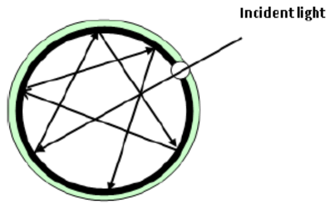
Bose-Einstein Gases
Chapter 18.1, .2

The amount of heat radiation emitted by a body depends on three things:

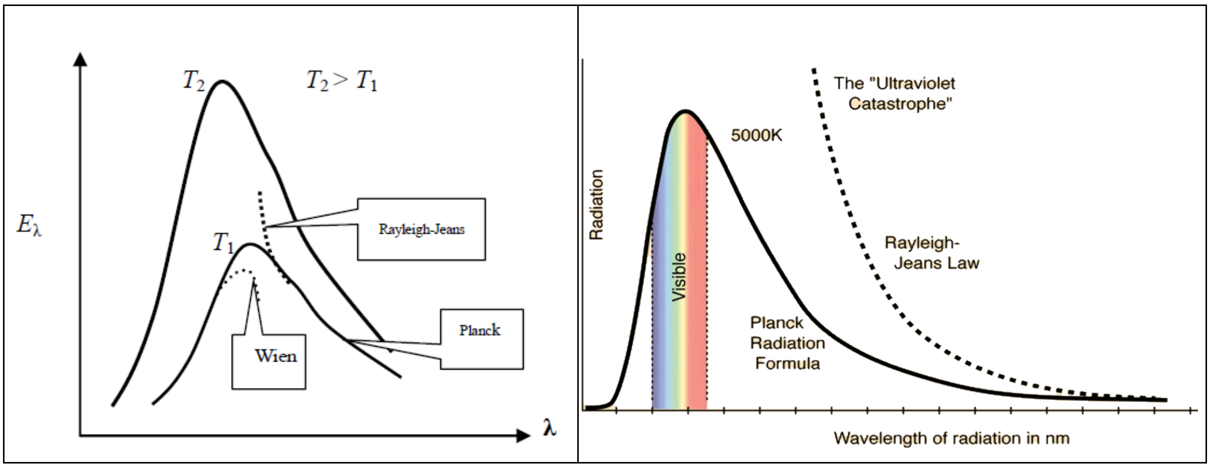
I- the surface area of the body,
II- the type of surface, and
III- the temperature of the body.



Ideal model for the black body

Comments and laws of black body radiation:

1- Electromagnetic radiation in thermal equilibrium inside an enclosure.
2- Black surfaces are the best emitters and absorbers of radiation at a given temperature.
3- The distribution of the energy flux over the wavelength spectrum does not depend on the nature of the body but does depend on its temperature.
4- The maxima of the curves tend towards short wavelengths at higher temperature.
5- The area between any curve and the wave length axis gives the total energy emitted by the body at that temperature (σT^4) Stefan's law.
6- The curves at lower temperature lie completely inside those of higher temperature.
7- Stefan's law: The total energy flux, φ, (total energy emitted by a black body per unit area of surface per second) is proportional to the fourth power of the body's absolute temperature (T), $\varphi = \sigma T^4$, where $\sigma = 5.67\times10^{-8}$ W m^{-2} K^{-4} is the Stefan's constant.
8- Wien's displacement law: $\lambda_{max} T = 2.9\times10^{-3}$ m K. λ_{max} is the wavelength at which most energy is emitted, that is the peak of the curve. Energy emitted at this wavelength is proportional to T^5.



8.1 BLACKBODY RADIATION

Statistical thermodynamics is applicable to radiant energy as well as material particles. It is a familiar observation that a hot body loses heat by radiation. The energy loss is attributable to the emission of electromagnetic waves from the body. The distribution of the energy flux over the wavelength spectrum does not depend on the nature of the body but does depend on its temperature.

Here we are concerned with the thermodynamic properties of electromagnetic radiation in thermal equilibrium. The radiation can be regarded as a photon gas. We consider an enclosure or cavity of volume V at a constant temperature T . The walls of the cavity are thermally insulated and perfectly reflecting. Since the system is isolated, it has a fixed energy *U*. However, the photons emitted by one energy level may be absorbed at another, so the number of photons is not constant. This means that the restriction $\delta N = \sum \delta n_i = 0$ does not apply. Correspondingly, the Lagrange multiplier α that was determined by this condition is zero and $e^{-\alpha} = 1$.

