

### CLASSICAL LIMIT

In classical mechanics we can specify simultaneously both  $q_i$  and  $p_i$  for a particle. In quantum mechanics the uncertainty principle prevents this. A classical description is a reasonable approximation only when the effect of  $h$  is negligible, i.e.

$$\Delta q \Delta p \gg h$$

Consider the motion of a molecule in a gas. If  $p_{av}$  denotes the mean momentum and  $r_{av}$  its mean separation from other identical molecules, then a classical description is valid when

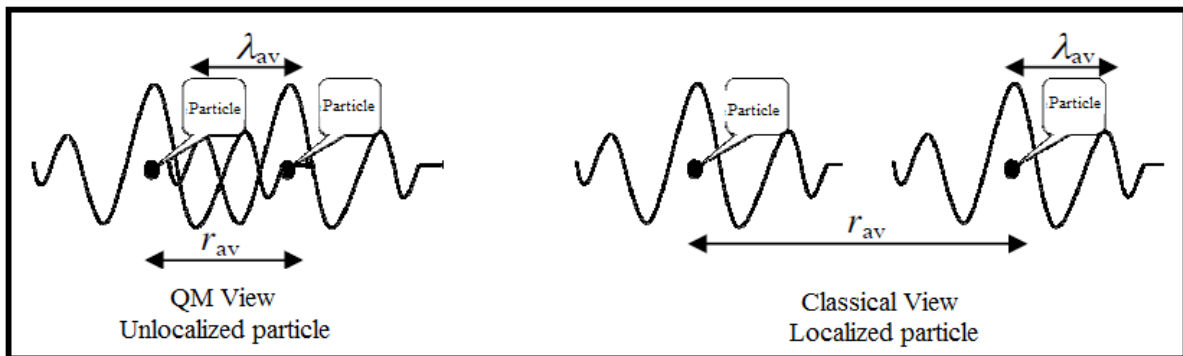
$$r_{av} p_{av} \gg h$$

or, Using De Broglie wavelength,  $p = h / \lambda$ , when

$$r_{av} \gg \lambda_{av} \quad (\text{classical limit});$$

$$r_{av} \ll \lambda_{av} \quad (\text{quantum limit});$$

Since  $\lambda_{av}$  is a measure of the spread of molecule in space, it means that when  $r_{av} \gg \lambda_{av}$  holds, the molecular wave functions do not overlap and therefore they are distinguished by their position.



To give a physical content to  $r_{av} \gg \lambda_{av}$ , we imagine that each particle occupies a tiny cube of side  $r_{av}$  and these cubes fill the volume  $V$ ,

$$r_{av}^3 N = V, \quad r_{av} = \left( \frac{V}{N} \right)^{1/3}$$

If we anticipate and relate the temperature  $T$  with the average energy  $\bar{\epsilon}$  by:

$$\frac{p_{av}^2}{2m} \approx \bar{\epsilon} = \frac{3}{2} k_B T, \quad p_{av} \approx (3mk_B T)^{1/2}$$

Where  $k_B$  is the Boltzmann constant, then

$$\lambda_{av} \approx \frac{h}{(3mk_B T)^{1/2}}$$

Therefore, the classical condition becomes

$$\left( \frac{V}{N} \right)^{1/3} \gg \frac{h}{(3mk_B T)^{1/2}}$$

This means the classical description is valid when:

- 1-  $N$  is small (dilute gas),
- 2-  $T$  is large, and
- 3-  $m$  is not too small.

### Examples:

- (i) Molecules in a gas at NTP

Using  $\rho_{molecules} = 10^{25} \text{ molecules m}^{-3}$ ,

We have the available volume  $V_{molecule} = 10^{-25} \text{ m}^3$ ,

Using  $r_{molecule} = 10^{-10} \text{ m}$

We have the actual volume  $V_{molecule} = 10^{-30} \text{ m}^3$

So, in gases, the volume of a molecule is usually being much smaller than the volume available to it, we can, in principle, identify each molecule in the gas. Therefore, the molecules are localized and distinguishable.

