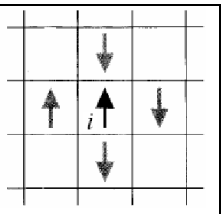
The Mean Field Approximation (Schroeder page 343)

This is a very crude approximation, which can be used to "solve" the Ising model in any dimensionality. This approximation won't be very accurate, but it does give some qualitative insight into what's happening and why the dimensionality matters. Let's concentrate on just a single dipole, somewhere in the middle of the lattice and label it "i" (see Figure 1), so its alignment is s_i which:

$$s_i = \begin{cases} +1 & \text{when the dipole is pointing up} \\ -1 & \text{when the dipole is pointing down} \end{cases}.$$

Let $n = 4$ be the number of nearest neighbors that this dipole has.

Figure 1. The four neighbors of this particular dipole have an average $\bar{s}$ value of $\frac{(1-3)}{4} = -\frac{1}{2}$. If the central dipole points up, the energy due to its interactions with its neighbors is $n\bar{s} = +2\varepsilon$, while if it points down, the energy is -2ε.



Imagine that the alignments of these neighboring dipoles are temporarily frozen, but that our dipole "i" is free to point up or down. In general, we have the interaction energy between this dipole and its neighbors is

$$E = -\varepsilon n s_i \bar{s} \Rightarrow \begin{cases} E_\uparrow = -\varepsilon n\bar{s} & s_i = +1 \\ E_\downarrow = \varepsilon n\bar{s} & s_i = -1 \end{cases}$$

If s_i points up, then the interaction energy between this dipole and its neighbors is

$$E_\uparrow = -\epsilon \sum_{\text{neighbors}} s_{\text{neighbor}} = -\epsilon\, n\bar{s}, \qquad (8.46)$$

where $\bar{s}$ is the *average* alignment of the neighbors and"ε" is a positive quantity represents the dipole-dipole interaction. Similarly, if s_i points down, then the interaction energy is

$$E_\downarrow = +\epsilon\, n\bar{s}. \qquad (8.47)$$

The partition function for just this dipole is therefore ($Z_i = \sum_i e^{-\beta E_i}$)

$$Z_i = e^{\beta\epsilon n\bar{s}} + e^{-\beta\epsilon n\bar{s}} = 2\cosh(\beta\epsilon n\bar{s}), \qquad (8.48)$$

and the average expected value of its spin alignment is ($\bar{s}_i = \frac{1}{Z_i}\sum_i s_i\, e^{-\beta E_i}$)

$$\bar{s}_i = \frac{1}{Z_i}\left[(1)e^{\beta\epsilon n\bar{s}} + (-1)e^{-\beta\epsilon n\bar{s}}\right] = \frac{2\sinh(\beta\epsilon n\bar{s})}{2\cosh(\beta\epsilon n\bar{s})} = \tanh(\beta\epsilon n\bar{s}). \qquad (8.49)$$

Now look at both sides of this equation (8.49). On the left is $\bar{s}_i$, the thermal average value of the alignment of any typical dipole (except those on the edge of the lattice, which we'll neglect). On the right is $\bar{s}$, the average of the actual instantaneous alignments of this dipole's *n* neighbors. The idea of the mean field approximation is to assume (or pretend) that these two quantities are the same: $\bar{s}_i = \bar{s}$. In other words, we assume that at every moment, the alignments of all the dipoles are such that every neighborhood is "typical"-there are no fluctuations $\left(s_i - \bar{s}_i\right)$ that cause the magnetization in any neighborhood to be more or less than the expected thermal average.

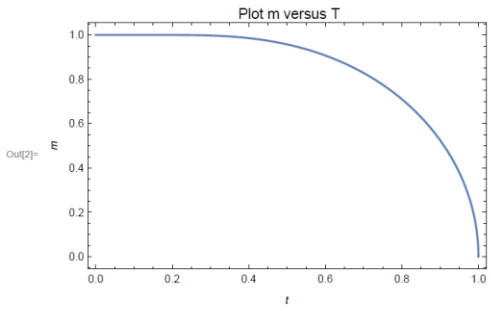
In the mean field approximation $\left(\text{when } \bar{s}_i \to \bar{s}\right)$, then, we have the relation

$$\bar{s} = \tanh(\beta\epsilon n\bar{s}), \qquad (8.50)$$

where $\bar{s}$ is now the average dipole alignment over the entire system. This is a transcendental equation, so we can't just solve for $\bar{s}$ in terms of $\beta\varepsilon n$.

```
In[1]:= (*    Plot the two-dimensional square lattice,n = 4 so equation
         m =tanh(n m/t) where t=kT/ϵ. *)

In[2]:= Plot[FindRoot[m == Tanh[1 m / T], {m, .5}][[1, 2]], {T, 0, 1}, PlotRange → All,
        Frame → True, PlotLabel → " Plot m versus T", FrameLabel → {t, m}]
```

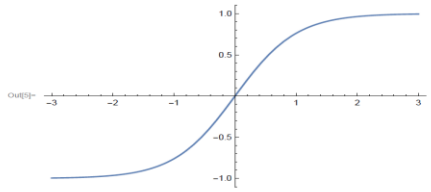


Mathematica program to plot equation $m = \tanh\left[\beta\varepsilon nm\right]$.

