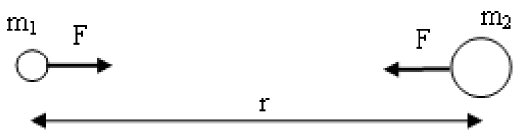
Gravitation

This is one of the most interesting, challenging, and mysteries subject in Physics, old and modern era.

13-1 NEWTON'S LAW OF GRAVITATION

Newton's Universal Law of Gravitation (first stated by Newton):
Any two masses m_1 and m_2 exert an attractive gravitational force on each other according to the rule:

$$|\vec{F}| = G\frac{m_1\, m_2}{r^2}\ , \qquad (13.1)$$



This applies to all masses, not just big ones.

$$G \;=\; \textbf{gravitational constant} \;=\; 6.67\times 10^{-11}\ \mathrm{N\,m^2/kg^2} \qquad (13.2)$$

(G is very small, so it is very difficult to measure!)

- Don't confuse G with g: "Big G" and "little g" are totally different things.

Newton showed that the force of gravity must act according to this rule in order to produce the observed motions of the planets around the sun, of the moon around the earth, and of projectiles near the earth. He then had the great insight to realize that this same force acts between *all* masses. [That gravity acts between all masses, even small ones, was experimentally verified in 1798 by Cavendish]

Newton couldn't say *why* gravity acted this way, only *how*. Einstein (1915) General Theory of Relativity, explained why gravity acted like this.

Example: Force of attraction between two humans: 2 people with masses $m_1 \cong m_2 \cong 70$ kg, distance r = 1 m apart.
Answer:

$$|\vec{F}| = G\frac{m_1\, m_2}{r^2} = \frac{(6.67\times10^{-11})(70)^2}{1^2} = 3.3\times10^{-7}\ \mathrm{N}$$

This is a very tiny force! It is the weight of a mass of 3.4×10^{-5} gram. A hair weighs 2×10^{-3} grams – the force of gravity between two people talking is about 1/60 the weight of a single hair.

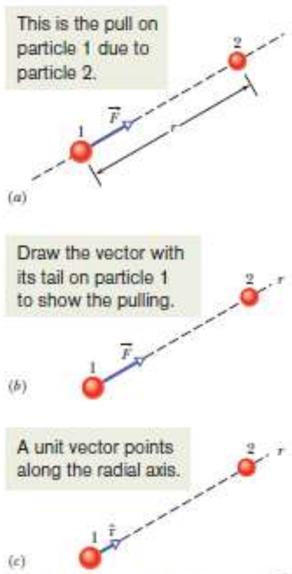


Figure 13-2 (*a*) The gravitational force $\vec{F}$ on particle 1 due to particle 2 is an attractive force because particle 1 is attracted to particle 2. (*b*) Force $\vec{F}$ is directed along a radial coordinate axis *r* extending from particle 1 through particle 2. (*c*) $\vec{F}$ is in the direction of a unit vector $\hat{r}$ along the *r* axis.

In Fig. 13-2*a*, $\vec{F}$ is the gravitational force acting on particle 1 (mass m_1) due to particle 2 (mass m_2). The force is directed toward particle 2 and is said to be an *attractive force* because particle 1 is attracted toward particle 2. The magnitude of the force is given by Eq. 13-1. We can describe $\vec{F}$ as being in the positive direction of an *r* axis extending radially from particle 1 through particle 2 (Fig. 13-2*b*). We can also describe $\vec{F}$ by using a radial unit vector $\hat{r}$ (a dimensionless vector of magnitude 1) that is directed away from particle 1 along the *r* axis (Fig. 13-2*c*). From Eq. 13-1, the force on particle 1 is then

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}. \tag{13-3}$$

The gravitational force on particle 2 due to particle 1 has the same magnitude as the force on particle 1 but the opposite direction. These two forces form a third-law force pair, and we can speak of the gravitational force *between* the two particles as having a magnitude given by Eq. 13-1. This force between two particles is not altered by other objects, even if they are located between the particles. Put another way, no object can shield either particle from the gravitational force due to the other particle.

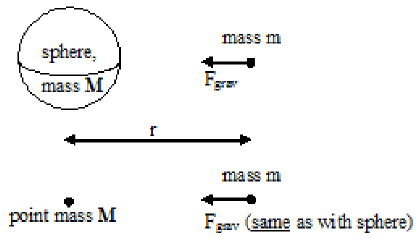
The strength of the gravitational force—that is, how strongly two particles with given masses at a given separation attract each other—depends on the value of the gravitational constant *G*. If *G*—by some miracle—were suddenly multiplied by a factor of 10, you would be crushed to the floor by Earth's attraction. If *G* were divided by this factor, Earth's attraction would be so weak that you could jump over a building.

Nonparticles. Although Newton's law of gravitation applies strictly to particles, we can also apply it to real objects as long as the sizes of the objects are small relative to the distance between them. The Moon and Earth are far enough apart so that, to a good approximation, we can treat them both as particles—but what about an apple and Earth? From the point of view of the apple, the broad and level Earth, stretching out to the horizon beneath the apple, certainly does not look like a particle.

Newton solved the apple–Earth problem with the *shell theorem:*

Notes:

1- A **uniform spherical shell** of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its center.



2- Important fact about the gravitational force from spherical masses: a **uniform spherical body** exerts a gravitational force on surrounding bodies that is the same as if all the sphere's mass were concentrated at its center. This is difficult to prove.

3- Symbolically, the force $\vec{F}_{ij}$ represents the gravitational force on particle i due to particle j.

4- $\left|\vec{F}_{ij}\right| = \left|\vec{F}_{ji}\right|$, but in the opposite direction as the Newton's third law.

5- The force between two particles is not altered by other objects, even if they are located between the particles. Put another way, no object can shield either particle from the gravitational force due to the other particle.

13-2 GRAVITATION AND THE PRINCIPLE OF SUPERPOSITION

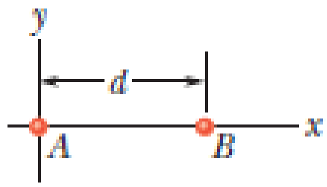
For n interacting particles, we can write the principle of superposition for the gravitational forces on particle 1 as

$$\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}. \quad (13\text{-}4)$$

Here $\vec{F}_{1,net}$ is the net force on particle 1 due to the other particles and, for example, $\vec{F}_{13}$ is the force on particle 1 from particle 3. We can express this equation more compactly as a vector sum:

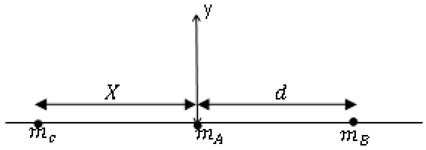
$$\vec{F}_{1,net} = \sum_{i=2}^{n} \vec{F}_{1i} \quad (13\text{-}5)$$

Example1: In **Figure**, two point particles are fixed on an x axis separated by distance d = 3.50 m. Particle A, located at the origin, has mass m_A = 1.00 kg and particle B has mass m_B = 3.00 kg. A third particle C, of mass m_C = 75.0 kg is to be placed on the x axis and near particles A and B. At what x coordinate should C be placed so that the net gravitational force on particle A from particles B and C is zero?