- (iii) Conduction electrons in metal

Using  $\rho_{electrons} = 10^{28} \text{ electrons m}^{-3}$ ,

We have the available volume  $V_{electron} = 10^{-28} \text{ m}^3$ ,

Using  $r_{electron} = \frac{h}{p} = \frac{h}{\sqrt{2mE (1eV)}} = 10^{-9} \text{ m}$

We have the actual volume  $V_{electron} = 10^{-27} \text{ m}^3$

So, in metals, the volume of the electron is larger than the available volume. Hence the electron's wave functions overlap considerably. We can not localize the electrons, they are indistinguishable, and quantum statistics must be applied.

- (iii) We can also use the above analysis to estimate a transition temperature for the onset of quantum effects. For example, we take He atoms in a fluid phase and conduction electrons in an alkali

metal. From  $\left(\frac{V}{N}\right)^{1/3} \gg \frac{h}{(3mk_B T)^{1/2}}$  we get:

$$T = \left(\frac{h^2}{3mk_B}\right) \left(\frac{N}{V}\right)^{2/3}$$

$$= \begin{cases} 1.2 \times 10^{-14} (N/V)^{2/3} \text{ K m}^2, \text{ electrons} \\ 1.6 \times 10^{-18} (N/V)^{2/3} \text{ K m}^2, \text{ He atoms} \end{cases}$$

for He atoms  $\frac{V}{N} \approx 5 \times 10^{-29} \text{ m}^3 \text{ atom}^{-1} \Rightarrow \underline{\underline{T \approx 10 \text{ K}}}$

for electrons  $\frac{V}{N} \approx 1 \times 10^{-29} \text{ m}^3 \text{ electron}^{-1} \Rightarrow \underline{\underline{T \approx 3 \times 10^5 \text{ K}}}$

So we can see that at room temperature a classical treatment of atomic fluids is allowable whereas we must use a quantum description of electrons in a metal up to a very high temperature. Gases at room temperature have a density 1000 times less than a fluid and so will have a transition temperature even lower than 10 K. They can clearly be treated as classical under most conditions.

**Symmetry of wave functions:** Matter waves are described by a complex valued wave function  $\Psi(\mathbf{r}, t)$  obeying the Schrodinger equation

$$\hat{H}\Psi(\mathbf{r}, t) = E\Psi(\mathbf{r}, t), \quad \hat{H} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right], \quad E = i\hbar\frac{\partial}{\partial t}$$

which is the fundamental equation in quantum mechanics. The simplest of  $N$  identical noninteracting particles could be described by the Schrodinger equation

$$\hat{H}\Psi = E\Psi, \quad \hat{H} = \sum_{i=1}^N \frac{p_i^2}{2m}, \quad \Psi = \Psi(r_1, r_2, \dots, r_N) \quad \#$$

where  $p_i^2 = \mathbf{p}_i \cdot \mathbf{p}_i$ , and  $\mathbf{p}_i$  is the momentum operator of the  $i$ th particle. The Hamiltonian is independent of the positions of the particles or any other coordinates. e.g., spin, if any.

Introduce the *permutation operator*  $\hat{P}$ , which interchanges particle  $i$  with particle  $j$ , i.e.  $r_i \leftrightarrow r_j$ . For example:

$$\hat{P}_{12}\Psi_a(1)\Psi_b(2) = \Psi_a(2)\Psi_b(1) \text{ or } \hat{P}_{12}\Psi(1, 2) = \Psi(2, 1)$$

The Hamiltonian  $\hat{H}$  is invariant under all permutation of the identical particles:

$$\hat{P}\hat{H}\hat{P}^{-1} = \hat{H} \Rightarrow \hat{P}\hat{H}\hat{P}^{-1}\hat{P} = \hat{H}\hat{P} \Rightarrow [\hat{H}, \hat{P}] = 0$$

The eigenstates of  $\hat{H}$  should also be eigenstates of  $\hat{P}$ . In general, the single product wave function

$$\Psi = \Psi(r_1, r_2, \dots, r_N) = \prod_i^N \Psi_k(r_i)$$

is not an eigenstate of  $\hat{P}$ . An eigenstate must satisfy

$$\hat{P}\Psi(r) = p\Psi(r) \Rightarrow \hat{P}^2\Psi(r) = p\hat{P}\Psi(r) = p^2\Psi(r)$$

$$\hat{P}^2 = \hat{1} \Rightarrow p^2 = 1 \Rightarrow p = \pm 1$$

then

$$\hat{P}\Psi(r) = \Psi(r) \Rightarrow \text{Symmetric wavefunction}$$

$$\hat{P}\Psi(r) = -\Psi(r) \Rightarrow \text{Antisymmetric wavefunction}$$

It is a *law of nature* that the symmetry ( $S$ ) or antisymmetric ( $A$ ) under the interchange of two particles is a property of the particles themselves.

