is **decreasing** over the interval $\left[-\frac{\pi}{12}, \frac{11\pi}{12}\right]$

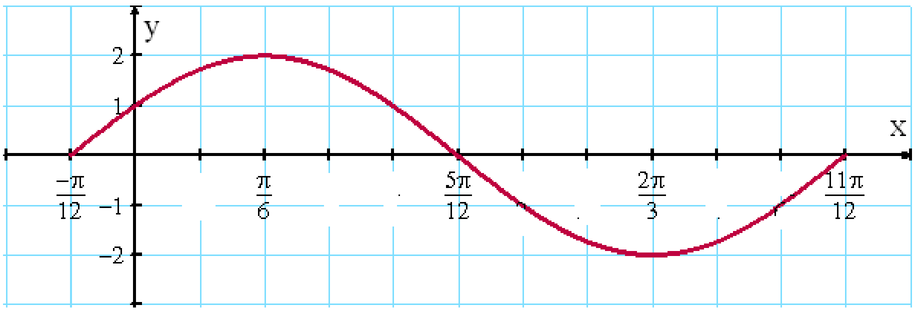
(c): Find the interval where the graph is above the *x*-axis over the interval $\left[-\frac{\pi}{12}, \frac{11\pi}{12}\right]$

(d): Find the interval where the graph is below the x-axis over the interval $\left[-\frac{\pi}{12}, \frac{11\pi}{12}\right]$

Solution (a): $f(x) = \cos 2x + \sqrt{3}\sin 2x = \sqrt{1^2 + \left(\sqrt{3}\right)^2}\sin(2x+\alpha) = 2\sin(2x+\alpha)$

$$\left.\begin{array}{l}\sin\alpha = \frac{1}{2} \\ \cos\alpha = \frac{\sqrt{3}}{2}\end{array}\right\} \Rightarrow \alpha = \frac{\pi}{6} \quad \Rightarrow \quad f(x) = 2\sin\left(2x + \frac{\pi}{6}\right)$$

$$0 \le 2x + \frac{\pi}{6} \le 2\pi \quad \Rightarrow \quad -\frac{\pi}{6} \le 2x \le 2\pi - \frac{\pi}{6} \quad \Rightarrow \quad -\frac{\pi}{12} \le x \le \frac{11\pi}{12}$$



(b): Decreasing on $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right]$ **(c):** The graph is above the *x*-axis $\left(-\frac{\pi}{12}, \frac{5\pi}{12}\right)$

Q6. (5 points) (7.1 Textbook Exercises 105-112): Prove the following identities (**Show all steps**)

(a): $\ln|\tan x \sin x| = 2\ln|\sin x| + \ln|\sec x|$

(b): $\ln|\tan x| + \ln|\cot x| = 0$

(c): $e^{\sin^2 x} e^{\tan^2 x} = e^{\sec^2 x} e^{-\cos^2 x}$

(d): $e^{x + 2\ln|\sin x|} = e^x \sin^2 x$

(e): $x e^{\ln x^2} = x^3$

Solution:

(a): $LHS = \ln|\tan x \sin x| = \ln\left[|\tan x||\sin x|\right] = \ln|\tan x| + \ln|\sin x| = \ln\left|\frac{\sin x}{\cos x}\right| + \ln|\sin x|$

$= \ln|\sin x| - \ln|\cos x| + \ln|\sin x| = 2\ln|\sin x| + \ln|\cos x|^{-1} = 2\ln|\sin x| + \ln|\sec x| = RHS$

(b): $LHS = \ln|\tan x| + \ln|\cot x| = \ln\left[|\tan x||\cot x|\right] = \ln 1 = 0 = RHS$

(c): $LHS = e^{\sin^2 x} e^{\tan^2 x} = e^{1-\cos^2 x} e^{\sec^2 x - 1} = e^{1-\cos^2 x + \sec^2 x - 1} = e^{-\cos^2 x + \sec^2 x} = e^{-\cos^2 x} e^{\sec^2 x} = RHS$

(d): $e^{x + 2\ln|\sin x|} = e^x \cdot e^{2\ln|\sin x|} = e^x \cdot e^{\ln|\sin x|^2} = e^x \cdot |\sin x|^2 = e^x \sin^2 x = RHS$

(e): $x e^{\ln x^2} = x \cdot e^{\ln x^2} = x \cdot x^2 = x^3 = RHS$

Q7. (5 points): Whenever possible, find the value of each of the following:

