Oscillations

PREVIEW

An object such as a pendulum or a mass on a spring is *oscillating* or vibrating if it is moving in a repeated path at regular time intervals. We call this type of motion ***harmonic motion*** or ***periodic motion***. For an object to continue oscillating there must be a *restoring force* continually trying to restore it to its equilibrium position. For, the force exerted by an *ideal spring* obeys *Hooke's law*. As an object in simple harmonic motion oscillates, its energy is repeatedly converted from potential energy to kinetic energy, and vice – versa.

QUICK REFERENCE

Important Terms

amplitude
: maximum displacement from equilibrium position; the distance from the midpoint of a wave to its crest or trough.

equilibrium position
: the position about which an object in harmonic motion oscillates; the center of vibration

frequency
: the number of vibrations per unit of time

Hooke's law
: law that states that the restoring force applied by a spring is proportional to the displacement of the spring and opposite in direction

ideal spring
: any spring that obeys Hooke's law and does not dissipate energy within the spring.

mechanical resonance
: condition in which natural oscillation frequency equals frequency of a driving force

period
: the time for one complete cycle of oscillation

periodic motion
: motion that repeats itself at regular intervals of time

restoring force
: the force acting on an oscillating object which is proportional to the displacement and always points toward the equilibrium position.

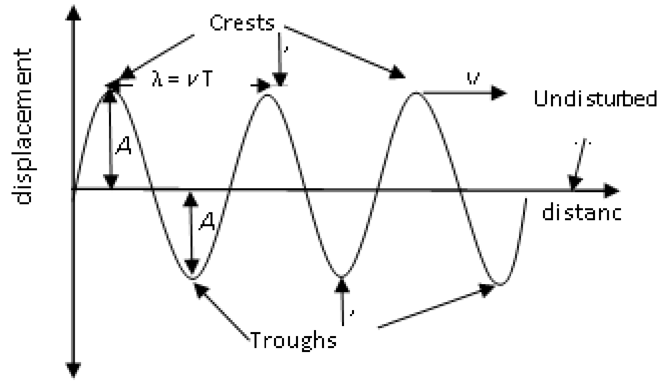
simple harmonic motion

regular, repeated, friction-free motion in which the restoring force has the mathematical form F = - kx.

Equations and Symbols

$\mathbf{F}_s = -k\mathbf{x}$ $PE_{elastic} = \frac{1}{2}kx^2$ $x = A\cos\omega t$ $\omega = \frac{2\pi}{T} = 2\pi f$ $T_S = 2\pi\sqrt{\frac{m}{k}}$ $T_P = 2\pi\sqrt{\frac{l}{g}}$ $T = \frac{1}{f}$	F_s = the restoring force of the spring k = spring constant x = displacement from equilibrium position $PE_{elastic}$ = elastic (spring) potential energy A = *amplitude* ω = angular frequency T = period f = frequency m = mass T_P = period of a pendulum T_S = period of a mass on a spring g = acceleration due to gravity

Harmonic motion is characterized by:



1- **Amplitude** (A, $[A]=\text{m}$): "*the maximum displacement of a particle in a medium on either side of its undisturbed position*". This is the **size** of the wave measured from the middle position.
2- **Wavelength** (λ-the Greek letter lambda,$[\lambda]=\text{m}$) "the distance between two successive crests or two successive troughs" (or the distance between two successive points moving at the same phase, i.e. corresponding points on the wave). In term of the wavelength, we can define the wave number "*k*" as $k=2\pi/\lambda$, $[k]=\text{rad/m}$.
3- **Periodic Time** (T, $[T]$ = s) "time taken to produce one complete oscillation".
4- **Frequency** ($f=1/T$, $[[f]=\text{s}^{-1}\equiv\text{Hz}$ (Hertz)) "It is the number of complete waves passing a given point per second". Also, the **angular frequency**, ω, is defined by $\omega=2\pi f$, and has a units of rad/s.

It is important to note that the variables $f, \lambda,$ and ω are all positive quantities.

15-1 SIMPLE HARMONIC MOTION

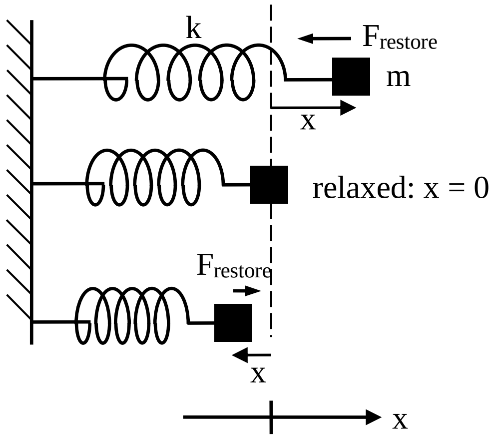
A pendulum, a mass on a spring, and many other kinds of oscillators exhibit a special kind of oscillatory motion called Simple Harmonic Motion (SHM).

SHM occurs whenever:

i. there is a restoring force proportional to the displacement from equilibrium: $\text{F} \propto -\text{x}$
ii. the potential energy is proportional to the square of the displacement: $\text{PE} \propto \text{x}^2$
iii. the period T or frequency $f=1/T$ is <u>independent</u> of the <u>amplitud</u>e of the motion.
iv. the position *x*, the velocity *v*, and the acceleration *a* are all sinusoidal in time. (***Sinusoidal* means sine, cosine, or anything in between**.)

As we will see, any one of these four properties guarantees the other three. If one of these 4 things is true, then the oscillator is a simple harmonic oscillator and all 4 things must be true. Not every kind of oscillation is SHM. For instance, a perfectly elastic ball bouncing up and down on a floor: the ball's position (height) is oscillating up and down, but none of the 4 conditions above is satisfied, so this is not an example of SHM.

A mass on a spring is the simplest kind of Simple Harmonic Oscillator.



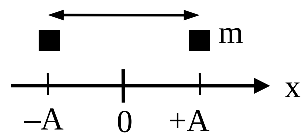
Hooke's Law: $\mathbf{F}_{spring} = -k\,\mathbf{x}$

(–) sign because direction of $\mathbf{F}_{spring}$ is opposite to the direction of displacement vector **x** (**bold** font indicates vector)

k = spring constant = stiffness,
units [k] = N / m

Big k = stiff spring

Definition: *amplitude* A = $|x_{max}|$ = $|x_{min}|$.



Mass oscillates between extreme positions x = +A and x = –A

Notice that:

1- Hooke's Law (F = – kx) is **condition i** : restoring force proportional to the displacement from equilibrium.
2- We showed previously (Work and Energy Chapter) that for a spring obeying Hooke's Law, the potential energy is $U = (1/2)kx^2$, which **is condition ii**.
3- Also, in the chapter on Conservation of Energy, we showed that F = –dU/dx, from which it follows that condition ii implies condition i.

Thus, Hooke's Law and quadratic PE ($U \propto x^2$) are equivalent.

Question: In 1D , drive the *differential equation* for Hooke's Law .
Answer: A differential equation is simply an equation containing a derivative. Start with Hooke's and Newton's laws, we have

$$F_{net} = ma \quad \text{and} \quad F_{net} = -kx \quad \Rightarrow \quad ma = -kx$$

With the definition $a = dv/dt = d^2x/dt^2$, one can have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x, \quad \omega^2 = k/m$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{(Equation of SHM)}$$

This is the *differential equation* for SHM.

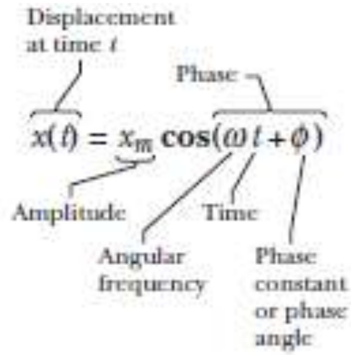
The way you solve differential equations is the same way you solve integrals: you *guess* the solution and then check that the solution works. Based on observation, we guess a sinusoidal solution:

$$x(t) = A\cos(\omega t + \varphi), \qquad A = x_m,$$

where $A = x_m(t)$, φ are <u>any</u> constants and $\omega = \sqrt{k/m}$.

A = amplitude: x oscillates between +A and –A

φ = phase constant (depends on the initial information, t=0, about x and v)



- **Danger:** ωt and φ have units of radians (not degrees). So set your calculators to radians when using this formula.