**Example**, for two-particle system we have

$$\Psi(1, 2) = \Psi_a(1)\Psi_b(2) \quad (\text{classical}) \quad \#$$

$$\Psi^{(S)}(1, 2) = \Psi^{(S)}(2, 1) = \frac{1}{\sqrt{2}}[\Psi_a(1)\Psi_b(2) + \Psi_a(2)\Psi_b(1)] \quad (\text{symmetric}) \quad \#$$

$$\Psi^{(A)}(1, 2) = -\Psi^{(A)}(2, 1) = \frac{1}{\sqrt{2}}[\Psi_a(1)\Psi_b(2) - \Psi_a(2)\Psi_b(1)] \quad (\text{antisymmetric}) \quad \#$$

For  $N$  fermions, we have the Slater's determinant

$$\Psi^{(A)}(1, 2) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \Psi_a(1) & \Psi_a(2) & \dots & \dots & \Psi_a(N) \\ \Psi_b(1) & \Psi_b(2) & \dots & \dots & \Psi_b(N) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \Psi_z(1) & \Psi_z(2) & \dots & \dots & \Psi_z(N) \end{vmatrix} \quad \#$$

Due to Pauli, with interchanges all the coordinates of the particles, then identical particles could be divided into two groups, Bose gas and Fermi gas. The following table shows the main properties of each system plus the unnatural gas, Boltzmann gas.

**Maxwell-Boltzmann statistics** This is the classical statistics and is applicable for a system of independent, distinguishable (modified by  $N!$  for indistinguishable) and identical particles. No restriction about the symmetry, spins, etc.

**Bose-Einstein statistics** is applicable for a system of independent, indistinguishable and identical particles of integral spin (like photons). There is no restriction on the number of particles present in any given quantum state.

**Fermi-Dirac statistics** is applicable for a system of independent, indistinguishable and identical particles of half-integral spin (like electrons). In this statistics, a given energy state can be occupied at best by one particle only.

	<b>B.E. (Bosons)</b>	<b>F.D. (Fermions)</b>	<b>M.B</b>
Particles	Indistinguishable	Indistinguishable	Distinguishable
Wave-functions	Symmetric	Antisymmetric	Any
Spin	$0, \hbar, 2\hbar, \dots$	$\frac{1}{2}\hbar, \frac{3}{2}\hbar, \dots$	Any
Examples	photons, $\pi$ -meson, ${}^4\text{He}$	electron, proton, ${}^3\text{He}$	Any
$n_i$	0, 1, 2, .....	0, 1	0, 1, 2, .....
$\omega(i)$	$\omega_{BE} = \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$	$\omega_{FD} = \frac{g_i!}{n_i!(g_i - n_i)!}$	$\omega_{MB} = \frac{g_i^{n_i}}{n_i!}$ (*)
$\Omega$	$\Omega_{BE} = \prod_{i=1}^r \omega_{BE}$	$\Omega_{FD} = \prod_{i=1}^r \omega_{FD}$	$\Omega_{MB} = \prod_{i=1}^r \omega_{MB}$
$f(\epsilon_i) = \frac{N_i}{g_i}$	$\frac{1}{e^{-\alpha+\beta\epsilon_i} - 1}$	$\frac{1}{e^{-\alpha+\beta\epsilon_i} + 1}$	$e^{-\alpha-\beta\epsilon_i}$
Applications	Photons of radiation, gas molecules at very low temperature.)	Free electrons in metal and semiconductor (except at very High temperature.)	Gas molecules (except near 0 K), electrons at Extremely High temperature.

(\*) corrected by "correct Boltzmann counting" i.e. divided by  $N!$  This correction does not correspond to any physical property of the particles in the system. It is just a rule that defines the mathematical model for Indistinguishable particles.

\*\* Define the dilute gas, dilute means that for all energy levels, the occupation numbers are very small compared with the available number of quantum states (i.e. most quantum states are empty). We assume that

$$N_j \ll g_j \quad \text{for all } j.$$

This condition holds for real gases except at very low temperatures.