Photons are bosons of spin 1 and hence obey Bose-Einstein statistics. The number of photons per quantum state is therefore given by with μ set equal to zero (recall that $\alpha = \mu / k_B T$):

$$f_j = \frac{N_j}{g_j} = \frac{1}{e^{\beta\varepsilon_j} - 1}$$

For a continuous spectrum of energies,

$$f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{\beta\varepsilon} - 1} \qquad (1)$$

The energy of a photon is $h\nu$, so this equation can be written

$$f(\nu) = \frac{N(\nu)}{g(\nu)} = \frac{1}{e^{\beta h\nu} - 1} \qquad (2)$$

Here $g(\nu)d\nu$ is the number of quantum states with frequencies in the range ν to $\nu + d\nu$ and is:

$$g(\nu)d\nu = \left(\frac{V}{c^3}\right) 8\pi\, \nu^2 d\nu \qquad (3)$$

where c in this case is the speed of light.

Planck radiation formula

The energy $u(\nu)d\nu$ in the range ν to $\nu + d\nu$ is the number of photons in this range times the energy $h\nu$ of each:

$$u(\nu)d\nu = h\nu N(\nu)d\nu = h\nu g(\nu) f(\nu)d\nu = 8\pi h\left(\frac{V}{c^3}\right)\left[\frac{\nu^3 d\nu}{e^{\beta h\nu} - 1}\right] \qquad \text{(A)}$$

This is the **Planck** radiation formula. It gives the spectral distribution of the radiant energy inside the enclosure, i.e., the energy per unit frequency.

The blackbody spectrum is often expressed in terms of the wavelength. Then
$u(\nu)d\nu \propto u(\lambda)d\lambda$
And

$$d\nu = d\left(\frac{c}{\lambda}\right) = -\left(\frac{c}{\lambda^2}\right)d\lambda \quad \text{or} \quad |d\nu| = \frac{c}{\lambda^2}|d\lambda|$$

The wavelength spectrum is therefore

$$u(\lambda)d\lambda = 8\pi hcV\left[\frac{d\lambda}{\lambda^5\left(e^{hc/\lambda k_B T} - 1\right)}\right] \qquad \text{(B)}$$

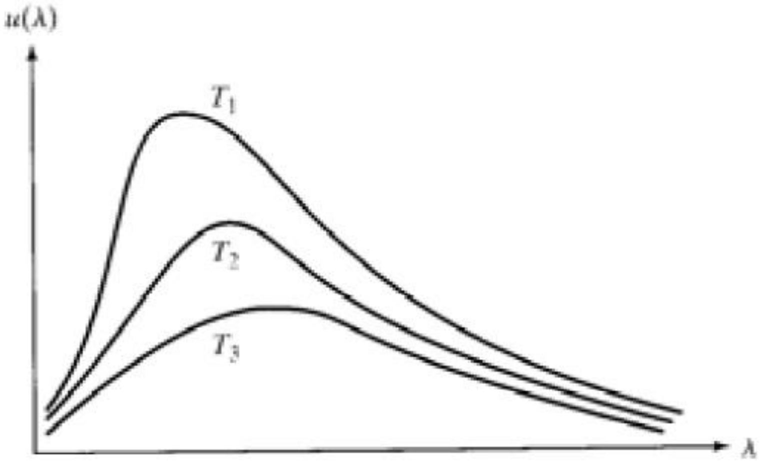


Figure 1 The wavelength spectrum of blackbody radiation energy for three temperatures: $T1 > T2 > T3$.

(Figure 1). Here $u(\lambda)$ is the energy per unit wavelength. Prior to Planck's analysis, various empirical formulas existed. All of them can be found from Equations (A) or (B).