Question: Prove that the guessed sinusoidal function is a solution of the SHM equation:
Proof: Taking the first derivative dx/dt , we get

$$v(t) = \frac{dx}{dt} = -A\omega\,\sin(\omega t + \varphi).$$

- Here, we've used the Chain Rule:

$$\frac{d}{dt}\cos(\omega t + \varphi) = \frac{d\cos(\theta)}{d\theta}\frac{d\theta}{dt}, \qquad (\theta = \omega t + \varphi)$$

$$= -\sin\theta \cdot \omega = -\omega\sin(\omega t + \varphi)$$

Taking a second derivative, we get

$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \varphi) = -\omega^2 x(t)$$

This is the SHM equation, with $\omega^2 = k/m$, $\omega = \sqrt{k/m}$

We have also shown condition iv: x, v, and a are all sinusoidal functions of time:

$$x(t) = A\cos(\omega t + \varphi)$$

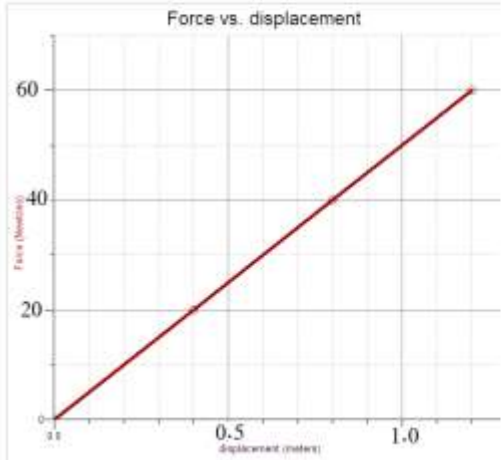
$$v(t) = -A\omega\sin(\omega t + \varphi) \quad \Rightarrow \quad v_{max} = A\omega$$

$$a(t) = -A\omega^2\cos(\omega t + \varphi) \quad \Rightarrow \quad a_{max} = A\omega^2$$

The period T is given by $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$.

We see that **T does not depend on the amplitude A**.

Example 1: A mass $m = 5.0$ kg oscillates on the end of a spring on a horizontal surface with negligible friction according to the equation $x = A\cos(\omega t)$. The graph of *F vs. x* for this motion is shown below. The last data point corresponds to the maximum displacement of the mass. Determine the



(a) Angular frequency *ω* of the oscillation,
(b) Frequency *f* of oscillation,
(c) Amplitude *A* of oscillation,
(d) Displacement from equilibrium position (*x* = 0) at a time of 2 s.

Solution:

(a) We know that the spring constant $k = \frac{60}{1.2} = 50$ N/m from when we looked at this graph earlier. So,

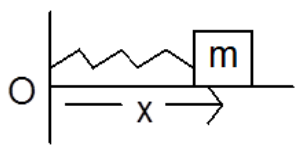
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\,\text{N/m}}{5.0\,\text{kg}}} = 10\,\frac{\text{rad}}{\text{s}}$$

(b) $f = \frac{\omega}{2\pi} = \frac{10\,\text{rad/s}}{2\pi} = 1.6\,\text{Hz}$

(c) The amplitude corresponds to the last displacement on the graph, *A* = 1.2 m.

(d) $x = A\cos(\omega t) = (1.2\,m)\cos[(10\,\text{rad/s})(2\,\text{s})] = 0.5\,\text{m}$

Example: A block of mass 0.02 kg is attached to a horizontal spring with spring constant of 25 N/m. The other end of the spring is fixed. The block is pulled a distance 10 cm from its equilibrium position (x = 0) on a frictionless horizontal table and released. The frequency of the resulting simple harmonic motion is:



Answer:

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{25}{0.02}} = \underline{5.6\text{ Hz}}.$$

Example: A 0.25-kg mass is attached to a spring with force constant k = 36 N/m. The mass is displaced 5 cm from its equilibrium position and released from rest.

☞ *What is the frequency and period of oscillation?*

$$f = \frac{1}{2\pi}\sqrt{\frac{36N/m}{0.25kg}} = 1.91Hz, \;\; T = \frac{1}{f} = 0.52\,s$$

☞ *What is the maximum speed of the mass?*

$$v_{max} = \omega A = 2\pi fA = 2\pi(1.91Hz)(0.05m) = \underline{0.6\,m/s} \text{ (occurs at x = 0)}$$

☞ *What is the magnitude of the maximum acceleration*?

$$a_{max} = \omega^2 A = (2\pi f)^2 A = (2\pi(1.91Hz))^2(0.05m) = \underline{7.2\,m/s^2}$$

(occurs at x = ± 5 cm)

☞ *How long does it take for the mass to first return to x = 0*?

$$t = \tfrac{1}{4}T = \tfrac{1}{4}(0.52\,s) = \underline{0.13\,s}$$

Example: The displacement of a particle oscillating along the x-axis is given as a function of time according to the equation:

$$x(t) = (0.5\,\text{m})\cos(\pi t + \pi/2)..$$

Calculate the magnitude of the maximum acceleration of the particle.

Answer: It is given that: $\omega = \pi, \quad A = 0.5$

$$a_{\max} = \omega^2 \times A = \pi^2 \times 0.5 = \underline{4.9\text{ m/s}^2}.$$

Example: The maximum speed of a 3.00-kg object executing simple harmonic motion is 6.00 m/s. The maximum acceleration of the object is 5.00 m/s^2. What is its period of oscillations?

Answer:

$$a_{\max} = \omega^2 A, \quad \text{v}_{\max} = \omega A \quad \Rightarrow \quad a_{\max} = \text{v}_{\max}\omega = \text{v}_{\max}\frac{2\pi}{T}$$

$$T = \frac{2\pi \text{v}_{\max}}{a_{\max}} = \frac{2 \times \pi \times 6}{5} = 7.54\text{ s}$$

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A 0.500 kg mass attached to a spring of force constant 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate the time it takes the mass to move from *x* = 0 to *x* = 10.0 cm.

Ans:

$$\text{Time taken} = \frac{T}{4}$$

$$\text{where } T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.5}{8}} = 1.57\text{ s}$$

$$\therefore t = \frac{1.57}{4}\text{ s} = 0.393\text{ s}$$

15-2 ENERGY IN SIMPLE HARMONIC MOTION

Recall $PE_{elastic}$ = (1/2) k x^2 = work done to compress or stretch a spring by distance x. If there is no friction, then the mechanical energy:

$$E_{tot} = KE + PE = K(t) + U(t) = \text{constant}$$

during oscillation. The value of E_{tot} depends on initial conditions – where the mass is m and how fast it is moving initially. But once the mass is set in motion, E_{tot} stays constant (assuming no dissipation.)

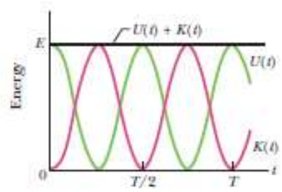
At any position x, speed v is such that $\frac{1}{2}m v^2 + \frac{1}{2}k x^2 = E_{tot}$.

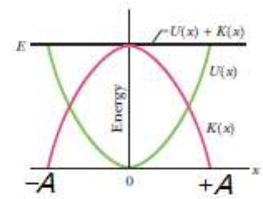
When |x| = A, then v = 0, and all the energy is PE: $\underset{0}{KE} + \underset{(1/2)kA^2}{PE} = E_{tot}$

So total energy $E_{tot} = \frac{1}{2}k A^2$

When x = 0, v = v_{max}, and all the energy is KE: $\underset{(1/2)m v_{max}^2}{KE} + \underset{0}{PE} = E_{tot}$

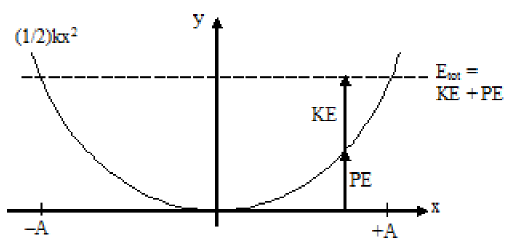
So, total energy $E_{tot} = \frac{1}{2}m v_{max}^2$.





So, we can relate v_{max} to amplitude A :

$$PE_{max} = KE_{max} = E_{tot} \Rightarrow \frac{1}{2}k A^2 = \frac{1}{2}m v_{max}^2 \quad \Rightarrow \quad v_{max} = \sqrt{\frac{k}{m}}\,A = \omega A$$



Homework1: A mass m on a spring with spring constant k is oscillating with amplitude A. Derive a general formula for the speed v of the mass when its position is x.

Answer: $v(x) = A\sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{x}{A}\right)^2}$

Homework2: Prove that at PE = KE, the $x = A/\sqrt{2}$.

Be sure you understand the following things:
Periodic time is T = 2 (time taken from $x = -A$ to $x = A$)

$x = -A$	$x = 0$	$x = A$
$v = 0$	$v = \pm\omega A$	$v = 0$
$a = +\omega^2 A$	$a = 0$	$a = -\omega^2 A$
$K = 0$	$K = \frac{1}{2}kA^2$	$K = 0$
$U = \frac{1}{2}kA^2$	$U = 0$	$U = \frac{1}{2}kA^2$
$E = \frac{1}{2}kA^2$	$E = \frac{1}{2}kA^2$	$E = \frac{1}{2}kA^2$

Example: A block attached to an ideal horizontal spring undergoes a simple harmonic motion about the equilibrium position (x = 0) with an amplitude A = 10 cm. The mechanical energy of the system is 16 J. What is the kinetic energy of the block when x = 5.0 cm?