For dilute gas one finds

$$\omega_{FD} = \omega_{BE} = \omega_{MB} = \prod_{i=1}^r \frac{g_i^{N_i}}{N_i!}$$

Simply we can have

$$f_j \equiv \frac{n_j}{g_j} = \frac{1}{e^{(\epsilon_j - \mu)/k_B T} + a}, \quad a = \begin{cases} +1 & \text{for FD statistics} \\ -1 & \text{for BE statistics} \\ 0 & \text{for MB statistics} \end{cases}$$

**H.W.** Use  $\omega(i)$  for Fermi-Dirac and Bose-Einstein to prove the above distribution.

**Example:** Compare the Fermi-Dirac, Bose-Einstein, and Maxwell-Boltzmann statistics when four particles are arranged in two energy levels. Three particles are at energy level  $\varepsilon_1$  having a degeneracy  $g_1 = 4$  and one particle at energy level  $\varepsilon_2$  having a degeneracy  $g_2 = 2$ .

**Solution:**

$$W_{MB} = N! \prod_{i=1}^n \frac{g_i^{N_i}}{N_i!} = 4! \frac{4^3 \times 2^1}{3!1!} = 512$$

$$W_{BE} = \prod_{i=1}^n \frac{(N_i + g_i - 1)!}{N_i! (g_i - 1)!} = \frac{(3 + 4 - 1)! (1 + 2 - 1)!}{3!3! 1!1!} = 40$$

$$W_{FD} = \prod_{i=1}^n \frac{g_i!}{N_i! (g_i - N_i)!} = \frac{4! 2!}{3!1! 1!1!} = 4$$

**Example:** Consider the case of  $N = 3, k = 2, n_1 = 2$  in  $\varepsilon_1, n_2 = 1$  in  $\varepsilon_2, g_1 = 2$  and  $g_2 = 1$ . Then

$$\prod_{i=1}^n \frac{g_i!}{N_i! (g_i - 1)!} = \frac{(2+2-1)!}{2!1!} \times \frac{(1+1-1)!}{1!0!} = 3;$$

$$n_2 = 1 \quad \text{—————} \quad \varepsilon_2 = \Delta$$

$$\prod_{i=1}^n \frac{(N_i + g_i - 1)!}{N_i! (g_i - 1)!} = \frac{2!}{2!0!} \times \frac{1!}{1!0!} = 1;$$

$$n_1 = 2 \quad \text{=====} \quad \varepsilon_1 = 0$$

$$N! \prod_{i=1}^n \frac{g_i^{n_i}}{n_i!} = 3! \frac{2^2 \times 1^1}{2!1!} = 12;$$

**Example:** Consider a system consisting of two particles A and B in a volume  $V$ , each of which can be in any one of three quantum states of respective energies,  $\epsilon_1 = 0, \epsilon_2 = 1\epsilon$ , and  $\epsilon_3 = 3\epsilon$ . The system is in contact with a heat reservoir at temperature  $T$ . Write an expression for the **partition function "Z"** if the particles obey:

- (a) the classical MB statistics and are considered *distinguishable*.
- (b) the classical MB statistics and are considered *undistinguishable*.
- (c) the particles obey BE statistics.
- (d) the particles obey FD statistics.

R	Configuration			$E_{\text{total}}(\epsilon)$	No. of microstates		
	0	$\epsilon$	$3\epsilon$		MB	BE	FD
1	AB			0	1	1	0
2	A	B		1	2	1	1
3		AB		2	1	1	0
4	A		B	3	2	1	1
5		A	B	4	2	1	1
6			AB	6	1	1	0

$$Z = \sum_R e^{-\beta \sum_i n_i E_i}$$

$$a - Z_{\text{dis}}(MB) = e^{-\beta(0+0)\epsilon} + 2e^{-\beta(0+1)\epsilon} + e^{-\beta(1+1)\epsilon} + 2e^{-\beta(0+3)\epsilon} + 2e^{-\beta(1+3)\epsilon} + e^{-\beta(3+3)\epsilon}$$

$$= 1 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon} + 2e^{-4\beta\epsilon} + e^{-6\beta\epsilon}$$

$$b - Z_{\text{undis}}(MB) = 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon} + e^{-6\beta\epsilon}$$

$$c - Z(BE) = 1 + e^{-2\beta\epsilon} + e^{-6\beta\epsilon} + e^{-\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon}$$

$$d - Z(FD) = e^{-\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon}$$

Notes:  $N =$  number of particles = 2, number of states = 3.