Answer:

First, we have to check for the position of m_C that gives the net gravitational force on particle A from particles B and C is zero. The place will be in the left of m_A, Why????



m_A $\to F_{AC}$ $\to F_{AB}$ m_C m_B; m_A $\to F_{AC}$ $\to F_{AB}$ m_B m_C; m_C $F_{AC} \leftarrow m_A \to F_{AB}$ m_B

Second, Use the condition of equilibrium for particle A:

$$\sum F_A = 0 \Rightarrow |F_{CA}| = |F_{AB}| \qquad \frac{Gm_c m_A}{x^2} = \frac{Gm_A m_B}{d^2} \Rightarrow \frac{\sqrt{m_c}}{X} = \frac{\sqrt{m_B}}{d}$$

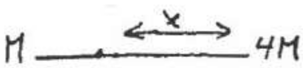
$$|X| = d\sqrt{\frac{m_c}{m_B}} = 3.5 \times \sqrt{\frac{75}{3}} = 3.5 \times 5 = 17.5$$

So the position of particle C will be at -17.5 m from point A.

Example2: Two particles with masses M and 4M are separated by a distance D. What is the shortest distance from the 4M mass for which the net gravitational field due to the two masses is zero?

Answer: Suppose the distance between the two masses is D, and the test mass is M. The condition for the net gravitational field due to the two masses is zero is given as:

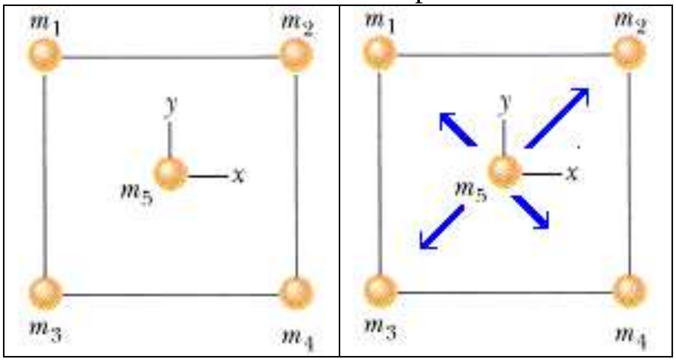
$|\vec{F}_{M,M}| = |\vec{F}_{M,4M}|$, i.e



$$G\frac{4MM}{x^2} = G\frac{MM}{(D-x)^2}$$

$$2(D-x) = \pm x \Rightarrow D = \frac{3}{2}x \Rightarrow x = \frac{2}{3}D$$

Example3: In Fig. 13-32, a square of edge length 20.0 cm is formed by four spheres of masses m_1 = 5.00 g, m_2 = 3.00 g, m_3 = 1.00 g, and m_4 = 5.00 g. In unit-vector notation, what is the net gravitational force from them on a central sphere with mass m_5 = 2.50 g?



Answer: From the superposition principle, we have:

$$F_{\text{net}} = \vec{F}_{51} + \vec{F}_{52} + \vec{F}_{53} + \vec{F}_{54}\,.$$

The gravitational forces $|\vec{F}_{51}| = |\vec{F}_{54}|$ on m_5 from the two 5.00 g masses m_1 and m_4 cancel each other. Contributions to the net force on m_5 come from the remaining two masses:

$$F_{\text{net}} = \vec{F}_{52} + \vec{F}_{53} = G\frac{m_5}{\left(\sqrt{2}\times 10^{-1}\text{ m}\right)^2}(m_2 - m_3)$$

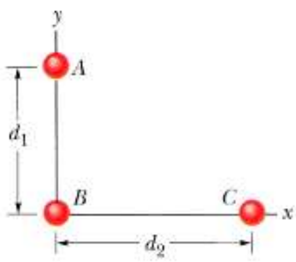
$$= \frac{\left(6.67\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2\right)\left(2.50\times10^{-3}\text{ kg}\right)\left(3.00\times10^{-3}\text{ kg} - 1.00\times10^{-3}\text{ kg}\right)}{\left(\sqrt{2}\times10^{-1}\text{ m}\right)^2}$$

$$= 1.67\times10^{-14}\text{ N.}$$

The force is directed along the diagonal between m_2 and m_3, towards m_2. In unit-vector notation, we have

$$\vec{F}_{\text{net}} = F_{\text{net}}(\cos 45°\hat{\text{i}} + \sin 45°\hat{\text{j}}) = (1.18\times10^{-14}\text{ N})\,\hat{\text{i}} + (1.18\times10^{-14}\text{ N})\,\hat{\text{j}}$$

Example4: In Fig., three 5.00 kg spheres are located at distances d_1 = 0.300 m and d_2 = 0.400 m. What are the (a) magnitude and (b) direction (relative to the positive direction of the *x* axis) of the net gravitational force on sphere *B* due to spheres *A* and *C*?



Answer: Using $F = GmM/r^2$, we find that the topmost mass pulls upward on the one at the origin with 1.9 × 10^{-8} N, and the rightmost mass pulls rightward on the one at the origin with 1.0 × 10^{-8} N. Thus, the (*x*, *y*) components of the net force, which can be converted to polar components (here we use magnitude-angle notation), are

$$\vec{F}_{\text{net}} = \left(1.04\times10^{-8}, 1.85\times10^{-8}\right) \Rightarrow \left(2.13\times10^{-8}\ \angle 60.6°\right).$$

(a) The magnitude of the force is 2.13 × 10^{-8} N.

(b) The direction of the force relative to the +*x* axis is $60.6°$.

13-3 GRAVITATION NEAR EARTH'S SURFACE

Table 13-1 Variation of a_g with Altitude

Altitude (km)	a_g (m/s^2)	Altitude Example
0	9.83	Mean Earth surface
8.8	9.80	Mt. Everest
36.6	9.71	Highest crewed balloon
400	8.70	Space shuttle orbit
35 700	0.225	Communications satellite

Gravitation Near Earth's Surface

Let us assume that Earth is a uniform sphere of mass M. The magnitude of the gravitational force from Earth on a particle of mass m, located outside Earth a distance r from Earth's center, is then given by Eq. 13-1 as

$$F = G\frac{Mm}{r^2}. \tag{13-9}$$

If the particle is released, it will fall toward the center of Earth, as a result of the gravitational force $\vec{F}$, with an acceleration we shall call the **gravitational acceleration** $\vec{a}_g$. Newton's second law tells us that magnitudes F and a_g are related by

$$F = ma_g. \tag{13-10}$$

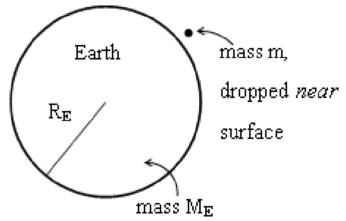
Now, substituting F from Eq. 13-9 into Eq. 13-10 and solving for a_g, we find

$$a_g = \frac{GM}{r^2}. \tag{13-11}$$

Table 13-1 shows values of a_g computed for various altitudes above Earth's surface. Notice that a_g is significant even at 400 km.