Answer: It is given that $E = 16$ J; at A = 0.1 m

$$E = 16 = \frac{1}{2}kA^2 = \frac{1}{2}k \times 0.1^2 \quad \Rightarrow \quad k = 3200 \text{ N/m};$$

At x = 5.0 cm,

$$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 3200 \times 0.05^2 = \underline{4.0 \text{ J}} \quad \Rightarrow \quad K = E - U = 16 - 4.0 = \underline{12 \text{ J}}$$

Example: A block of mass 2.0 kg attached to a spring oscillates in simple harmonic motion along the x axis. The limits of its motion are x = −20 cm and x = + 20 cm and it goes from one of these extremes to the other in 0.25 s. The mechanical energy of the block-spring system is:

Answer: The periodic time is given by: $T = 2.0 \times 0.25 = 0.5$ s

$$k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2 = 2.0 \times \left(\frac{2\pi}{0.5}\right)^2 = 316 \text{ N/m};$$

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 316 \times 0.2^2 = \underline{6.3 \text{ J}}$$

Example: A 0.25 kg block oscillates on the end of the spring with a spring constant of 200 N/m. If the system has an energy of 6.0 J, then the maximum speed of the block is:

Answer: $v_{max} = \omega A$

$$E = \frac{1}{2}kA^2 = \frac{1}{2} \times 200 \times A^2 = 6 \text{ J} \quad \Rightarrow \quad A = 0.24 \text{ m}$$

Then,

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}} A = \sqrt{\frac{200}{0.25}} (0.245) = 6.9 \text{ m/s}$$

Example: A 2.0-kg mass connected to a spring of force constant 8.0 N/m is displaced 5.0 cm from its equilibrium position and released. It oscillates on a horizontal, frictionless surface. Find the speed of the mass when it is at 3.0 cm from its equilibrium position.

Answer:

$$E = \begin{cases} \frac{1}{2} k x_m^2 \\ \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \end{cases} \Rightarrow \frac{1}{2} k x_m^2 = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$\Rightarrow v = \sqrt{\frac{k}{m}\left[x_m^2 - x^2\right]} = \sqrt{\frac{8}{2}\left[(0.05)^2 - (0.03)^2\right]} = \underline{0.08 \text{ m/s}}$$

Example: 10 kg mass hanging on a spring has characteristic frequency 2 Hz. How much will the length of the spring change when the mass is detached? *Equivalent question:* How much is it stretched, when hanging in the equilibrium position?

Answer:

In this position	Mg = kx (equilibrium situation),	$\therefore$ x = Mg/k.
Now,	ω^2 = k/M,	
and from above,	$\omega = 2\pi f$, $= 4\pi\ s^{-1}$, $\omega^2 = 16\pi^2\ s^{-2}$	
	$\therefore$ x = g(M/k) = g/ω^2 = 9.8 meter/s^2 x 0.0063 s^2 = 6.2 cm.	

Note that we don't need to know the mass M - the answer is the same for all M.

Example: A block is in SHM on the end of a spring, with position given by:

$$x = x_m \cos(\omega t + \pi/6 \text{ rad}),$$

where t is in seconds. At t = 0, calculate the ratio of the potential energy U to the total mechanical energy E, i.e. U/E of the system.

Answer:

At t=0,

$$U = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k \left[x_m \cos(\pi/6)\right]^2, \quad E = \tfrac{1}{2} k \left[x_m\right]^2$$

$$\frac{U}{E} = \left[x_m \cos(\pi/6)\right]^2 = 0.75$$

Extra Problems

Q: A block of mass 20 g is attached to a horizontal spring with spring constant of 25 N/m. The other end of the spring is fixed. The block is pulled a distance 10 cm from its equilibrium position (x = 0) on a frictionless horizontal table and released. The frequency of the resulting simple harmonic motion is:

Answer:

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{25}{20/1000}} = \underline{5.6\text{ Hz}}.$$

Q: A simple pendulum consists of a mass m = 6.00 kg at the end of a light cord of length L. The angle θ between the cord and the vertical is given by $\theta = 0.08\cos\left[\left(4.43\,t+\pi\right)\right]$, where t is in second and θ is in radian. Find the length L.

Answer: From the equation, one finds $\omega = 4.43$ rad/s.

$$T = \begin{cases} \frac{2\pi}{\omega} \\ 2\pi\sqrt{\frac{L}{g}} \end{cases} \Rightarrow \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \Rightarrow L = \frac{g}{\omega^2} = \frac{9.8}{\left(4.43\right)^2} = \underline{0.5\text{ m}}.$$

Q: A 0.20 kg object attached to a horizontal spring whose spring constant is 500 N/m executes simple harmonic motion. If its maximum speed is 5.0 m/s, the amplitude of its oscillation is:

Answer:

$$v_{\max} = \omega x_m \Rightarrow x_m = \frac{v_{\max}}{\omega} = \frac{v_{\max}}{\sqrt{k/m}} = \frac{5}{\sqrt{500/0.2}} = \underline{0.10\text{ m}}.$$

Q: A 3.0 kg block, attached to a spring, executes simple harmonic motion according to the relation:

x = 2.0 cos(50 t) ,

where x is in m and t is in s. The spring constant of the spring is: (Ans: 7.5×10^3 N/m)

Answer: $\omega = 50; x_m = 2;$

$$k = m\omega^2 \Rightarrow k = 3.0\times 50^2 = \underline{7.5\times 10^3\text{ N/m}}.$$

Q: A particle is in simple harmonic motion along the x axis. The amplitude of the motion is x_m When it is at x = 10 cm, its kinetic energy K = 6.0 J and its potential energy U = 4.0 J (measured with U = 0 at x = 0). When it is at x = -5.0 cm, the kinetic and potential energies are: (Ans: K = 9.0 J and U = 1.0 J)

Answer: It given that $E = 6+4 = 10$ J; at x = 0.1 m

$$U = 4 = \frac{1}{2}kx^2 = \frac{1}{2}k\,0.1^2 \Rightarrow k = 800\text{ N/m};$$

At x = - 5.0 cm,

$$U = \frac{1}{2}kx^2 = \frac{1}{2}\times 800\times 0.05^2 = \underline{1.0\text{ J}}$$

$$K = 10 - 1.0 = \underline{9.0\text{ J}}$$

Q: The displacement of a particle oscillating along the x-axis is given as a function of time according to the equation: x(t)=0.50cos(πt+π/2). The magnitude of the maximum acceleration of the particle is:

Answer: It is given that: $\omega = \pi, \quad x_m = 0.5$

$$a_{\max} = \omega^2 \times A = \pi^2 \times 0.5 = \underline{4.9\text{ m/s}^2}.$$

Q: A block of mass 2.0 kg attached to a spring oscillates in simple harmonic motion along the x axis. The limits of its motion are x = −20 cm and x = + 20 cm and it goes from one of these extremes to the other in 0.25 s. The mechanical energy of the block-spring system is:

Answer: $T = 2.0\times 0.25 = 0.5\text{ s}$

$$k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2 = 2.0\times\left(\frac{2\pi}{0.5}\right)^2 = 316\text{ N/m};$$

$$E = \frac{1}{2}kA^2 = \frac{1}{2}\times 316\times 0.2^2 = \underline{6.3\text{ J}}$$

Q: A 2.0-kg mass connected to a spring of force constant 8.0 N/m is displaced 5.0 cm from its equilibrium position and released. It oscillates on a horizontal, frictionless surface. Find the speed of the mass when it is at 3.0 cm from its equilibrium position. (Ans: 0.08 m/s)

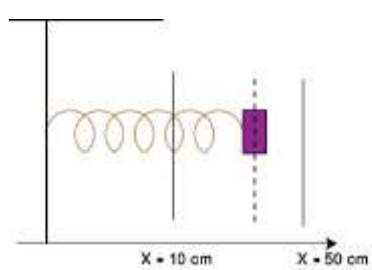
Answer:

$$E = \begin{cases} \frac{1}{2}kA^2 \\ \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \end{cases} \Rightarrow \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\Rightarrow v = \sqrt{\frac{k}{m}\left[A^2 - x^2\right]} = \sqrt{\frac{8}{2}\left[(0.05)^2 - (0.03)^2\right]} = \underline{0.08\text{ m/s}}$$

Q2:

A block attached to a spring oscillates in simple harmonic motion along the x axis. The limits of its motion are x = 10 cm and x = 50 cm and it goes from one of these extremes to the other in 0.25 s. What is the maximum speed (in SI unit) of block?