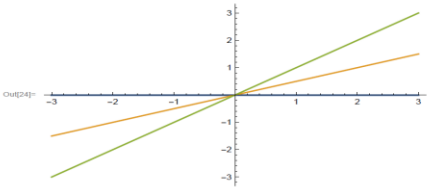
Plot the one-dimensional lattice, n = 2, $m = \tanh\left(m/t\right)$,

$t = k_B T/\varepsilon$.

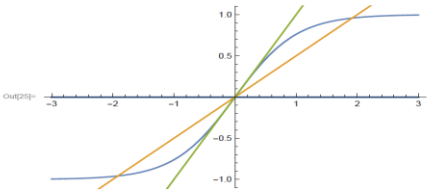
```
In[3]:= (*Critical behavior of the spontaneous magnetization S=
In[4]:= y1 = Tanh[x]; y2 = t x;
In[5]:= p1 = Plot[y1, {x, -3, 3}]
```




```
In[24]:= p2 = Plot[Evaluate[Table[y2, {t, 0, 1, 0.5}]], {x, -3, 3}]
```




```
In[25]:= Show[p1, p2]
```



The best approach is to plot both sides of the equation and look for a graphical solution (see Figure 2). Notice that the larger the value of $\beta\varepsilon n$, the steeper the slope of the hyperbolic tangent function near $\bar{s} = 0$. This means that our equation can have either one solution or three, depending on the value of $\beta\varepsilon n$.

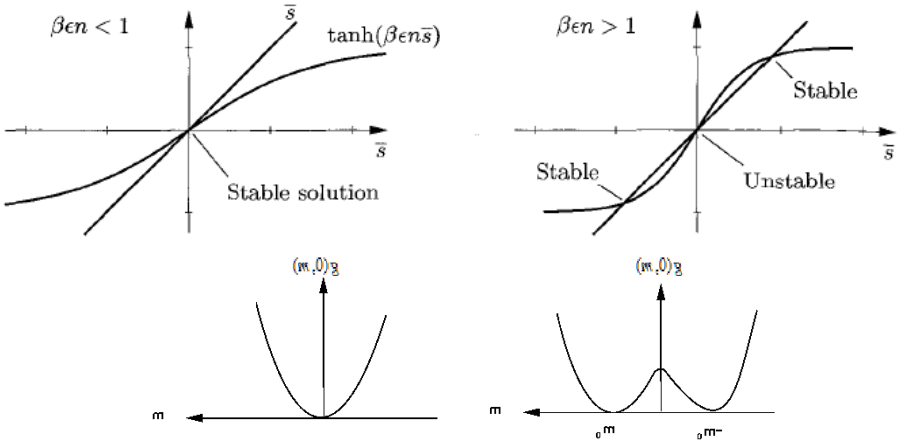


Figure 2. Graphical solution of equation 8.50. The slope of the tanh function at the origin is $\beta\varepsilon n$. When this quantity is less than 1, there is only one solution, at $\bar{s} = 0$; when this quantity is greater than 1, the $\bar{s} = 0$ solution is unstable but there are also two nontrivial stable solutions.

When $\beta\varepsilon n < 1$, that is, when $kT > n\varepsilon$ (high temperature), the only solution is at $\bar{s} = 0$; there is no net magnetization. If a thermal fluctuation were to momentarily increase the value of $\bar{s}$, then the hyperbolic tangent function, which dictates what $\bar{s}$ *should* be, would be less than the current value of $\bar{s}$, so $\bar{s}$ would tend to decrease back to zero. The solution $\bar{s} = 0$ is stable.

When $\beta\varepsilon n > 1$, that is, when $kT < n\varepsilon$ (low temperature), we still have a solution at $\bar{s} = 0$ and we also have two more solutions, at positive and negative values of $\bar{s}$. But ***the solution at*** $\bar{s} = 0$ ***is unstable***: A small positive fluctuation of $\bar{s}$ would cause the hyperbolic tangent function to exceed the current value of $\bar{s}$, driving $\bar{s}$ to even higher values. The stable solutions are the other two, which are symmetrically located because the system has no inherent tendency toward positive or negative magnetization. Thus, the system will acquire a net nonzero magnetization, which is equally likely to be positive or negative. When a system has a built-in symmetry such as this, yet must choose one state or another at low temperatures, we say that the symmetry is **spontaneously broken**.

The critical temperature T_c below which the system becomes magnetized is

$$\beta_c n\varepsilon = 1 \quad \Rightarrow \quad kT_c = n\varepsilon \tag{8.51}$$

proportional to both the neighbor-neighbor interaction energy and to the number of neighbors. This result is no surprise: The more neighbors each dipole has, the greater the tendency of the whole system to magnetize. Notice, though, that even a one-dimensional Ising model should magnetize below a temperature of $2\varepsilon/k$, according to this analysis. Yet we already saw from the exact solution that there is no abrupt transition in the behavior of a one-dimensional Ising model; it magnetizes only as the temperature goes to zero. Apparently, the mean field approximation is no good at all in one dimension. *Fortunately, the accuracy improves as the dimensionality increases.

d	$k_B T_c = \varepsilon n$	Exact	Comment
1	2ε	0 (No PT)	Wrong
2	4ε	2.269ε	overestimated
3	6ε	4.511ε	better

Here we cannot calculate the critical exponent.