Computation of g

We can now *compute* the acceleration of gravity g! (Before, g was experimentally determined, and it was a mystery why g was the same for all masses.)



$F_{grav} = m a = m g$ (i)

$F_{grav} = G \dfrac{M_E m}{R_E^{\,2}}$ (ii)

(since $r = R_E$ is distance from m to center of Earth). Equating (i) and (ii) implies:

m's cancel ! $\Rightarrow$ $g = \dfrac{G M_E}{R_E^{\,2}}$

If you plug in the numbers for G, M_E, and R_E, you get $g = 9.8$ m/s^2.

Newton's Theory explains why all objects near the Earth's surface fall with the same acceleration (because the m's cancel in $F_{grav} = \dfrac{G M m}{R^2} = m a$.) Newton's theory also makes a quantitative prediction for the value of g, which is correct.

Example: g on Planet X. Planet X has the same mass as earth ($M_X = M_E$) but has ½ the radius ($R_X = 0.5\, R_E$). What is g_x , the acceleration of gravity on planet X?

Answer:

Method I, Planet X is denser than earth, so expect g_x larger than g.

$$g_x = \frac{G M_X}{R_X^{\,2}} = \frac{G M_E}{(R_E/2)^2} = \frac{1}{(1/2)^2}\underbrace{\frac{G M_E}{R_E^{\,2}}}_{\text{g of earth}} = 4g.$$

☺ Don't need values of G, M_E, and R_E!

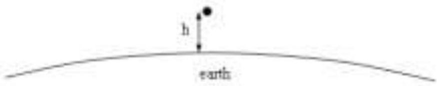
Method II, set up a ratio:

$$\frac{g_x}{g_E} = \frac{\left(\cancel{G} M_X / R_X^{\,2}\right)}{\left(\cancel{G} M_E / R_E^{\,2}\right)} = \frac{M_X}{M_E}\left(\frac{R_E}{R_X}\right)^2 = 1\cdot 2^2 = 4\ , \quad g_X = 4 g_E$$

Notes:

At height h above the surface of the earth, g is less, since we are further from the surface, further from the earth's center.

$r = R_E + h \Rightarrow \quad g = \frac{G M_E}{r^2} = \frac{G M_E}{(R_E + h)^2}$



The space shuttle orbits earth at an altitude of about 200 mi × 1.6 km/mi ≅ 320 km. Earth's radius is R_E = 6380 km. So the space shuttle is only about 5% further from the earth's center than we are. If r is 5% larger, then r^2 is about 10% larger, and

$$F_{grav}(\text{on mass m in shuttle}) = G\frac{M_E\, m}{(R_E + h)^2} \cong \text{about 10\% less than on earth's surface}$$

We assumed that *g* has the constant value 9.8 m/s^2 any place on Earth's surface. However, any *g* value measured at a given location will differ from the a_g value calculated with Eq. 13-11 for that location for three reasons:

1) Earth's mass is not distributed uniformly,
2) Earth is not a perfect sphere, and
3) Earth rotates.

Moreover, because *g* differs from a_g, the same three reasons mean that the measured weight *mg* of a particle differs from the magnitude of the gravitational force on the particle is given by:

$$g = a_g - \omega^2 R_E$$

Extra Problems

Q11 Two concentric shells of uniform density having masses M1 and M2 and Radii R_1 =2.0 m, R_2 = 4.0 m are situated as shown in FIGURE 4. Find the gravitational FORCE on a particle of mass m placed at point B at a distance of 3.0 m from the center :A1 (G*M1*m)/9.

Answer: The only affected force is the one due to the inner shell.

$$F = F_1 + F_2 = G\frac{mM1}{3^2} + 0 = G\frac{mM1}{9}.$$

FIGURE 4

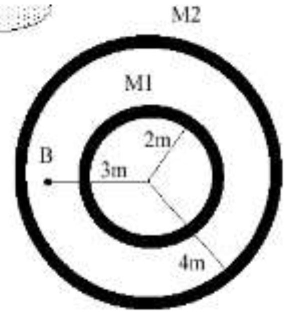


Figure 13-4*a* shows an arrangement of three particles, particle 1 of mass m_1 = 6.0 kg and particles 2 and 3 of mass $m_2 = m_3$ = 4.0 kg, and distance a = 2.0 cm. What is the net gravitational force $\vec{F}_{1,net}$ on particle 1 due to the other particles?

KEY IDEAS (1) Becau tude of the gravitatio ther of the other par Gm_1m_2/r^2). (2) The

force on particle 1 is toward the particle responsible for it. (3) Because the forces are not along a single axis, we *cannot* simply add or subtract their magnitudes or their components to get the net force. Instead, we must add them as vectors.

Calculations: From Eq. 13-1, the magnitude of the force $\vec{F}_{12}$ on particle 1 from particle 2 is

$$F_{12} = \frac{Gm_1m_2}{a^2} = \frac{(6.67\times10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)(6.0\ \text{kg})(4.0\ \text{kg})}{(0.020\ \text{m})^2} = 4.00\times10^{-6}\ \text{N}.$$

Similarly, the magnitude of force $\vec{F}_{13}$ on particle 1 from particle 3 is

$$F_{13} = \frac{Gm_1m_3}{(2a)^2} = \frac{(6.67\times10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2)(6.0\ \text{kg})(4.0\ \text{kg})}{(0.040\ \text{m})^2} = 1.00\times10^{-6}\ \text{N}.$$

Force $\vec{F}_{12}$ is directed in the positive direction of the y axis (Fig. 13-4*b*) and has only the y component F_{12}. Similarly, $\vec{F}_{13}$ is directed in the negative direction of the x axis and has only the x component $-F_{13}$.

To find the net force $\vec{F}_{1,net}$ on particle 1, we must add the two forces as vectors. We can do so on a vector-capable calculator. However, here we note that $-F_{13}$ and F_{12} are actually the x and y components of $\vec{F}_{1,net}$.

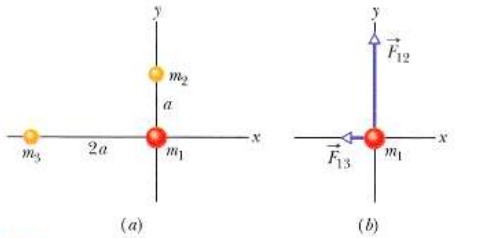


FIG. 13-4 (*a*) An arrangement of three particles. (*b*) The forces acting on the particle of mass m_1 due to the other particles.