Answer: Note that, the periodic time T = 2 t, t = 0.25 s

$$v_m = \omega x_m = \frac{2\pi}{T} x_m = \frac{2\pi}{2\times 0.25}\,\frac{(50-10)\times 10^{-2}}{2}$$
$$= \underline{2.5 \text{ m/s}}.$$

Q1:
A certain spring elongates 9:0 mm when it is suspended vertically and a block of mass M is hung on it. The natural angular frequency of this block-spring system:
Answer:

$$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{\frac{\cancel{m}g}{x}}{\cancel{m}}} = \sqrt{\frac{g}{x}} = \sqrt{\frac{9.8}{9\times 10^{-3}}} = 33 \text{ rad/s}$$

Q2:
A 3-kg block, attached to a spring, executes simple harmonic motion according to
x = 2 cos (50 t)
where x is in meters and t is in seconds. The mechanical energy of the block-spring system is:
(**Ans:**

Find the mechanical energy of the block spring system:

$$E = K_{\max} = \frac{1}{2} m v_{\max}^2 = \frac{1}{2} m\omega^2 x_m^2 = \frac{1}{2}\times 3\times 50^2\times 2^2$$
$$= \underline{1.5\times 10^4 \text{ J}}.$$

Q3:
The period of a simple pendulum is 1 s on Earth. When brought to a planet where g is one-tenth that on Earth, its period becomes:

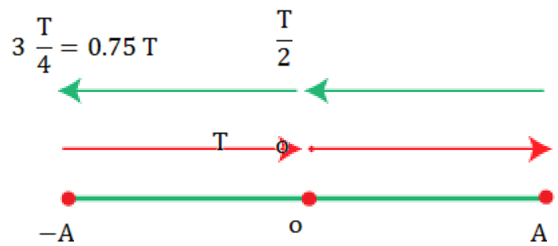
Ans:

$$T = 2\pi\sqrt{\frac{\ell}{g}}, \quad T' = 2\pi\sqrt{\frac{\ell}{g'}} = 2\pi\sqrt{\frac{\ell}{0.1g}} = \sqrt{\frac{1}{0.1}}T = \sqrt{\frac{1}{0.1}}(1)$$
$$= \underline{\sqrt{10} \text{ s}}.$$

Q1:

A particle is in simple harmonic motion with period T. **At time t = 0 it is at the equilibrium point**. Of the following times, at which time is it furthest from the equilibrium point?

A. 0.5T
B. 0.75T
C. T
D. 1.5T
E. 2.0T



Ans:

B ; The particle will at the furthest point at $t=\frac{T}{4}$ and $3\frac{T}{4}$, $3\frac{T}{4}$ is between the choices.

Q2:
A particle moves in simple harmonic motion according to

$$x(t) = 3\cos[60t + 2],$$

where x is in meters and t is in seconds. Find the maximum velocity (in m/s) of the particle. (Give your answer in three significant figures form)

Ans:

$$v(t) = \frac{dx}{dt} = -3 \times 60 \sin(60\,t + 2)$$

Maximum velocity $v_{max} = A w = 3 \times 60 = 180\ m/s$

Q3:
A 0.50 kg object attached to a spring whose spring constant is 600 N/m executes simple harmonic motion. If its maximum linear acceleration is 5.0 m/s^2, find the amplitude (in m) of its oscillation. (Give your answer in three significant figures form)

Ans:

$$m = 0.5\ kg;\ \ k = 600\ N/m;\ \ a_m = 5 m/s^2\ ;\ \ A = ?$$

$$a_m = A\omega^2;\ \text{where}\ \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow a_m = A\frac{k}{m}$$

$$\Rightarrow A = \frac{m a_m}{k} = \frac{0.5 \times 5}{600} = 4.17 \times 10^{-3}\ m$$

Q2:
A 0.25 kg block oscillates on the end of the spring with a spring constant of k. The amplitude of the motion is 10 cm. If the system has kinetic energy of 6.0 J at x = 4 cm, find the spring constant of k (in N/m). (Give your answer in three significant figures form)

Ans:

$$x_m = 0.1 \text{ m}; \quad m = 0.25 \text{ kg}$$

$$k = 6 \text{ J at } x = 0.04 \text{ m}; \quad k = ?$$

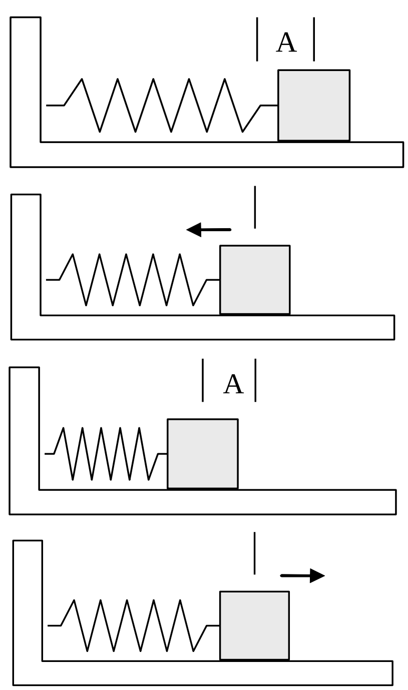
$$E = k + U$$

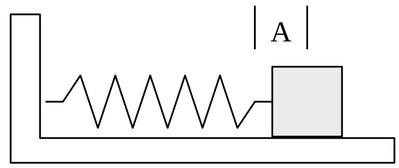
$$\frac{1}{2}kx_m{}^2 = K + \frac{1}{2}kx^2$$

$$\Rightarrow k = \frac{2K}{(x_m{}^2 - x^2)}$$

$$\Rightarrow k = \frac{2 \times 6}{0.1^2 - 0.04^2} = 1428 \approx 1.42 \times 10^3 \text{ N/m}$$

The oscillation of a Hooke's law spring is called Simple Harmonic Motion (SHM).





The energy of the oscillating mass is given by

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

At t = 0, v = 0 and x = A, so $E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$

Solving for v, we have $v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$

So, when x = ±A, v = 0. And when x = 0, $v = \pm\sqrt{\frac{k}{m}}A$

Since $F = ma = -kx$, then $a = -\frac{k}{m}x$.

Thus, the acceleration is always opposite to the displacement and its magnitude is a maximum when the displacement is a maximum.

Assuming that the mass is released from rest at x = A, the expressions for x, v, and a are given by

$$x = x_{max}\cos(\omega t) = A\cos(2\pi\, ft)$$

$$v = -v_{max}\sin(\omega t) = -v_{max}\sin(2\pi\, ft)$$

$$a = a_{max}\cos(\omega t) = a_{max}\cos(2\pi\, ft)$$

The maximum v and a are given by

$$v_{max} = \omega A \qquad a_{max} = \omega^2 A$$

The frequency (f), angular frequency (ω), and period (T) are related by

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Review Problems
Phase Shift problems:

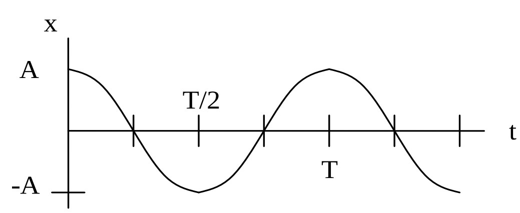
Be sure you understand the following things:
Periodic time is T = 2 (time taken from $x=-A$ to $x=A$)

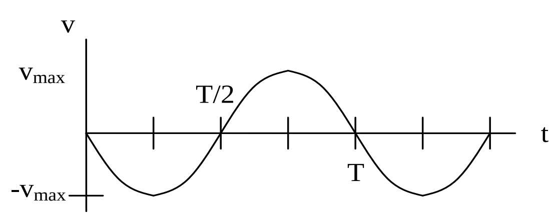
$x=-A$	$x=0$	$x=A$
$v=0$	$v=\pm\omega A$	$v=0$
$a=+\omega^2 A$	$a=0$	$a=-\omega^2 A$
$K=0$	$K=\frac{1}{2}kA^2$	$K=0$
$U=\frac{1}{2}kA^2$	$U=0$	$U=\frac{1}{2}kA^2$
$E=\frac{1}{2}kA^2$	$E=\frac{1}{2}kA^2$	$E=\frac{1}{2}kA^2$

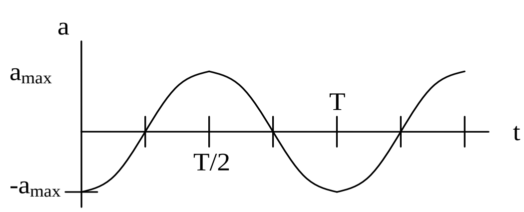
$$x(t) = A\cos(\omega t+\varphi), \qquad A = x_m$$