Stefan-Boltzmann's law

The Stefan-Boltzmann law states that the total radiation energy is proportional to T^4 (the area under the curves of Figure 1).
The total energy density (energy per unit volume) is

$$\frac{U}{V} = 8\pi hc\int_0^\infty \frac{d\lambda}{\lambda^5\left(e^{hc/\lambda k_B T} - 1\right)} \qquad \text{(C)}$$

H.W. By setting $x = \frac{hc}{\lambda k_B T}$, $so \quad dx = -\frac{hc}{\lambda^2 k_B T}d\lambda$, Check the following:

$$\frac{U}{V} = 8\pi hc \int_0^\infty \frac{d\lambda}{\lambda^5\left(e^{hc/\lambda kT}-1\right)} = 8\pi hc \int_0^\infty \frac{1}{\lambda^5\left(e^{hc/\lambda kT}-1\right)} \frac{\lambda^2 kT}{hc} dx$$
$$= 8\pi kT \int_0^\infty \frac{1}{\lambda^3\left(e^{hc/\lambda kT}-1\right)} dx = 8\pi kT \int_0^\infty \left(\frac{kTx}{hc}\right)^3 \frac{1}{\left(e^x-1\right)} dx$$
$$= \frac{8\pi}{(hc)^3}(kT)^4 \int_0^\infty \frac{x^3 dx}{e^x-1} = aT^4,$$

where the numerical value of the constant *a* is

$$a = \frac{8\pi^5 k_B^4}{15c^3h^3} = 7.55\times10^{-16} \text{ J m}^{-3}\text{K}^{-4}$$ **Stefan-Boltzmann's constant**

Hence:

$$c_v = \left(\frac{\partial u}{\partial T}\right)_V = 4aT^3$$

Energy flux

Since radiation inside the cavity is continually absorbed and emitted by the inner surface, we can relate the energy per unit volume, which moves at the speed of light *c*, to the energy emitted per unit area of the surface per unit time. The latter is the power per unit area or energy flux. From elementary kinetic theory we found the particle flux to be $\langle \text{v}\rangle\langle \text{n}\rangle/4$, where $\langle \text{v}\rangle$ is the mean speed and $\langle \text{n}\rangle$ is the number of particles per unit volume. In a similar way, the energy flux *e* is

$$\text{energy flux } e = \frac{\text{power}}{\text{area}} = \frac{\text{energy}}{\text{time}\times\text{area}} = \frac{\text{energy}\times\text{length}}{\text{area}\times\text{length}\times\text{time}}$$
$$= \frac{\text{energy}\times\text{length}}{\text{volume}\times\text{time}} = \left(\frac{\text{energy}}{\text{volume}}\right)\times\left(\frac{\text{length}}{\text{time}}\right) = \left(\frac{U}{V}\right)\left(\frac{c}{4}\right).$$

Wien displacement law

The wavelength $\lambda_{\max}$ at which $u(\lambda)$ is a maximum satisfies a relation known as **Wien's** displacement law. It can be found by setting the derivative of $u(\lambda)$ equal to zero or, equivalently, by minimizing the denominator in Equation (B):

$$\frac{d}{d\lambda}\left[\lambda^5\left(e^{hc/\lambda k_B T}-1\right)\right] = 0$$

Or

$$0 = \frac{d\lambda}{dx}\frac{d}{d\lambda}\left\{\left(\frac{hc}{xkT}\right)^5\left(e^x-1\right)\right\} = \frac{d}{dx}\left\{x^{-5}\left(e^x-1\right)\right\}$$

Where $x = \frac{hc}{\lambda k_B T}, \quad so \quad dx = -\frac{hc}{\lambda^2 k_B T} d\lambda$

$$\frac{d}{dx}\left\{x^{-5}\left(e^{x}-1\right)\right\}=-5x^{-6}\left(e^{x}-1\right)+x^{-5}e^{x}=0 \quad\Rightarrow\quad 5\left(e^{x}-1\right)=xe^{x} \quad\Rightarrow\quad \left(1-e^{-x}\right)=\frac{x}{5}$$

This is a transcendental equation whose numerical solution is 4.96. Hence
$\lambda_{\max}T$ = constant ,

$$\lambda_{\max}T=\frac{hc}{4.96\,k_B}=2.9\times10^{-3}\ \text{ m K}$$

Last equation is Wien's displacement law. The experimental values of the constant on the right-hand side and of the Stefan-Boltzmann constant can be used to determine the values of *h* and *k*, assuming that *c* is known.

