King Fahd University of Petroleum and Minerals

Prep-Year Math Program

Math 002 - Term 132

Reduction Identity

Solved by Sayed Omar

$$a\sin x + b\cos x = k\sin(x+\alpha)$$

where $k=\sqrt{a^2+b^2}$ and α is determined by: $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

Or $\tan\alpha = \dfrac{b}{a}$ where α can be detrmined from the quadrant that contains the point (a,b)

Question 1: Given the function $f(x) = 2\sin\frac{x}{3} - 2\sqrt{3}\cos\frac{x}{3}$

a) Rewrite $f(x)$ in the form $f(x) = k\sin(bx+\alpha)$

b) Find the amplitude, the phase shift, the period, and the range for the graph of $f(x)$.

c) Sketch the graph of the function $f(x) = 2\sin\frac{x}{3} - 2\sqrt{3}\cos\frac{x}{3}$ over two periods.

Solution:

(a): $f(x) = a\sin\frac{x}{3} + b\cos\frac{x}{3} = k\sin\left(\frac{x}{3}+\alpha\right)$

$a=2$, $b=-2\sqrt{3}$ $\Rightarrow$ $(2,-2\sqrt{3})$ is in Quadrant IV.

$k=\sqrt{a^2+b^2}=\sqrt{2^2+(-2\sqrt{3})^2}=\sqrt{4+12}=4$

$$\left.\begin{aligned}\sin\alpha &= \frac{b}{k} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}\\ \cos\alpha &= \frac{a}{k} = \frac{2}{4} = \frac{1}{2}\end{aligned}\right\} \Rightarrow \alpha \text{ is in Quadrant IV and } \alpha = -\frac{\pi}{3} \text{ OR } \alpha = \frac{5\pi}{3}$$

$f(x) = 4\sin\left(\frac{x}{3} - \frac{\pi}{3}\right)$ OR $f(x) = 4\sin\left(\frac{x}{3} + \frac{5\pi}{3}\right)$

(b): Amplitude $=4$

$\text{Phase shift} = -\dfrac{-\frac{\pi}{3}}{\frac{1}{3}} = \pi$ OR $\text{Phase shift} = -\dfrac{\frac{5\pi}{3}}{\frac{1}{3}} = 5\pi$

$\text{Period} = \dfrac{2\pi}{1/3} = 6\pi$ $\quad$ Range $=[-4,4]$

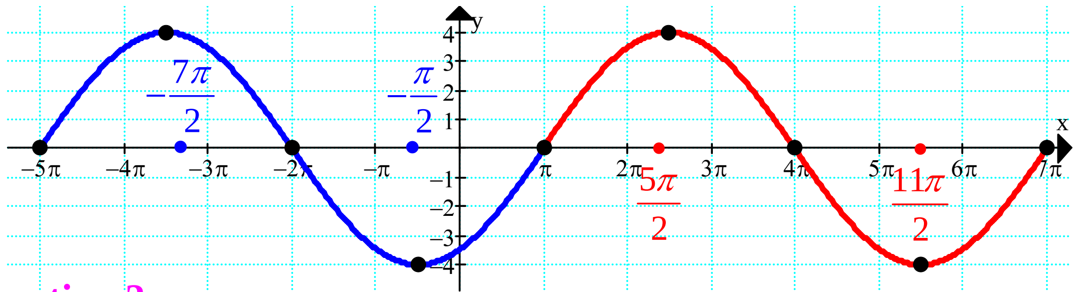
Beginning key point $=$ Phase shift $=\pi$ OR Beginning key point $=$ Phase shift $=-5\pi$

First quarter key point $=$ phase shift $+\frac{1}{4}$period $=\pi+\frac{1}{4}(6\pi)=\pi+\frac{3\pi}{2}=\frac{5\pi}{2}$

The key points are $\pi = \frac{2\pi}{2}$, $\frac{5\pi}{2}$, $\frac{8\pi}{2} = 4\pi$, $\frac{11\pi}{2}$, $\frac{14\pi}{2} = 7\pi$

The key points are -5π , $-5\pi + \frac{1}{4}Period = -5\pi + \frac{3\pi}{2} = -\frac{7\pi}{2}$, $-\frac{4\pi}{2}$, $-\frac{\pi}{2}$

(c): Graph of $f(x) = 4\sin\left(\frac{x}{3} + \frac{5\pi}{3}\right)$ is: Graph of $f(x) = 4\sin\left(\frac{x}{3} - \frac{\pi}{3}\right)$ is:



Question 2:

If $\sin 20^\circ - \sqrt{3}\cos 20^\circ = k\sin\theta,\ 0^\circ < \theta < 90^\circ$. Then k and θ are equal to

A) $-2,\ 40^\circ$

B) $2,\ 20^\circ$

C) $1-\sqrt{3},\ 20^\circ$

D) $-2,\ 20^\circ$

E) $-2,\ 30^\circ$

Solution: $\sin 20^\circ - \sqrt{3}\cos 20^\circ = \sqrt{a^2+b^2}\sin\left(20^\circ + \alpha\right) = k\sin\theta,\ 0^\circ < \theta < 90^\circ$

$$\sqrt{a^2+b^2} = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\left.\begin{aligned}\sin\alpha &= \frac{b}{k} = \frac{-\sqrt{3}}{2}\\ \cos\alpha &= \frac{a}{k} = \frac{1}{2}\end{aligned}\right\} \Rightarrow \begin{array}{l}\alpha \in \text{IV}\\ \boxed{\alpha = -60^\circ}\ \text{OR}\ \boxed{\alpha = 300^\circ}\end{array}$$

$\alpha = -60^\circ$**:**
$$\begin{aligned}\sin 20^\circ - \sqrt{3}\cos 20^\circ &= k\sin\left(20^\circ + \alpha\right)\\ &= 2\sin\left(20^\circ - 60^\circ\right)\\ &= 2\sin\left(-40^\circ\right)\\ &= -2\sin 40^\circ = k\sin\theta \quad \Rightarrow \quad \boxed{k = -2},\ \boxed{\theta = 40^\circ}\end{aligned}$$

OR $\alpha = 300°$**:**
$$\begin{aligned}\sin 20^\circ - \sqrt{3}\cos 20^\circ &= k\sin\left(20^\circ + \alpha\right)\\ &= 2\sin\left(20^\circ + 300^\circ\right)\\ &= 2\sin\left(320^\circ\right)\\ &= -2\sin\left(-320^\circ\right)\\ &= -2\sin\left(-320^\circ + 360°\right)\\ &= -2\sin\left(40^\circ\right)\end{aligned}$$