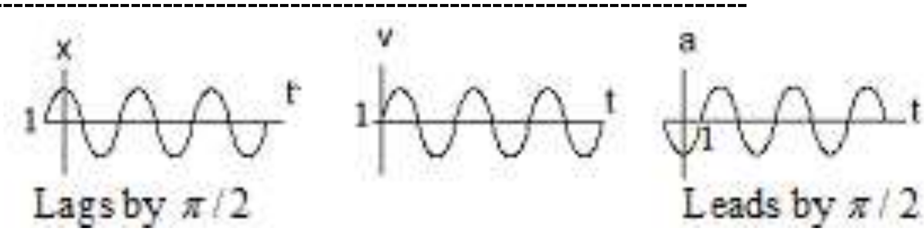
$$v(t) = -A\omega \sin(\omega t+\varphi) \Rightarrow v_{max} = A\omega$$

$$a(t) = -A\omega^2 \cos(\omega t+\varphi) \Rightarrow a_{max} = A\omega^2$$







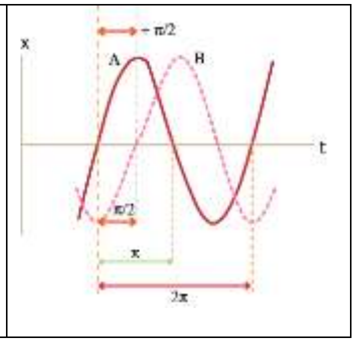


Q1:

The figure below show the displacements x(t) of a pair of simple harmonic oscillators (A and B) that are identical **except for phase**. What phase shift is needed to **shift the curve for A** to **coincide** with the curve for B?

A. $+\pi$
B. $-\pi$
C. $+\pi/2$
D. $-\pi/2$
E. 0

Ans:
D



T122_ Q27.

A particle executes simple harmonic motion on a horizontal frictionless surface, with the equilibrium position at $x = 0$. At $t = 0$, it is released from rest at a displacement $x = 0.5$ m. If the frequency of oscillation is 5 Hz, find the displacement x at $t = 0.02$ s.

Answer: The given boundary conditions are:

$v(0) = 0$ (I)

$x(0) = 0.5$ (II)

$$x(t) = x_m \cos[\omega t + \phi], \quad v(t) = -\omega x_m \sin[\omega t + \phi],$$

The boundary condition I gives:

$$\Rightarrow v(0) = -\omega x_m \sin[0 + \phi] = 0 \Rightarrow \phi = 0$$

The boundary condition II gives:

$$x(0) = 0.5 = x_m \cos[0 + 0] \Rightarrow x_m = 0.5 \text{ m}$$

Finally,

$$x(t) = 0.5\cos[\omega t] = 0.5\cos[2\pi \times 5 \times 0.02] = \underline{0.4 \text{ m}}$$

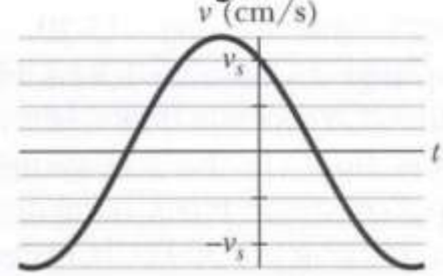
T113_Q26.

What is the phase constant for the harmonic oscillator with the velocity function v(t) given in Figure 5 if the position function x(t) has the form

$$x(t) = x_m \cos[\omega t + \phi]\,?$$

The vertical axis scale is set by $v_s = 4.00$ cm/s.

Fig# 5



Answer:

Start with the equations:

$x(t) = x_m \cos[\omega t + \phi], \quad v(t) = -\omega x_m \sin[\omega t + \phi] = -v_m \sin[\omega t + \phi]$, From the figure, one finds $v_m = 5.0$ cm/s. Using the boundary value $v(0) = 4.0$ cm/s., and the equation $v(0) = -v_m \sin[0 + \phi]$, one has

$$4 = -5\sin[\phi] \Rightarrow \sin[\phi] = -\frac{4}{5} \Rightarrow \phi = \underline{-53.1^\circ}$$

T133_Q27.

What is the phase constant for the harmonic oscillator with the position function given in **Figure 9** if the position function has the form

$$x(t) = x_m \cos[\omega\, t + \phi]\,?$$

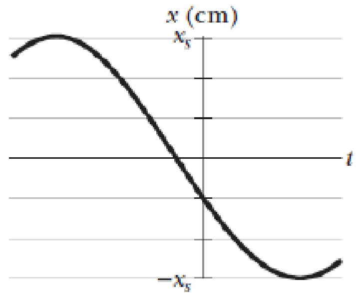
The vertical axis scale is set by $x_s = 6.0$ cm .

Ans:

$$x(0) = 6 \times cos(\varphi) = -2$$

$$\dot{x}(0) = -6 \times \omega \times sin(\varphi) < 0$$

$$\rightarrow \varphi = +1.91\ rad$$



Q2:

The displacement of an object oscillating on a spring is given by

$$x(t) = x_m \cos[\omega t + \phi]$$

If the initial displacement is zero and the initial velocity is in the negative x direction, Calculate the phase constant ϕ.

Answer: The given boundary conditions are:

$$x(0) = 0 \qquad \text{(I)}$$

$$v(0) = -v_m \qquad \text{(II)}$$

Using BC I, we have:

$$x(0) = x_m \cos[0 + \phi] = 0 \Rightarrow \phi = \pi / 2$$

Check the BC II, the initial velocity is the change in displacement with time.

$$v(t) = -v_m \sin[\omega t + \phi] \quad \Rightarrow \quad v(0) = -v_m \sin[\phi] = -v_m$$

So, the phase constant is π/2 rad.

Q1:

The acceleration of a body executing simple harmonic motion leads the velocity by what phase?

A. 0
B.π/8
C. π /4 rad
D. π /2 rad
E. π rad

Ans:

D

Extra Problems

T142_Q28.

A simple harmonic oscillator has amplitude of 0.035 m and a maximum speed of 28.0 cm/s. What is its speed when its displacement is 0.0175 m?

Answer: Given that $x_m = 0.035$ m, $v_m = \omega x_m = 0.28$ m/s $\Rightarrow \omega = 0.28/0.035 = 8$ rad/s.

Use the relation:

$$E = KE + PE = \frac{1}{2}kx_m^2 \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2 - \frac{1}{2}kx^2$$

$$\Rightarrow v^2 = \frac{k}{m}\left(x_m^2 - x^2\right) = \omega^2\left(x_m^2 - x^2\right)$$

$$v = \omega\sqrt{\left(x_m^2 - x^2\right)} = 8\sqrt{0.035^2 - 0.0175^2} = \underline{0.242 \text{ m/s}}$$

T113_Q27.

If the phase angle ϕ for a block-spring system in SHM is π/6 rad and the block's position is given by

$$x(t) = x_m \sin[\omega t + \phi]\ ,$$

what is the ratio of the kinetic energy to the potential energy at time t = 0?

Answer:

$$x(t) = x_m \sin[\omega t + \phi], \quad v(t) = \omega x_m \cos[\omega t + \phi]$$

$$U(t=0) = \tfrac{1}{2}kx^2(0) = \tfrac{1}{2}k\,x_m^2 \sin^2[\phi],$$

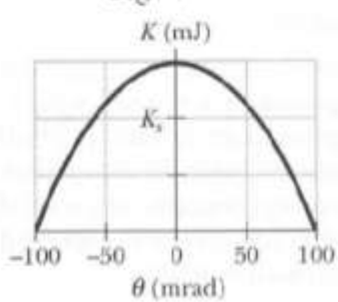
$$K(t=0) = \tfrac{1}{2}mv^2(0) = \tfrac{1}{2}m\omega^2\,x_m^2\cos^2[\phi] = \tfrac{1}{2}k\,x_m^2\cos^2[\phi]$$

$$\therefore \frac{K(t=0)}{U(t=0)} = \frac{\cos^2[\phi]}{\sin^2[\phi]} = \frac{\cos^2[\pi/6]}{\sin^2[\pi/6]} = 3$$

T113_Q30.

Figure 7 shows the kinetic energy K of a simple pendulum versus its angle θ from the vertical. The vertical axis scale is set by K_s = 20.0 mJ. The pendulum bob has mass 0.30 kg. What is the length of the pendulum?

Fig# 7



Ans:

$$\theta = 100 \times 10^{-3}\text{rad} = 5.73°$$

$$k_{max} = U_{max} = mgL(1 - \cos\theta)$$

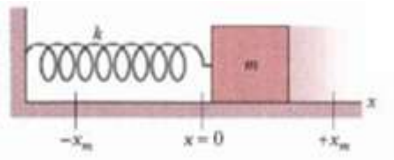
$$\therefore 30 \times 10^{-3} = (0.30)(9.8)(L)(1 - \text{Cos}5.73°)$$

$$30 \times 10^{-3} = (0.30)(9.8)(L)(1 - 0.995)$$

$$L = \frac{30 \times 10^{-3}}{(0.30)(9.8)(0.00499)}$$

$$L = 2044 \times 10^{-3}\ \text{m} = 2.04\ \text{m}$$

T112_Q14.