Therefore, we can use Eq. 3-6 to find first the magnitude and then the direction of $\vec{F}_{1,net}$. The magnitude is

$$F_{1,net} = \sqrt{(F_{12})^2 + (-F_{13})^2} = \sqrt{(4.00\times10^{-6}\ \text{N})^2 + (-1.00\times10^{-6}\ \text{N})^2} = 4.1\times10^{-6}\ \text{N}. \quad \text{(Answer)}$$

Relative to the positive direction of the x axis, Eq. 3-6 gives the direction of $\vec{F}_{1,net}$ as

$$\theta = \tan^{-1}\frac{F_{12}}{-F_{13}} = \tan^{-1}\frac{4.00\times10^{-6}\ \text{N}}{-1.00\times10^{-6}\ \text{N}} = -76°.$$

Is this a reasonable direction? No, because the direction of $\vec{F}_{1,net}$ must be between the directions of $\vec{F}_{12}$ and $\vec{F}_{13}$. Recall from Chapter 3 (Problem-Solving Tactic 3) that a calculator displays only one of the two possible answers to a $\tan^{-1}$ function. We find the other answer by adding 180°:

$$-76° + 180° = 104°, \quad \text{(Answer)}$$

which *is* a reasonable direction for $\vec{F}_{1,net}$.

Figure 13-5*a* shows an arrangement of five particles, with masses $m_1 = 8.0$ kg, $m_2 = m_3 = m_4 = m_5 = 2.0$ kg, and with $a = 2.0$ cm and $\theta = 30°$. What is the net gravitational force $\vec{F}_{1,net}$ on particle 1 due to the other particles?

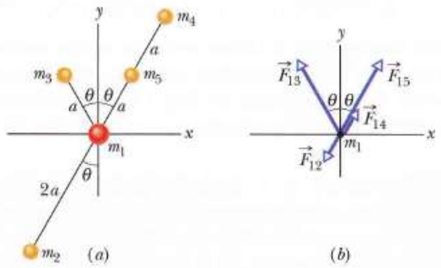


FIG. 13-5 (*a*) An arrangement of five particles. (*b*) The forces acting on the particle of mass m_1 due to the other four particles.

KEY IDEAS (1) Because we have particles, the magnitude of the gravitational force on particle 1 due to either of the other particles is given by Eq. 13-1 ($F = Gm_1m_2/r^2$). (2) The direction of a gravitational force on particle 1 is toward the particle responsible for the force. (3) We can use symmetry to eliminate unneeded calculations.

Calculations: For the magnitudes of the forces on particle 1, first note that particles 2 and 4 have equal masses and equal distances of $r = 2a$ from particle 1. Thus, from Eq. 13-1, we find

$$F_{12} = F_{14} = \frac{Gm_1m_2}{(2a)^2}. \qquad (13\text{-}7)$$

Similarly, since particles 3 and 5 have equal masses and are both distance $r = a$ from particle 1, we find

$$F_{13} = F_{15} = \frac{Gm_1m_3}{a^2}. \qquad (13\text{-}8)$$

We could now substitute known data into these two equations to evaluate the magnitudes of the forces, indicate the directions of the forces on the free-body diagram of Fig. 13-5*b*, and then find the net force either (1) by resolving the vectors into *x* and *y* components, finding the net *x* and net *y* components, and then vectorially combining them or (2) by adding the vectors directly on a vector-capable calculator.

Instead, however, we shall make further use of the symmetry of the problem. First, we note that $\vec{F}_{12}$ and $\vec{F}_{14}$ are equal in magnitude but opposite in direction; thus, those forces *cancel.* Inspection of Fig. 13-5*b* and Eq. 13-8 reveals that the *x* components of $\vec{F}_{13}$ and $\vec{F}_{15}$ also *cancel,* and that their *y* components are identical in magnitude and both act in the positive direction of the *y* axis. Thus, $\vec{F}_{1,net}$ acts in that same direction, and its magnitude is twice the *y* component of $\vec{F}_{13}$:

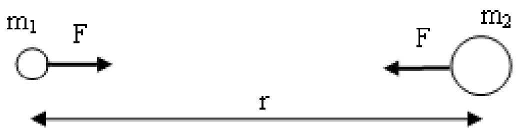
$$\begin{aligned} F_{1,net} &= 2F_{13}\cos\theta = 2\frac{Gm_1m_3}{a^2}\cos\theta \\ &= \frac{2(6.67\times10^{-11}\ \mathrm{m^3/kg\cdot s^2})(8.0\ \mathrm{kg})(2.0\ \mathrm{kg})}{(0.020\ \mathrm{m})^2}\cos 30° \\ &= 4.6\times10^{-6}\ \mathrm{N}. \qquad \text{(Answer)} \end{aligned}$$

Note that the presence of particle 5 along the line between particles 1 and 4 does not alter the gravitational force on particle 1 from particle 4.

Gravitation

Summary (13-1, -2,-3)
Newton's Universal Law of Gravitation (first stated by Newton): *any two masses m_1 and m_2 exert an attractive gravitational force on each other according to the rule:*

$$F = G\frac{m_1 m_2}{r^2}, \qquad (13.1)$$



G = **gravitational constant** = 6.67×10^{-11} N m^2 / kg^2

$$a_g = G\frac{M}{R^2} \qquad (13\text{-}11) \qquad \text{and on earth } g = \frac{G M_E}{R_E{}^2}$$

13-5 GRAVITATIONAL POTENTIAL ENERGY

i- Measurement of Big G

The value of G ("big G") was not known until 1798. In that year, Henry Cavendish (English) measured the very tiny F_{grav} between 2 lead spheres, using a device called a **torsion balance**.

$$F_{grav} = G\frac{m_1 m_2}{r^2} \Rightarrow G = \frac{F_{grav} r^2}{m_1 m_2} \qquad \text{(If } F_{grav}\text{, r, and m's known, can compute G.)}$$

Before Cavendish's experiment, g and R_E were known, so using $g = \frac{G M_E}{R_E{}^2}$, one could compute the product $G \cdot M_E$, but G and M_E could not be determined separately. With Cavendish's measurement of G, one could then compute M_E. Hence, Cavendish "weighed the earth".

ii- Gravitational Potential Energy

Previously, we showed for the gravity that PE = mgh. But to derive PE = mgh, we assumed that F_{grav} = mg = constant, which is only true near the surface of the Earth. In general, $F_{grav} = G\frac{M m}{r^2} \neq$ constant (it depends on r). We now show that for the general case,

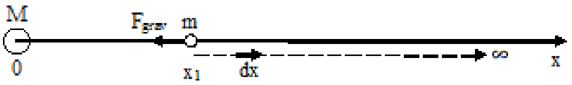
$$PE_{grav} = U_{grav}(r) = -\frac{G M m}{r}, \qquad [U_{grav}(r = \infty) = 0]$$

This is the gravitational potential energy for two masses, M and m, separated by a distance r. By convention, the zero of gravitational potential energy is set at r = ∞. [We will use the common notation U(r), instead of PE.] Recall the general definition of

PE: $$\Delta PE_F \equiv -W_F = -\int_{x1}^{x2} F(x)\,dx\,.$$

Here, we have used the definition of work for the case of 1D motion:

$$W_F \equiv \int_i^r \vec{F}\cdot d\vec{r} \underset{(1D)}{=} \int_{x1}^{x2} F(x)\cdot dx\,.$$



Consider a mass M at the origin and a mass m at position x_1, as shown in the diagram. We compute the work done by the force of gravity as the mass m moves from $x = x_1$ to $x = \infty$. The attractive force $F_{grav}(x)$ of mass M on mass m is in the negative direction of x. Here, the work done by gravity is negative, since force and displacement are in opposite directions:

$$W_{grav} = \int_{x1}^{\infty} F_{grav}(x)\,dx = -\int_{x1}^{\infty} \frac{GMm}{x^2}\,dx = +\left.\frac{GMm}{x}\right|_{x1}^{\infty} = -\frac{GMm}{x_1}$$

From the definition of PE_{grav},

$$\Delta PE_{grav} = \Delta U_{grav} = \underbrace{U_{grav}(x{=}\infty)}_{0} - U_{grav}(x_1) = -W_{grav} = +\frac{GMm}{x_1}\,.$$

Calling the initial position r (instead of x_1), we have $U_{grav}(r) = U(r) = -\dfrac{G\,M\,m}{r}$.