- 1- Number of microstates =  $3^2 = 9$  (MB) (number of states)<sup>number of particles</sup>
- =  $3 \times 2 = 6$  (BE) number of states x number of particles
- = 3 (FD) number of states

We can not use BE and FD because they are not exact.

2- Define

$$P(\text{name}) = \frac{\text{Probability that two particles are found in the same states}}{\text{Probability that two particles are found in different states}}$$

	M.B	B.E.	F.D.
Probability that two particles are found in the same states	3	3	0
Probability that two particles are found in different states	6	3	3
$P(\text{name})$	$\frac{1}{2}$	1	0

Then

$$P(MB) = 3/6 = 1/2$$

$$P(BE) = 3/3 = 1$$

$$P(FD) = 0/3 = 0$$

Thus in the BE case there is a great relative tendency for particles to bunch together in the same states (e.g. Einstein condensation) than in classical statistics. On the other hand, in the FD case there is a great relative tendency for particles to remain apart in different states than there is in classical statistics.

**Example:** Consider a system of *two* identical and non-interacting particles in a volume  $V$ . Each particle has three accessible energy levels  $\epsilon_1 = 0$ ,  $\epsilon_2 = 1\epsilon$ , and  $\epsilon_3 = 2\epsilon$ . The lowest energy level is doubly degenerate. Determine the partition function and the mean energy of the system if:

- 1- the particles obey Fermi-Dirac statistics.
- 2- the particles obey Bose-Einstein statistics.
- 3- Determine the high temperature limit of the mean energy of the above three cases. Comparing the results what can you conclude about the behavior of fermions and bosons in this limit?

**Solution:**

**1- Fermi-Dirac Statistics:** We have the following distributions:

Energy / R	Microstates					
	1	2	3	4	5	6
$2\epsilon$	0	0	0	a	a	a
$\epsilon$	0	a	a	0	0	a
0	a   a	a   0	0   a	0   a	0   a	0   0
Total Energy =	0	$\epsilon$	$\epsilon$	$2\epsilon$	$2\epsilon$	$3\epsilon$

The partition function is:

$$Z_{FD} = e^{-\beta\epsilon_0} + 2e^{-\beta\epsilon_1} + 2e^{-\beta\epsilon_2} + e^{-\beta\epsilon_3} = 1 + 2e^{-\beta\epsilon} + 2e^{-2\beta\epsilon} + e^{-3\beta\epsilon}$$

$$U_{FD} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \frac{2\epsilon e^{-\beta\epsilon} + 4\epsilon e^{-2\beta\epsilon} + 3\epsilon e^{-3\beta\epsilon}}{1 + 2e^{-\beta\epsilon} + 2e^{-2\beta\epsilon} + e^{-3\beta\epsilon}} \rightarrow \frac{3}{2}\epsilon \quad (as T \rightarrow \infty)$$

**2- Bose-Einstein Statistics:** We have the following distributions:

Energy / R	Macrostates				
	1	2	3	4	5
2ε	0	0	0	0	0
ε	0	0	0	a	a
0	aa 0	0 aa	a a	a 0	0 a
Total Energy =	0	0	0	ε	ε

Energy / R	Microstates				
	6	7	8	9	10
2ε	a	2	0	a	aa
ε	0	0	aa	a	0
0	a 0	0 a	0 0	0 0	0 0
Total Energy =	2ε	2ε	2ε	3ε	4ε

The partition function is:

$$Z_{BE} = 3e^{-\beta\epsilon_0} + 2e^{-\beta\epsilon_1} + 3e^{-\beta\epsilon_2} + e^{-\beta\epsilon_3} + e^{-\beta\epsilon_4} = 3 + 2e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon}$$

$$U_{BE} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \frac{2\epsilon e^{-\beta\epsilon} + 6\epsilon e^{-2\beta\epsilon} + 3\epsilon e^{-3\beta\epsilon} + 4\epsilon e^{-4\beta\epsilon}}{3 + 2e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + e^{-3\beta\epsilon} + e^{-4\beta\epsilon}} \rightarrow \frac{3}{2} \epsilon \quad (\text{as } T \rightarrow \infty)$$

**3-Note that:** At high temperature limit ( $T \rightarrow \infty$ ,  $\beta \rightarrow 0$ ) fermions and bosons behave as classical Boltzmann particles.