In **Figure 7**, the horizontal block-spring system has a kinetic energy of K = 5.0 J and an elastic potential energy of U = 3.0 J, when the block is at x = + 2.0 cm. What are the kinetic and elastic potential energy when the block is at x = – xm?

Answer:

$$(K+U)_{x=2} = (K+U)_{x=\pm x_m}$$

$$\Rightarrow 5+3 = 0+U \Rightarrow U = \underline{8\text{ J}}$$

T121_Q30.

Figure 12 shows plots of the kinetic energy *K* versus position *x* for three linear simple harmonic oscillators that have the **same** mass. Rank the plots according to the corresponding **period** of the oscillator, greatest first.

Answer:

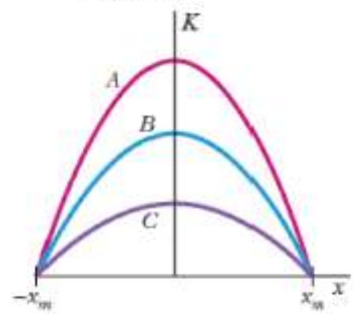
A) C, B, A
B) A, B, C
C) B, A, C
D) A, C, B
E) B, C, A

$$K_{\max}(x=0) = E = \frac{1}{2}k\,x_m^2 = \frac{1}{2}m\,\omega^2 x_m^2,$$

m And x_m are constant, $\Rightarrow K_{\max} \propto \omega^2 \propto \dfrac{1}{T^2}$

The answer is A.

Figure 12



T131_ Q25.

A particle executes simple harmonic motion in one dimension described by:

$$x = (10\text{ cm})\sin[\pi t] = x_m \sin[\pi t]$$

where t is in seconds. At what time is the potential energy of the particle equal to its kinetic energy?

Answer: The boundary condition is $U(x) = K(x)$.

Uusing the total energy as:

$$E = K + U = U + U = 2U$$

$$\Rightarrow \frac{1}{2}k\,x_m^2 = 2\frac{1}{2}k\,x^2$$

$$x = x_m/\sqrt{2} \qquad (1)$$

$$x = x_m \sin[\pi t] \qquad (2)$$

Equating (1) and (2), one finds:

$$x = \begin{cases} \dfrac{x_m}{\sqrt{2}} \\ x_m \sin[\pi t] \end{cases} \Rightarrow \frac{x_m}{\sqrt{2}} = x_m \sin[\pi t]$$

Last equation implies:

$$t = \frac{1}{\pi}\sin^{-1}\left[\frac{1}{\sqrt{2}}\right] = \frac{1}{\pi}45^o = \underline{0.25\text{ s}}.$$

T151_Q24.
A block-spring system, moving with simple harmonic motion on a horizontal frictionless surface, has an amplitude x_m. When the kinetic energy of the block equals twice the potential energy stored in the spring, what is the position x of the block? Ans: A) $x_m/\sqrt{3}$

Answer: The boundary condition is $2U(x) = K(x)$.

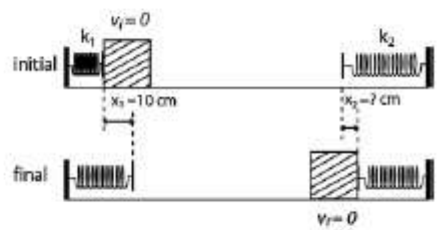
$$E = K + U = 2U + U \quad \Rightarrow \quad E = 3U(x)$$

$$\Rightarrow \frac{1}{2}kx_m^2 = 3\frac{1}{2}kx^2$$

Solve for x, we found:

$$x = \underline{x_m/\sqrt{3}}$$

Q: Two springs of spring constants k_1 = 40 N/m and k_2 = 160 N/m are fixed opposite to each other on a frictionless floor as shown in Fig. 4. A 0.50 kg block, not attached to any of the springs, oscillates between the two springs. If the block compresses the first spring by a maximum distance of 10 cm then it will compress the second spring by a maximum distance of:



Answer: The force on each one will be the same, which implies that

$$k_1 x_1^2 = k_2 x_2^2 \Rightarrow x_2 = \sqrt{\frac{k_1}{k_1}}x_1 = \sqrt{\frac{40}{160}}10 = \underline{5.0\text{ cm}}$$

Oscillations

Summary of Equations:

$$\frac{d^2x}{dt^2} = -\omega^2 x \text{ (SHE)}, \ \omega = \sqrt{k/m}, \ \omega = 2\pi f = \frac{2\pi}{T}, \ T = 2\pi\frac{1}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$x(t) = A\cos(\omega t + \varphi), \qquad A = x_m$$

$$\begin{aligned} x(t) &= A\cos(\omega t + \varphi) \\ v(t) &= -A\omega\sin(\omega t + \varphi) \quad \Rightarrow \quad v_{max} = A\omega \\ a(t) &= -A\omega^2\cos(\omega t + \varphi) \quad \Rightarrow \quad a_{max} = A\omega^2 \end{aligned}$$

15-4 PENDULUMS, CIRCULAR MOTION

The Physical Pendulum

A real pendulum, usually called a **physical pendulum,** can have a complicated distribution of mass. A physical pendulum is simply a rigid object which swings freely about some pivot point. The physical pendulum may be compared with a simple pendulum, which consists of a small mass suspended by a (ideally massless) string.
Does it also undergo SHM? If so, what is its period?
To answer this question, let us consider the physical pendulum is a stick with length L, see figure 1, and it pivots about a fixed point a distance h from the center of mass. The period T of physical pendulum is given by

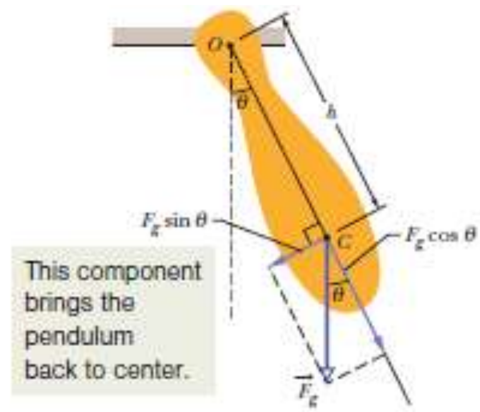
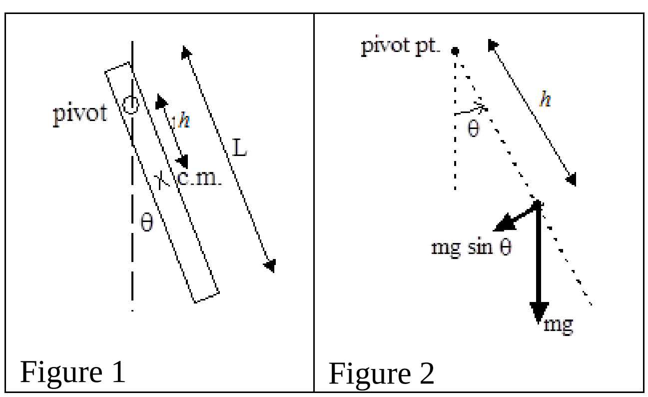


Figure 15-12 A physical pendulum. The restoring torque is $hF_g \sin\theta$. When $\theta = 0$, center of mass C hangs directly below pivot point O.

(1) $$T = 2\pi\sqrt{\frac{I}{mgh}},$$



where I is the moment of inertia about the pivot point, m is the total mass, and g is the acceleration of gravity.

To derive eq'n(1), recall the relation between torque τ and angular acceleration α,

(2) $$\tau = I\,\alpha .$$

In this case, the torque is due to the force of gravity, mg. To compute the torque, one can assume that the full weight mg of the object acts at the center of mass (This is one of the reasons that center-of-mass is such a useful concept.) The magnitude of the torque about the pivot point is $\tau = mgh\sin\theta$. Since $\alpha = \frac{d^2\theta}{dt^2}$, we have for the equation of motion (from eq'n (2))

(3) $$mgh\sin\theta = -I\,\frac{d^2\theta}{dt^2} .$$

Understanding the minus sign in eq'n(3) is a bit tricky. Torque has a sign, depending on the sense of rotation induced by the torque. If we choose the counterclockwise sense of rotation as positive for both the torque and the angle θ, then at the moment shown in the diagram above, the torque is negative and θ is positive and the two always have the opposite sign as the object swings back and forth.