- ☞ With the choice of the zero of potential energy at infinity distance where the force approaches zero, the gravitational potential energy is the work done to bring an object from infinity to radius r.
- ☞ The negative potential energy indicates a bound state. An object at radius r out from the earth is bound to the earth by energy $U(r)$, and would require the supply of extra energy equal to U to escape the earth's gravity.

Important points

i. Potential energy is a scalar quantity.
ii. *Unit* : Joule
iii. *Dimension* : $[ML^2T^{-2}]$
iv. Gravitational potential energy is always negative in the gravitational field because the force is always attractive in nature.
v. As the distance *r* increases, the gravitational potential energy becomes less negative i.e., it increases.
vi. If $r = \infty$ then it becomes zero (maximum)
vii. In case of discrete distribution of masses, Gravitational potential energy

$$U_{total} = \sum_{\substack{j>i\\ j\neq i}} U_{ij}, \qquad U_{ij} = -G\frac{m_i m_j}{r_{ij}}$$

viii. If the body of mass *m* is moved from a point at a distance r_1 to a point at distance r_2 ($r_1 > r_2$) then change in potential energy

$$\Delta U = \int_{r_1}^{r_2} \frac{GMm}{x^2} dx = -GMm\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \text{ or } \Delta U = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

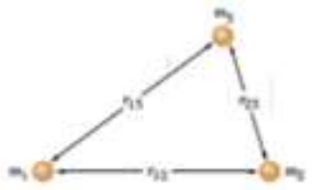
As r_1 is greater than r_2 , the change in potential energy of the body will be negative. It means that if a body is brought closer to earth it's potential energy decreases.

ix. Relation between gravitational potential energy and potential

$$U(r) = -\frac{GMm}{r} = -m\left[\frac{GM}{r}\right] = mV, \qquad V = \text{Gravitational potential.}$$

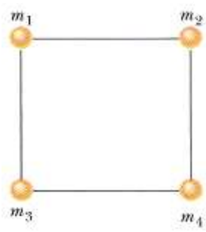
x. Gravitational potential energy of a body at height *h* from the earth surface is given by

$$U_h = -\frac{GMm}{R+h} = -\frac{gR^2m}{R+h} \equiv -\frac{mgR}{1+\frac{h}{R}}$$



Example1: Find the potential energy for a system of three particles placed at the corners' of a triangle (see figure).
Answer: the potential energy of the system is given by:

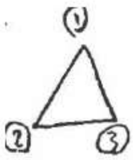
$$U_{total} = U_{12} + U_{13} + U_{23}$$



Example: Find the potential energy for four particles placed at the corners' of a square (see figure).
Answer: the potential energy of the system is given by:

$$U_{total} = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$$

Example: Three particles, each of mass m = 10^4 kg, each are placed at the corners' of an equilateral triangle with each side 10^2 m long. Calculate the potential energy of the system.
Answer:



$$U = U_{12} + U_{13} + U_{23}$$
$$= -Gm^2\left[\frac{3}{100}\right] = -2\times10^{-4} \text{ J}$$

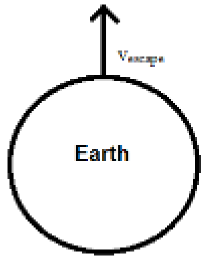
Example: How much work is done by the Moon's gravitational field in moving a 995 kg rock from infinity to the Moon's surface? [The Moon's radius and mass are 1.74×10^6 m and 7.36×10^{22} kg, respectively.]
Answer:

$$W = -\Delta U = -(U_f - U_i) = -\left(-\frac{GMm}{R} - \left[-\frac{GMm}{\infty}\right]\right) = \frac{GMm}{R}$$

$$\therefore \quad W = \frac{GMm}{R} = \frac{(6.67\times10^{-11})(7.36\times10^{22})(995)}{1.74\times10^{6}} = 2.8\times10^{9}\ \text{J.}$$

----

iii- Escape Speed v_{escape}



Throw a rock away from an (airless) planet with a speed v. If $v < v_{escape}$, the rock will rise to a maximum height and then fall back down. If $v > v_{escape}$, the rock will go to $r = \infty$, and will still have some speed left over and be moving away from the planet. If $v = v_{escape}$, the rock will have just enough initial KE to escape the planet: its distance goes to $r = \infty$ at the same time its speed approaches zero: $v \to 0$ as $r \to \infty$.

We can use conservation of energy to compute the escape speed v_{esc} (often called, incorrectly, the "escape velocity"). Initial configuration:
r = R (surface of planet), $v_i = v_{esc}$, $\underset{(+)}{KE} + \underset{(-)}{PE} = \underset{(+)\text{ or }(-)}{E_{tot}} = \text{constant}$

$$KE_i + PE_i = KE_f + PE_f \quad \Rightarrow \quad \tfrac{1}{2}\cancel{m} v_{esc}^2 - \frac{GM\cancel{m}}{R} = \tfrac{1}{2}\cancel{m} v_f^2 - \frac{GM\cancel{m}}{r}$$

$$\Rightarrow v_f = \sqrt{v_{esc}^2 - \frac{2GM}{R} + \frac{2GM}{r}}$$

As $v_f = 0$
$$\Rightarrow v_{esc} = \sqrt{\frac{2GM}{R} - \frac{2GM}{r}}$$

As: $r = \infty$.

$$\Rightarrow v_{esc} = \sqrt{\frac{2GM}{R}}$$

Note: If the rock is thrown with speed $v_i > v_{esc}$, it will go to $r = \infty$, and will have some KE left over, $v_f > 0$.

Example: A rocket is launched from the surface of a planet of mass M = 1.90 × 10^{27} kg and radius R = 7.15 × 10^{7} m. What minimum initial speed is required if the rocket is to rise to a height of 6R above the surface of the planet? (Neglect the effects of the atmosphere).