(a): $\cot^{-1}(-\sqrt{3})$

(b): $\sin^{-1}(\sin\pi)$

(c): $\cos\left(\cos^{-1}\frac{5}{4}\right)$

(d): $\sec\left(\tan^{-1}2\right)$

(e): $\cos^{-1}\left(\tan\frac{2\pi}{3}\right)$

Solution (a): $\cot^{-1}(-\sqrt{3})=\frac{\pi}{2}-\tan^{-1}(-\sqrt{3})=\frac{\pi}{2}-\left(-\frac{\pi}{3}\right)=\boxed{\frac{5\pi}{6}}$

(b): $\sin^{-1}(\sin\pi)=\sin^{-1}(0)=\boxed{0}$

(c): $\cos\left(\cos^{-1}\frac{5}{4}\right)=\boxed{\text{undefined}}$ because $\frac{5}{4}\notin D_{\cos^{-1}}=[-1,1]$

(d): Let $\theta=\tan^{-1}2$ where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ then $\tan\theta=2$, $\theta\in QI$

$\tan\theta=2=\frac{2}{1}\quad\Rightarrow\quad x=1,\ y=2$ and $r=\sqrt{5}$

$\sec\left(\tan^{-1}2\right)=\sec\theta=\frac{r}{x}=\frac{\sqrt{5}}{1}=\boxed{\sqrt{5}}$

(e): $\cos^{-1}\left(\tan\frac{2\pi}{3}\right)=\cos^{-1}\left(-\tan\frac{\pi}{3}\right)=\cos^{-1}\left(-\sqrt{3}\right)=\boxed{\text{undefined}}$ because $-\sqrt{3}\notin[-1,1]=D_{\cos^{-1}}$

Q8. (5 points) (7.1 Textbook Exercise 104): If $\sin^6\beta+\cos^6\beta=A+B\sin^2\beta\cos^2\beta$ then find A and B . (Show your work)

Solution: $\sin^6\beta+\cos^6\beta=\left(\sin^2\beta\right)^3+\left(\cos^2\beta\right)^3$

$$=(\sin^2\beta+\cos^2\beta)(\sin^4\beta-\sin^2\beta\cos^2\beta+\cos^4\beta)$$

$$=(1)(\underbrace{\sin^4\beta+\cos^4\beta+2\sin^2\beta\cos^2\beta}-2\sin^2\beta\cos^2\beta-\sin^2\beta\cos^2\beta)$$

$$=\left(\sin^2\beta+\cos^2\beta\right)^2-2\sin^2\beta\cos^2\beta-\sin^2\beta\cos^2\beta$$

$$=1-3\sin^2\beta\cos^2\beta$$

$$=A+B\sin^2\beta\cos^2\beta \text{ then } \boxed{A=1} \text{ and } \boxed{B=-3}$$

Another Method:

$$\sin^6\beta+\cos^6\beta=\left(\sin^2\beta\right)^3+\left(\cos^2\beta\right)^3$$