- For small θ, $\sin\theta \approx \theta$ (in rads!) , and we can write

(4) $$\frac{d^2\theta}{dt^2} \cong -\frac{mgh}{I}\theta .$$

This is the equation of a harmonic oscillator; the solution is

(5) $$\theta(t) = \theta_o \sin(\omega t + \phi) , \qquad \omega = \sqrt{\frac{mgh}{I}} ,$$

and ω is related to the period T by

(6) $$\omega = \frac{2\pi}{T},$$ which leads to eq'n (1).

- Although the total mass m appears in the expression (1) for the period T, the moment of inertia I is proportional to m, so the dependence on m cancels out and T is independent of the mass, just as with a simple pendulum. T depends only on the distribution of mass within the object, not on the total mass.

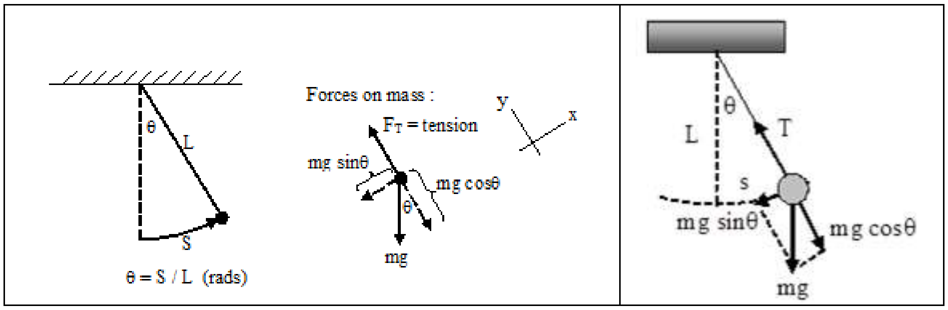
Example: A thin rod, of length 1.00 m, is pivoted from one end and is allowed to oscillate in a vertical plane like a pendulum. What is the period of oscillation of this system? Ignore air resistance and the friction at the pivot.

Answer: $T = 2\pi\sqrt{\frac{I}{mgh}}$, $I = mL^2/3, \quad h = L/2$

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mL^2/3}{mgL/2}} = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2\times 1}{3\times 9.8}} = \underline{1.64\ s}$$

Simple Pendulum

A simple pendulum consists of a small mass m suspended at the end of a massless string of length L. When displaced a small amount from equilibrium it will undergo simple harmonic motion, somewhat like a mass at the end of a Hooke's law spring. A pendulum executes SHM, <u>if</u> the amplitude is not too large, i.e $\sin\theta \cong \theta$.



The restoring force acting on the mass is the component of mg tangent to arc of the circle in which it swings.

$$F_s = -mg\sin\theta .$$

For small angles, $sin\theta \cong \theta$(in radians). So, using $\theta = s/L$,

$$F_s = -mg\,\theta = -mg\frac{s}{L} = -\left(\frac{mg}{L}\right)s$$

This is like Hooke's law, where k = mg/L. Thus,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

The periodic time is m independent.

☞ $I = mL^2$ $\Rightarrow T_{pend} = 2\pi\sqrt{\frac{I}{mgL}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$

<u>Example</u>: A 100-g mass oscillates at the end of a 50-cm string.

☞ *What is its period*?

$$T = 2\pi\sqrt{\frac{0.5m}{9.8m/s^2}} = \underline{1.42\,s}$$

☞ *What would be the length of a pendulum that had a period of* 1 s?

$$L = \frac{gT^2}{4\pi^2} = \frac{(9.8)(1)}{4\pi^2} = \underline{0.25m}$$

☞ *How would the period change if the mass were doubled*?

No change since T is independent of mass.

☞ *How would the period change is a pendulum were taken to the moon*?

T would increase since g(moon) < g(earth).

☞ *How would the period change if the amplitude (maximum swing) were increased*?

No change as long as θ is sufficiently small such that $\theta \simeq \sin\theta$.

Example: A thin uniform rod (mass = 0.50 kg) swings about an axis that passes through one end of the rod and is perpendicular to the plane of the swing. The rod swings with a period of 1.5 s and an angular amplitude of 10°. What is the length of the rod?

Answer: $T = 2\pi\sqrt{\frac{I}{mgh}}, \quad I = mL^2/3, \quad h = L/2$

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mL^2/3}{mgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

$$\Rightarrow L = \frac{T^2}{(2\pi)^2}(3g/2) = \frac{1.5^2}{(2\pi)^2}(3\times 9.8/2) = \underline{0.84 \text{ m}}$$

```
In[8]:= g = 9.8; T = 1.5;

In[9]:= Solve[T - 2 π √(2 L / (3 g)) == 0, L]

Out[9]= {{L → 0.8378}}
```

Example: In Fig. 15-44, a physical pendulum consists of a uniform solid disk (of radius R = 2.35 cm) supported in a vertical plane by a pivot located a distance d = 1.75 cm from the center of the disk. The disk is displaced by a small angle and released. What is the period of the resulting simple harmonic motion?

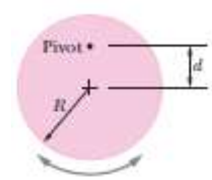


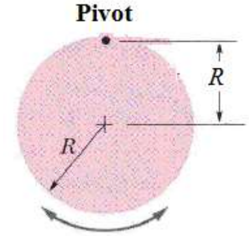
Figure 15-44
Problem 47.

Answer: We use Eq. 15-29 and the parallel-axis theorem $I = I_{cm} + mh^2$ where $h = d$. For a solid disk of mass m, the rotational inertia about its center of mass is $I_{cm} = mR^2/2$. Therefore,

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{mR^2/2 + md^2}{mgd}} = 2\pi\sqrt{\frac{R^2 + 2d^2}{2gd}} = 2\pi\sqrt{\frac{(2.35 \text{ cm})^2 + 2(1.75 \text{ cm})^2}{2(980 \text{ cm/s}^2)(1.75 \text{ cm})}}$$

$$= \underline{0.366 \text{ s}}.$$

Example: In Fig. below, a physical pendulum consists of a uniform solid disk (of radius R = 2.27 cm) supported in a vertical plane by a pivot located at the rim of the disk. The disk is displaced by a small angle and released. What is the period of the resulting simple harmonic motion?



Answer:

$$T = 2\pi\sqrt{\frac{I}{MgR}}, \quad I = I_{COM} + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{3R}{2g}} = \underline{0.370 \text{ s}}.$$

Sample Problem 15.05 Physical pendulum, period and length

In Fig. 15-13*a*, a meter stick swings about a pivot point at one end, at distance *h* from the stick's center of mass.

(a) What is the period of oscillation *T*?

KEY IDEA

The stick is not a simple pendulum because its mass is not concentrated in a bob at the end opposite the pivot point—so the stick is a physical pendulum.

Calculations: The period for a physical pendulum is given by Eq. 15-29, for which we need the rotational inertia *I* of the stick about the pivot point. We can treat the stick as a uniform rod of length *L* and mass *m*. Then Eq. 15-30 tells us that $I = \frac{1}{3}mL^2$, and the distance *h* in Eq. 15-29 is $\frac{1}{2}L$. Substituting these quantities into Eq. 15-29, we find

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{\frac{1}{3}mL^2}{mg(\frac{1}{2}L)}} \qquad (15\text{-}32)$$

$$= 2\pi\sqrt{\frac{2L}{3g}} \qquad (15\text{-}33)$$

$$= 2\pi\sqrt{\frac{(2)(1.00\text{ m})}{(3)(9.8\text{ m/s}^2)}} = 1.64\text{ s.} \qquad \text{(Answer)}$$

Note the result is independent of the pendulum's mass *m*.

(b) What is the distance L_0 between the pivot point *O* of the stick and the center of oscillation of the stick?

Calculations: We want the length L_0 of the simple pendulum (drawn in Fig. 15-13*b*) that has the same period as the physical pendulum (the stick) of Fig. 15-13*a*. Setting Eqs. 15-28 and 15-33 equal yields

$$T = 2\pi\sqrt{\frac{L_0}{g}} = 2\pi\sqrt{\frac{2L}{3g}}. \qquad (15\text{-}34)$$

You can see by inspection that

$$L_0 = \tfrac{2}{3}L \qquad (15\text{-}35)$$

$$= (\tfrac{2}{3})(100\text{ cm}) = 66.7\text{ cm.} \qquad \text{(Answer)}$$

In Fig. 15-13*a*, point *P* marks this distance from suspension point *O*. Thus, point *P* is the stick's center of oscillation for the given suspension point. Point *P* would be different for a different suspension choice.

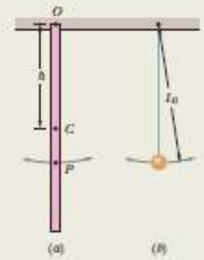
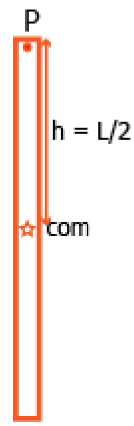


Figure 15-13 (*a*) A meter stick suspended from one end as a physical pendulum. (*b*) A simple pendulum whose length L_0 is chosen so that the periods of the two pendulums are equal. Point *P* on the pendulum of (*a*) marks the center of oscillation.