Answer:

$$v_{sc} = \left(\frac{2GM}{R_i}\right)^{1/2} = \sqrt{\frac{2\times6.65\times10^{-11}\times1.9\times10^{27}}{7.15\times10^{7}}} = \underline{5.51\times10^{4}\,\text{m/s}}.$$

Try to calculate

$$v_{sc} = \left(\frac{2GM}{R_i} - \frac{2GM}{6R_i}\right)^{1/2} = \underline{4.59\times10^{4}\,\text{m/s}}.$$

Some Escape Speeds

Body	Mass (kg)	Radius (m)	Escape Speed (km/s)
Ceres[a]	1.17×10^{21}	3.8×10^{5}	0.64
Earth's moon[a]	7.36×10^{22}	1.74×10^{6}	2.38
Earth	5.98×10^{24}	6.37×10^{6}	11.2
Jupiter	1.90×10^{27}	7.15×10^{7}	59.5
Sun	1.99×10^{30}	6.96×10^{8}	618
Sirius B[b]	2×10^{30}	1×10^{7}	5200
Neutron star[c]	2×10^{30}	1×10^{4}	2×10^{5}

[a]The most massive of the asteroids.
[b]A *white dwarf* (a star in a final stage of evolution) that is a companion of the bright star Sirius.
[c]The collapsed core of a star that remains after that star has exploded in a *supernova* event.

13-7 SATELLITES: ORBITS AND ENERGY

Key Ideas

- When a planet or satellite with mass m moves in a circular orbit with radius r, its potential energy U and kinetic energy K are given by

$$U = -\frac{GMm}{r} \quad \text{and} \quad K = \frac{GMm}{2r}.$$

The mechanical energy $E = K + U$ is then

$$E = -\frac{GMm}{2r}.$$

For an elliptical orbit of semimajor axis a,

$$E = -\frac{GMm}{2a}.$$

As a satellite orbits Earth in an elliptical path, both its speed, which fixes its kinetic energy K, and its distance from the center of Earth, which fixes its gravitational potential energy U, fluctuate with fixed periods. However, the mechanical energy E of the satellite remains constant. (Since the satellite's mass is so much smaller than Earth's mass, we assign U and E for the Earth–satellite system to the satellite alone.)

The potential energy of the system is given by

$$U = -\frac{GMm}{r}$$

(with $U = 0$ for infinite separation). Here r is the radius of the satellite's orbit, assumed for the time being to be circular, and M and m are the masses of Earth and the satellite, respectively. To find the kinetic energy of a satellite in a circular orbit, we write Newton's second law ($F = ma$) as

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \qquad (13\text{-}37)$$

where v^2/r is the centripetal acceleration of the satellite. Then, from Eq. 13-37, the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \frac{GMm}{2r}, \qquad (13\text{-}38)$$

which shows us that for a satellite in a circular orbit,

$$K = -\frac{U}{2} \quad \text{(circular orbit).} \qquad (13\text{-}39)$$

The total mechanical energy of the orbiting satellite is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

or

$$E = -\frac{GMm}{2r} \quad \text{(circular orbit).} \qquad (13\text{-}40)$$

This tells us that for a satellite in a circular orbit, the total energy E is the negative of the kinetic energy K:

$$E = -K \quad \text{(circular orbit).} \qquad (13\text{-}41)$$

For a satellite in an elliptical orbit of semimajor axis a, we can substitute a for r in Eq. 13-40 to find the mechanical energy:

$$E = -\frac{GMm}{2a} \quad \text{(elliptical orbit).} \qquad (13\text{-}42)$$

Equation 13-42 tells us that the total energy of an orbiting satellite depends only on the semimajor axis of its orbit and not on its eccentricity e. For example, four orbits with the same semimajor axis are shown in Fig. 13-15; the same satellite would have the same total mechanical energy E in all four orbits. Figure 13-16 shows the variation of K, U, and E with r for a satellite moving in a circular orbit about a massive central body. Note that as r is increased, the kinetic energy (and thus also the orbital speed) decreases.

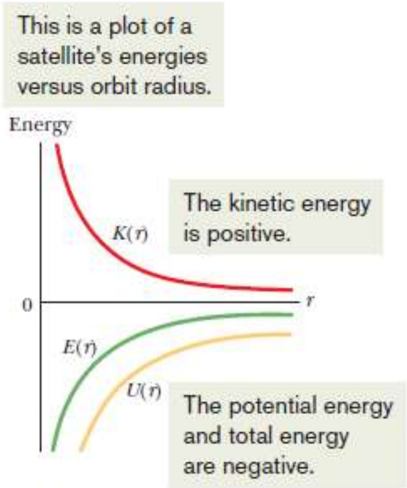


Figure 13-16 The variation of kinetic energy K, potential energy U, and total energy E with radius r for a satellite in a circular orbit. For any value of r, the values of U and E are negative, the value of K is positive, and $E = -K$. As $r \to \infty$, all three energy curves approach a value of zero.

Example: A 1000 kg satellite is in a circular orbit of radius = $2R_e$ about the Earth. How much energy is required to transfer the satellite to an orbit of radius = $4R_e$? (R_e = radius of Earth = 6.37×10^6 m, mass of the Earth = 5.98×10^{24} kg).

Answer:

$$\Delta E = E_f - E_i = -\frac{GMm}{2r_f} - \left(-\frac{GMm}{2r_i}\right) = -\frac{GMm}{2(4R_e)} - \left(-\frac{GMm}{2(2R_e)}\right) = \frac{GMm}{8R_e}$$

$$= \frac{6.67\times10^{-11}\times5.98\times10^{24}\times1000}{8\times6.37\times10^{6}} = \underline{7.8\times10^{9}\ \text{J}}.$$

Work Done Against Gravity

If the body of mess m is moved from the surface of earth to a point at distance h above the surface of earth, then change in potential energy or work done against gravity will be

$$W = \Delta U = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$\Rightarrow \quad W = GMm\left[\frac{1}{R} - \frac{1}{R+h}\right] \qquad [\text{As } r_1 = R \text{ and } r_2 = R+h]$$

$$\Rightarrow \quad W = \frac{GMmh}{R^2\left(1+\frac{h}{R}\right)} = \frac{mgh}{1+\frac{h}{R}} \qquad [\text{As } \frac{GM}{R^2} = g]$$

Extra Problems

13-4 GRAVITATION INSIDE EARTH

Gravitation Inside Earth

Newton's shell theorem can also be applied to a situation in which a particle is located *inside* a uniform shell, to show the following:

> A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Caution: This statement does *not* mean that the gravitational forces on the particle from the various elements of the shell magically disappear. Rather, it means that the *sum* of the force vectors on the particle from all the elements is zero.

If Earth's mass were uniformly distributed, the gravitational force acting on a particle would be a maximum at Earth's surface and would decrease as the particle moved outward, away from the planet. If the particle were to move inward, perhaps down a deep mine shaft, the gravitational force would change for two reasons. (1) It would tend to increase because the particle would be moving closer to the center of Earth. (2) It would tend to decrease because the thickening shell of material lying outside the particle's radial position would not exert any net force on the particle.