Rayleigh-Jeans formula

For long wavelengths, $hc/\lambda k_B T \ll 1$ in Equation (B), and the exponential can be approximated by the first two terms of its Taylor series expansion:
$e^{hc/\lambda k_B T} \approx 1+\left(hc/\lambda k_B T\right)+\cdots$

then

$$u(\lambda)d\lambda \approx V\frac{8\pi k_B T}{\lambda^4}d\lambda, \qquad \text{Rayleigh-Jeans law}$$

This is the so-called **Rayleigh-Jeans** formula, which exhibits an "ultraviolet catastrophe" (as the wavelength approaches zero $u(\lambda)$ becomes infinite)

Wien's law

For sort wavelengths, $e^{hc/\lambda k_B T} >> 1$ and

$$u(\lambda)d\lambda \approx V\left(\frac{8\pi k_B T}{\lambda^5}\right)e^{hc/\lambda k_B T}d\lambda, \qquad \text{Wien's law}$$

This is **Wien's** law, valid in the short wavelength region (Figure 2).

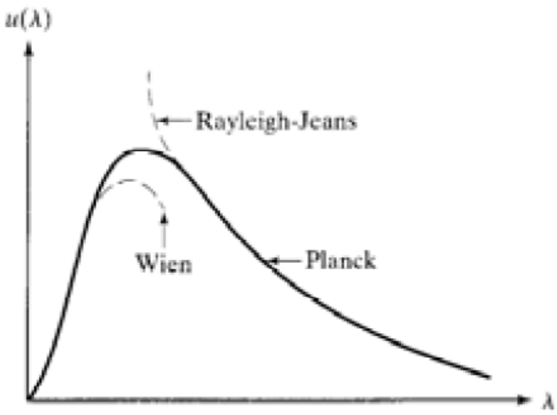


Figure 2 Sketch of Planck's law, Wien's law and the Rayleigh-Jeans law.

To summarize, the total blackbody radiant energy per unit volume increases with the fourth power of the temperature, and the wavelength of the peak of the radiation curve $u(\lambda)$ is inversely proportional to T.
The temperature of the Sun's surface
is approximately 6000 K, and λ_{max} is 483 nm, a wavelength in the visible range of the electromagnetic spectrum. At the surface temperature of the Earth, roughly 300 K, λ_{max} is about 10μ*m*, which is in the infrared region.

8.2 PROPERTIES OF A PHOTON GAS

It is instructive to ask: what is the average value of the ratio $h\nu/kT$ in a volume V at temperature T ? The number of photons having frequencies between ν to $\nu + d\nu$ is found by combining Equations (8.2) and (8.3):

$$N(\nu)d\nu = \frac{g(\nu)d\nu}{e^{\alpha+\beta\varepsilon}-1} = 8\pi\left(\frac{V}{c^3}\right)\frac{\nu^2 d\nu}{e^{h\nu/k_BT}-1}$$

The total number of photons in the cavity is determined by integrating this expression over the infinite range of frequencies. The result is

$$N = 8\pi V\left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2dx}{e^x-1}$$

where the substitution $x = h\nu/k_BT$ has been made. The integral has the numerical value 2.404. Hence

$$N = 2.02\times 10^7 T^3 V,$$

where T is in kelvin and V is in m^3. To find the mean energy of the photons in the cavity, we divide the total energy $U = 7.55 \times 10^{-16} T^4 V$ J (Equation (8.8)) by N The result is

$$\frac{U}{N} = \frac{7.55\times 10^{-16} T^4 V \text{ J}}{2.02\times 10^7 T^3 V} = 3.74\times 10^{-23} T \approx 2.7kT.$$

H.W.: Prove that in case no restrain on n_i , i.e. $\Delta n_i \neq 0$, the partition function is:

$$Z_{\text{photon}} = \prod_{i=1}^{\infty}\frac{1}{1-e^{\beta\varepsilon_i}}$$

Then,

$$F = -\frac{1}{3}bVT^4,$$

$$S = \frac{4}{3}bVT^3,$$

$$U = F + TS = bVT^4,$$

$$P = \frac{1}{3}bT^4 = \frac{1}{3}\frac{U}{V}$$