Example: The rotational inertia of a uniform thin rod about its end is $mL^2/3$, where M is the mass and L is the length. Such a rod is hung vertically from one end and set into small amplitude oscillation. If L = 100 cm this rod will have the same period as a simple pendulum of length L_o. Find L_o in **meters**.



Answer:

$$\text{Physical pendulum: } T = 2\pi\sqrt{\frac{I}{mgh}}$$

$$\text{where } I = ML^2/3 \text{ and } h = L/2\ ;\ \ L = 1\text{ m}$$

$$\text{Simple pendulum: } T_{pend} = 2\pi\sqrt{\frac{L_o}{g}}$$

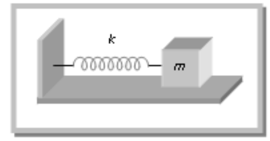
$$T = T_{pend} = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{L_o}{g}}$$

$$\Rightarrow \frac{I}{Mgh} = \frac{L_0}{g} \Rightarrow \frac{ML^2/3}{ML/2} = L_0 \Rightarrow L_0 = \frac{2L}{3}$$

$$\Rightarrow L_o = \frac{2\times 1}{3} = 0.667\text{ m}$$

Extra Problems

Spring Pendulum

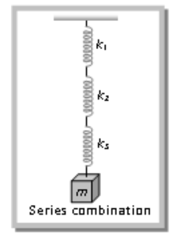


A point mass suspended from a mass less spring or placed on a frictionless horizontal plane attached with spring (fig.) constitutes a linear harmonic spring pendulum

Time period $T = 2\pi\sqrt{\frac{\text{inertia factor}}{\text{spring factor}}}$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad \text{Frequency } f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

- Series combination: If n springs of different force constant are connected in series having force constant $k_1, k_2, k_3.......$ respectively then



$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + ...$$

If all spring have same spring constant then

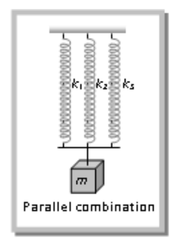
$$k_{eff} = \frac{k}{n}$$

- Parallel combination: If the springs are connected in parallel then

$$k_{eff} = k_1 + k_2 + k_3 + ...$$

If all spring have same spring constant then

$$k_{eff} = nk$$



Example: The end point of a spring oscillates with a period of 1.8 s when a block with mass m is attached to it. When this mass is increased by 2.0 kg, the period is found to be 3.0 s. Find m.

Answer: Using Δm = 2.0 kg, T_1 = 1.8 s and T_2 = 3.0 s, we write

$$T_1 = 2\pi\sqrt{\frac{m}{k}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{m+\Delta m}{k}}.$$

Dividing one relation by the other, we obtain

$$\frac{T_2}{T_1} = \sqrt{\frac{m+\Delta m}{m}}$$

which (after squaring both sides) simplifies to $m = \frac{\Delta m}{(T_2/T_1)^2 - 1} = 1.1$ kg.

Q (Hard): In Fig. 15-43, the pendulum consists of a uniform disk with radius r = 10.0 cm and mass 500 g attached to a uniform rod with length L = 500 mm and mass 270 g. (a) Calculate the rotational inertia of the pendulum about the pivot point. (b) What is the distance between the pivot point and the center of mass of the pendulum? (c) Calculate the period of oscillation.

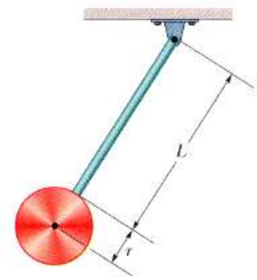


FIG. 15-41 Problem 43.

Answer:

(a) A uniform disk pivoted at its center has a rotational inertia of $\frac{1}{2}Mr^2$, where M is its mass and r is its radius. The disk of this problem rotates about a point that is displaced from its center by r+ L, where L is the length of the rod, so, according to the parallel-axis theorem, its rotational inertia is $\frac{1}{2}Mr^2+\frac{1}{2}M(L+r)^2$. The rod is pivoted at one end and has a rotational inertia of $mL^2/3$, where m is its mass. The total rotational inertia of the disk and rod is

$$I=\frac{1}{2}Mr^2+M(L+r)^2+\frac{1}{3}mL^2$$
$$=\frac{1}{2}(0.500\text{kg})(0.100\text{m})^2+(0.500\text{kg})(0.500\text{m}+0.100\text{m})^2+\frac{1}{3}(0.270\text{kg})(0.500\text{m})^2$$
$$=0.205\text{kg}\cdot\text{m}^2.$$

(b) We put the origin at the pivot. The center of mass of the disk is

$$\ell_d=L+r=0.500\text{ m}+0.100\text{ m}=0.600\text{ m}$$

away and the center of mass of the rod is $\ell_r=L/2=(0.500\text{ m})/2=0.250\text{ m}$ away, on the same line. The distance from the pivot point to the center of mass of the disk-rod system is

$$d=\frac{M\ell_d+m\ell_r}{M+m}=\frac{(0.500\text{ kg})(0.600\text{ m})+(0.270\text{ kg})(0.250\text{ m})}{0.500\text{ kg}+0.270\text{ kg}}=0.477\text{ m.}$$

(c) The period of oscillation is

$$T=2\pi\sqrt{\frac{I}{(M+m)gd}}=2\pi\sqrt{\frac{0.205\text{ kg}\cdot\text{m}^2}{(0.500\text{ kg}+0.270\text{ kg})(9.80\text{ m/s}^2)(0.447\text{ m})}}=1.50\text{ s}.$$

T112_Q: (Hard) In Figure 6, a stick of length L = 1.73 m oscillates as a physical pendulum. What value of x between the stick's center of mass and its pivot point O gives the least period?

Answer:

$$T = 2\pi\sqrt{\frac{I}{mgd}}\ ;\ d = x$$

$$I = \frac{1}{12}mL^2 + mx^2\ , \qquad T = 2\pi\left[\frac{\frac{1}{12}mL^2 + mx^2}{mgx}\right]^{\frac{1}{2}}$$

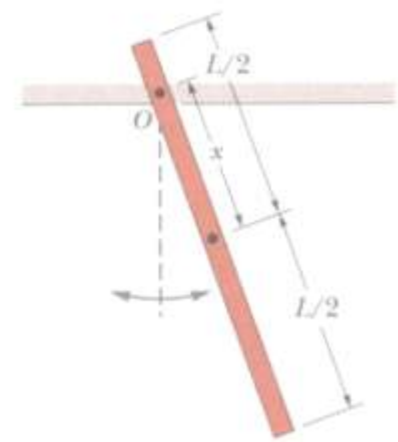
$$T^2 = 4\pi^2\left(\frac{\frac{1}{12}mL^2 + mx^2}{mgx}\right) = \frac{4\pi^2}{g}\left(\frac{1}{12}L^2x^{-1} + x\right)$$

$$2T\frac{dt}{dx} = \frac{4\pi^2}{g}\left(-\frac{1}{12}L^2x^{-1} + 1\right) = 0$$

$$\frac{L^2}{12x^2} = 1 \Rightarrow x^2 = \frac{L^2}{12} = \frac{(1.73)^2}{12}$$

$$\text{or } x^2 = 0.249 \Rightarrow x = 0.499\ \text{m}$$

Fig# 6



Q: Suppose that a simple pendulum consists of a small 60.0 g bob at the end of a cord of negligible mass. If the angle u between the cord and the vertical is given by

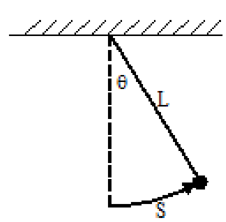
$$\theta(t) = (0.0800\ \text{rad})\cos\left[(4.43\ \text{rad/s})t + \phi\right]$$

what are
(a) The pendulum's length and
(b) Its maximum kinetic energy?

Answer

(a) Comparing the given expression to the equation $\theta(t) = \theta_{\max}\cos[\omega t + \phi]$ (after changing notation $x \to \theta$), we see that ω = 4.43 rad/s. Since $\omega = \sqrt{g/L}$ then we can solve for the length: L = 0.499 m.

Note: In physical pendulum, see the following figure:



$\theta = S / L$ (rads)

The amplitude is the maximum tangential distance $A = S = L\theta_{\max}$, where L is the length of the string.

(b) Since $v_m = \omega A = \omega L \theta_m$ = (4.43 rad/s)(0.499 m)(0.0800 rad) and m = 0.0600 kg, then we can find the maximum kinetic energy: $\frac{1}{2}mv_m^2 = 9.40 \times 10^{-4}$ J.