**Example:** Consider a system of *two* non-interacting and identical particles in a volume V. Each particle has three accessible energy levels  $\epsilon_1 = 0$ ,  $\epsilon_2 = 1\epsilon$ , and  $\epsilon_3 = 2\epsilon$ . Determine the partition function and the mean occupation numbers  $\overline{n_1}$ ,  $\overline{n_2}$ , and  $\overline{n_3}$  of the three quantum states if:

- the particles obey Bose-Einstein statistics.
- The mean energy  $U$  of the boson gas.
- Determine the low temperature limit of  $\overline{n_1}$ ,  $\overline{n_2}$ ,  $\overline{n_3}$ , and  $U$  of the above three cases. Comparing the results what can you conclude about the behavior the bosons in this limit?



**Solution::** For Bose-Einstein Statistics we have the following distributions:

Energy / R	Microstates					
	1	2	3	4	5	6
$\varepsilon_3 = 2\varepsilon$	0	0	1	0	1	2
$\varepsilon_2 = \varepsilon$	0	1	0	2	1	0
$\varepsilon_1 = 0$	2	1	1	0	0	0
Total Energy =	0	1 $\varepsilon$	2 $\varepsilon$	2 $\varepsilon$	3 $\varepsilon$	4 $\varepsilon$

$$Z_{BE} = \sum_{\{n_i\}} e^{-\beta \sum_s n_s \varepsilon_s} = \sum_{i(\text{energy levels})} g_i e^{-\beta \varepsilon_i}$$

$$Z = e^{-2\beta\varepsilon_1} + e^{-\beta(\varepsilon_1 + \varepsilon_2)} + e^{-\beta(\varepsilon_1 + \varepsilon_3)} + e^{-2\beta\varepsilon_2} + e^{-\beta(\varepsilon_2 + \varepsilon_3)} + e^{-2\beta\varepsilon_3}$$

$$= 1 + e^{-\beta\varepsilon} + 2e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon} = (1 + e^{-2\beta\varepsilon})(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})$$

Use  $\bar{n}_i = -\frac{1}{\beta Z} \left( \frac{\partial Z}{\partial \varepsilon_i} \right)_{T, \varepsilon_j \neq \varepsilon_i}$ .

a-

$$\bar{n}_1 = \frac{2e^{-2\beta\varepsilon_1} + e^{-\beta(\varepsilon_1 + \varepsilon_2)} + e^{-\beta(\varepsilon_1 + \varepsilon_3)}}{Z} = \frac{2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}{(1 + e^{-2\beta\varepsilon})(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})},$$

$$\bar{n}_2 = \frac{2e^{-2\beta\varepsilon_2} + e^{-\beta(\varepsilon_1 + \varepsilon_2)} + e^{-\beta(\varepsilon_2 + \varepsilon_3)}}{Z} = \frac{e^{-\beta\varepsilon} (1 + 2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})}{(1 + e^{-2\beta\varepsilon})(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})},$$

$$\bar{n}_3 = \frac{2e^{-2\beta\varepsilon_3} + e^{-\beta(\varepsilon_1 + \varepsilon_3)} + e^{-\beta(\varepsilon_2 + \varepsilon_3)}}{Z} = \frac{e^{-2\beta\varepsilon} (1 + e^{-\beta\varepsilon} + 2e^{-2\beta\varepsilon})}{(1 + e^{-2\beta\varepsilon})(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})}.$$

b-  $U = \bar{n}_1 \varepsilon_1 + \bar{n}_2 \varepsilon_2 + \bar{n}_3 \varepsilon_3 = \frac{\varepsilon e^{-\beta\varepsilon} (1 + 4e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon} + 4e^{-3\beta\varepsilon})}{(1 + e^{-2\beta\varepsilon})(1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon})}.$

c- Low temperature limit:  $T \rightarrow 0 \Rightarrow e^{-\beta\varepsilon} \Rightarrow 0$   
 $\bar{n}_1 \rightarrow 2, \bar{n}_2 \rightarrow 0, \bar{n}_3 \rightarrow 0, \bar{U} \rightarrow 0$