To find an expression for the gravitational force inside a uniform Earth, let's use the plot in *Pole to Pole*, an early science fiction story by George Griffith. Three explorers attempt to travel by capsule through a naturally formed (and, of course, fictional) tunnel directly from the south pole to the north pole. Figure 13-7 shows the capsule (mass m) when it has fallen to a distance r from Earth's center. At that moment, the *net* gravitational force on the capsule is due to the mass M_{ins} inside the sphere with radius r (the mass enclosed by the dashed outline), not the mass in the outer spherical shell (outside the dashed outline). Moreover, we can assume that the inside mass M_{ins} is concentrated as a particle at Earth's center. Thus, we can write Eq. 13-1, for the magnitude of the gravitational force on the capsule, as

$$F = \frac{GmM_{ins}}{r^2}. \qquad (13\text{-}17)$$

Because we assume a uniform density ρ, we can write this inside mass in terms of Earth's total mass M and its radius R:

$$\text{density} = \frac{\text{inside mass}}{\text{inside volume}} = \frac{\text{total mass}}{\text{total volume}},$$

$$\rho = \frac{M_{ins}}{\frac{4}{3}\pi r^3} = \frac{M}{\frac{4}{3}\pi R^3}.$$

Solving for M_{ins} we find

$$M_{ins} = \tfrac{4}{3}\pi r^3 \rho = \frac{M}{R^3} r^3. \qquad (13\text{-}18)$$

Substituting the second expression for M_{ins} into Eq. 13-17 gives us the magnitude of the gravitational force on the capsule as a function of the capsule's distance r from Earth's center:

$$F = \frac{GmM}{R^3} r. \qquad (13\text{-}19)$$

According to Griffith's story, as the capsule approaches Earth's center, the gravitational force on the explorers becomes alarmingly large and, exactly at the center, it suddenly but only momentarily disappears. From Eq. 13-19 we see that, in fact, the force magnitude decreases linearly as the capsule approaches the center, until it is zero at the center. At least Griffith got the zero-at-the-center detail correct.

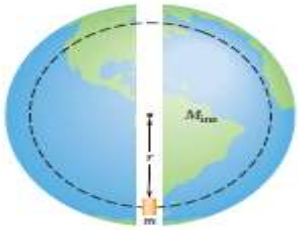


Figure 13-7 A capsule of mass m falls from rest through a tunnel that connects Earth's south and north poles. When the capsule is at distance r from Earth's center, the portion of Earth's mass that is contained in a sphere of that radius is M_{ins}.

Equation 13-19 can also be written in terms of the force vector $\vec{F}$ and the capsule's position vector $\vec{r}$ along a radial axis extending from Earth's center. Letting K represent the collection of constants in Eq. 13-19, we can rewrite the force in vector form as

$$\vec{F} = -K\vec{r}, \qquad (13\text{-}20)$$

in which we have inserted a minus sign to indicate that $\vec{F}$ and $\vec{r}$ have opposite directions. Equation 13-20 has the form of Hooke's law (Eq. 7-20, $\vec{F} = -k\vec{d}$). Thus, under the idealized conditions of the story, the capsule would oscillate like a block on a spring, with the center of the oscillation at Earth's center. After the capsule had fallen from the south pole to Earth's center, it would travel from the center to the north pole (as Griffith said) and then back again, repeating the cycle forever.

For the real Earth, which certainly has a nonuniform distribution of mass (Fig. 13-5), the force on the capsule would initially *increase* as the capsule descends. The force would then reach a maximum at a certain depth, and only then would it begin to decrease as the capsule further descends.

Sample Problem 13-5

An asteroid, headed directly toward Earth, has a speed of 12 km/s relative to the planet when the asteroid is 10 Earth radii from Earth's center. Neglecting the effects of Earth's atmosphere on the asteroid, find the asteroid's speed v_f when it reaches Earth's surface.

KEY IDEAS Because we are to neglect the effects of the atmosphere on the asteroid, the mechanical energy of the asteroid–Earth system is conserved during the fall. Thus, the final mechanical energy (when the asteroid reaches Earth's surface) is equal to the initial mechanical energy. With kinetic energy K and gravitational potential energy U, we can write this as

$$K_f + U_f = K_i + U_i. \quad (13\text{-}29)$$

Also, if we assume the system is isolated, the system's linear momentum must be conserved during the fall. Therefore, the momentum change of the asteroid and that of Earth must be equal in magnitude and opposite in sign. However, because Earth's mass is so much greater than the asteroid's mass, the change in Earth's speed is negligible relative to the change in the asteroid's speed. So, the change in Earth's kinetic energy is also negligible. Thus, we can assume that the kinetic energies in Eq. 13-29 are those of the asteroid alone.

Calculations: Let m represent the asteroid's mass and M represent Earth's mass (5.98×10^{24} kg). The asteroid is initially at distance $10R_E$ and finally at distance R_E, where R_E is Earth's radius (6.37×10^6 m). Substituting Eq. 13-21 for U and $\frac{1}{2}mv^2$ for K, we rewrite Eq. 13-29 as

$$\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}.$$

Rearranging and substituting known values, we find

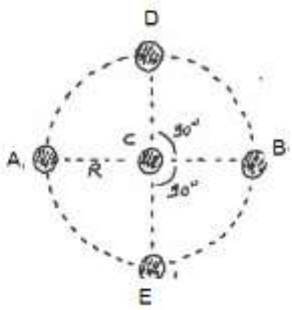
$$v_f^2 = v_i^2 + \frac{2GM}{R_E}\left(1 - \frac{1}{10}\right)$$
$$= (12 \times 10^3 \text{ m/s})^2 + \frac{2(6.67 \times 10^{-11} \text{ m}^3/\text{kg}\cdot\text{s}^2)(5.98 \times 10^{24} \text{ kg})}{6.37 \times 10^6 \text{ m}} 0.9$$
$$= 2.567 \times 10^8 \text{ m}^2/\text{s}^2,$$

and

$$v_f = 1.60 \times 10^4 \text{ m/s} = 16 \text{ km/s}. \quad \text{(Answer)}$$

At this speed, the asteroid would not have to be particularly large to do considerable damage at impact. If it were only 5 m across, the impact could release about as much energy as the nuclear explosion at Hiroshima. Alarmingly, about 500 million asteroids of this size are near Earth's orbit, and in 1994 one of them apparently penetrated Earth's atmosphere and exploded 20 km above the South Pacific (setting off nuclear-explosion warnings on six military satellites). The impact of an asteroid 500 m across (there may be a million of them near Earth's orbit) could end modern civilization and almost eliminate humans worldwide.

Example: Four stars (A, B, D, E), of equal mass, rotate in the same direction around a fifth star C of the same mass located at their common center of mass (see figure). The radius of the common orbit is R. What minimum speed would star A need **in order to depart from its companions for good?** (Express your answer in terms of G, M, R).



Answer: Apply the conservation of energy, where $E_i = E_f$

$$U_A = GMM\left[\frac{1}{R} + \frac{1}{\sqrt{2}R} + \frac{1}{2R} + \frac{1}{\sqrt{2}R}\right] = \frac{1}{2}Mv^2 \Rightarrow v = \sqrt{2\sqrt{2}+3}\left(\frac{GM}{R}\right)^{\frac{1}{2}}$$

Example: A satellite of mass 1300 kg is rotating around the earth in an orbit of radius 0.665×10^7 m. Then the satellite moves to a new orbit of radius 4.230×10^7 m. What is the change in its mechanical energy?