$$=\left(1-\cos^2\beta\right)^3+\left(\cos^2\beta\right)^3$$

$$=(1-3\cos^2\beta+3\cos^4\beta-\cos^6\beta)+\left(\cos^2\beta\right)^3$$

$$=1-3\cos^2\beta+3\cos^4\beta$$

$$=1-3\cos^2\beta(1-\cos^2\beta)$$

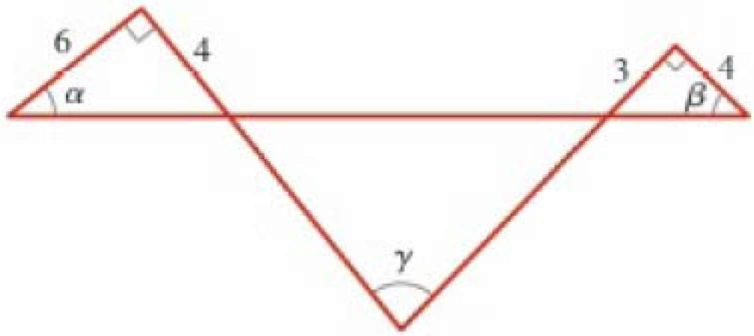
$$=1-3\cos^2\beta\sin^2\beta$$

$$=1-3\sin^2\beta\cos^2\beta$$

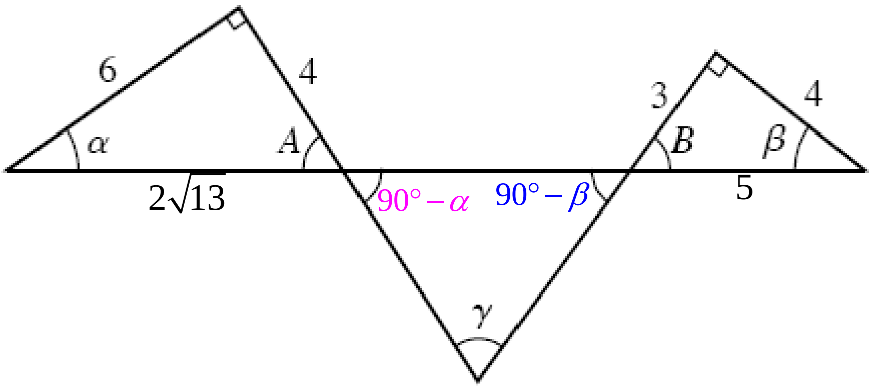
Another Method:

$$\sin^6\beta+\cos^6\beta=\left(\sin^2\beta\right)^3+\left(\cos^2\beta\right)^3$$
$$=(\sin^2\beta+\cos^2\beta)(\sin^4\beta-\sin^2\beta\cos^2\beta+\cos^4\beta)$$
$$=\sin^4\beta-\sin^2\beta\cos^2\beta+\cos^4\beta$$
$$=\sin^2\beta\sin^2\beta-\sin^2\beta\cos^2\beta+\cos^2\beta\cos^2\beta$$
$$=\sin^2\beta(1-\cos^2\beta)-\sin^2\beta\cos^2\beta+\cos^2\beta(1-\sin^2\beta)$$
$$=\sin^2\beta-\sin^2\beta\cos^2\beta-\sin^2\beta\cos^2\beta+\cos^2\beta-\sin^2\beta\cos^2\beta$$
$$=\sin^2\beta+\cos^2\beta-3\sin^2\beta\cos^2\beta$$
$$=1-3\sin^2\beta\cos^2\beta$$

Q9. (5 points): (7.2 Textbook Exercise 70): Refer to the figure. Find the exact value of $\gamma=?$.



Solution: Let $\angle A$ and $\angle B$ be the tw



$\angle B=90°-\beta$

$90°-\alpha+90°-\beta+\gamma=180°$

$180°-\alpha-\beta+\gamma=180°$

$\gamma=\alpha+\beta$

$$\tan\gamma=\tan(\alpha+\chi)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{\frac{4}{6}+\frac{3}{4}}{1-\frac{4}{6}\cdot\frac{3}{4}}=\frac{\frac{8}{12}+\frac{9}{12}}{1-\frac{1}{2}}=\frac{\frac{17}{12}}{\frac{1}{2}}$$

$$=\frac{17}{12}\cdot 2=\frac{17}{6}\quad\Rightarrow\quad\tan\gamma=\frac{17}{6}\quad\Rightarrow\quad\boxed{\gamma=\tan^{-1}\frac{17}{6}}\ \text{OR}\ \boxed{\gamma=\alpha+\beta=\cos^{-1}\frac{3}{\sqrt{13}}+\cos^{-1}\frac{4}{5}}$$

Q10. (5 points): Solve $4\sin 3\theta\cos 3\theta-2\sqrt{3}\sin 3\theta-2\sqrt{2}\cos 3\theta+\sqrt{6}=0$, where $0^\circ\le\theta<180^\circ$

Solution: $4\sin 3\theta\cos 3\theta-2\sqrt{3}\sin 3\theta-2\sqrt{2}\cos 3\theta+\sqrt{2}\sqrt{3}=0$

$2\sin 3\theta(2\cos 3\theta-\sqrt{3})-\sqrt{2}(2\cos 3\theta-\sqrt{3})=0$

$(2\cos 3\theta-\sqrt{3})(2\sin 3\theta-\sqrt{2})=0$

$2\cos 3\theta-\sqrt{3}=0$, $2\sin 3\theta-\sqrt{2}=0$

$\cos 3\theta=\frac{\sqrt{3}}{2}$, $\sin 3\theta=\frac{\sqrt{2}}{2}$

$3\theta=30°+k\,360°$, $3\theta=330°+k\,360°$, $3\theta=45°+k\,360°$, $3\theta=135°+k\,360°$ where k is an integer.

$\theta=10°+k\,120°$, $\theta=110°+k\,120°$, $\theta=15°+k\,120°$, $\theta=45°+k\,120°$

$k=0\ \Rightarrow\ \theta=10°$, $110°$, $15°$, $45°$

$k=1\ \Rightarrow\ \theta=130°$, $230°\notin[0°,180°)$, $135°$, $165°$ $SS=\{10°,15°,45°,110°,130°,135°,165°\}$