Answer:

$$\Delta E = E_2 - E_1 = -\frac{GM_E m}{2r_2} - \left(-\frac{GM_E m}{2r_1}\right) = -\frac{GM_E m}{2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$= \frac{6.67\times10^{-11} N.\frac{m^2}{kg^2} \times 5.98\times10^{24} kg \times 1300\, kg}{2}\left(\frac{1}{4.230\times10^7} - \frac{1}{0.665\times10^7}\right)$$

$$= 3.29\times10^{10}\,\mathrm{J}$$

Q5: An object is fired vertically upward from the surface of the Earth (Radius = R_e) with an initial speed of (Vesc)/2, where (Vesc = escape speed). Neglecting air resistance, how far above the surface of Earth will it reach?

Answer:

$$E_f = E_i = 0 \Rightarrow \frac{1}{2}mv_i^2 - \frac{GMm}{R_e} = \cancel{\frac{1}{2}mv_f^2} - \frac{GMm}{r}; \qquad r = R_e + h$$

$$\frac{1}{2}m\left(\frac{1}{2}\sqrt{\frac{2GM}{R_e}}\right)^2 - \frac{GMm}{R_e} = \cancel{\frac{1}{2}mv_f^2} - \frac{GMm}{r}$$

$$\Rightarrow r = \frac{4}{3}R_e \Rightarrow h = \underline{\frac{1}{3}R_e}.$$

Q. A 500 kg rocket is fired from earth surface with an escape speed. Find the rocket's speed when it is at a distance of $1.50\times10^5 km$ from the center of earth? [Neglect any friction and air resistance effects]

Ans:

$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{r_f}$$

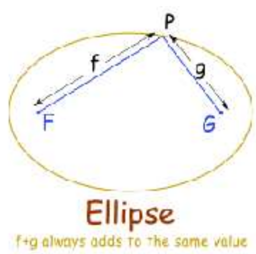
$$\Rightarrow v_f^2 = v_i^2 + 2GM_E\left(\frac{1}{r_f} - \frac{1}{R_E}\right)$$

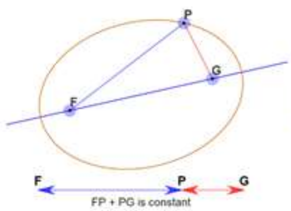
$$\Rightarrow v_f = 2.31\times10^3\ \mathrm{m/s}$$

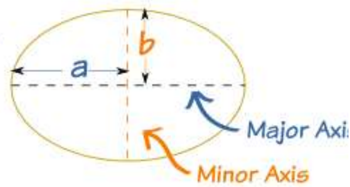
Gravitation

Summary of equations:

$$F = G\frac{m_1\ m_2}{r^2}, \quad g = a_g - \omega^2 R_E, \quad U(r) = -\frac{G M m}{r}, \quad v_{esc} = \sqrt{\frac{2 G M}{R}}$$







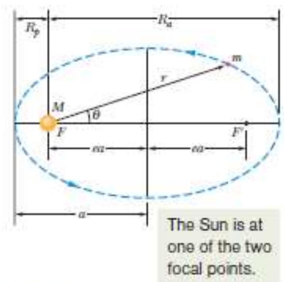
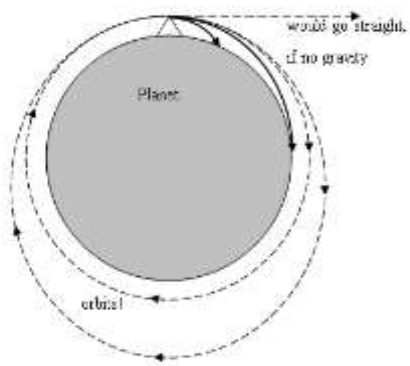


Figure 13-12 A planet of mass m moving in an elliptical orbit around the Sun. The Sun, of mass M, is at one focus F of the ellipse. The other focus is F', which is located in empty space. The semimajor axis a of the ellipse, the perihelion (nearest the Sun) distance R_p, and the aphelion (farthest from the Sun) distance R_a are also shown.

13-6 PLANETS AND SATELLITES: KEPLER'S LAWS

Examples of Orbits:



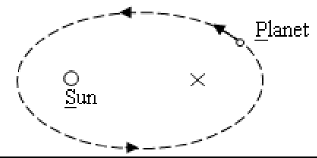
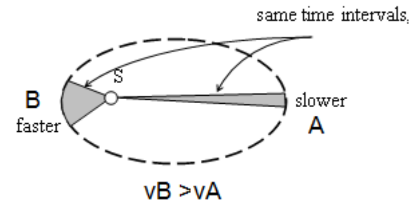
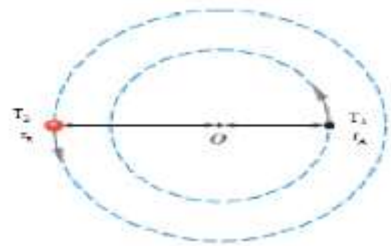
- Consider a planet like Earth, but with no air. Fire projectiles horizontally from a mountain top, with faster and faster initial speeds.
- The orbit of a satellite around the earth, or
- a planet around the sun

All orbits obey Kepler's 3 Laws.

Kepler, German (1571-1630) took the data that Danish astronomer Tycho Brahe ("Bra-hay") had spent his life collecting and used it (especially the information on Mars) to create three laws that apply to any object that is orbiting something else.

- Although Kepler's math was essentially wrong, the three laws he came up with were correct!
- It would be like you writing a test, and even though you did all the work on a question wrong, you somehow get the correct final answer.
- Kepler's Three Laws of Planetary Motion are still the basis for work done in the field of astronomy to this day.

THE THREE KEPLER'S LAWS

Law	Name	statement	Figure
I	**THE LAW OF ORBITS**	All planets move in elliptical orbits, with the Sun at one focus.	
II	**THE LAW OF AREAS**	A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals; that is, the rate dA/dt at which it sweeps out area A is constant. **Comment:** Area of the segment $A=\frac{1}{2}r^2 d\theta$, then $\frac{dA}{dt}=\frac{1}{2}r^2\frac{d(\theta)}{dt}=\frac{1}{2}r^2\omega=\frac{L}{2m}, \quad L=mr^2\omega$	
III	**THE LAW OF PERIODS**	The square of the period, T^2, of any planet is proportional to the cube of the semi-major, r^3, axis of its orbit, i.e. $\boxed{\frac{T^2}{r^3} = \text{constant}}$. **Comment:** Since in circular motion we have $F_G = F_c$. Then, $G\frac{M\not{m}}{r^2} = \not{m}\frac{v^2}{\not{r}}$; and we can have: i- $v^2 = G\frac{M}{r}$ ii- $v = 2\pi r / T$,	

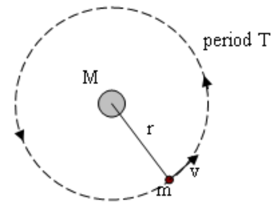
KIII: For planets around the sun, the period T and the mean distance r from the sun are related by $\frac{T^2}{r^3} = \text{constant}$. That is for any two planets A and B, $\frac{T_A^{\ 2}}{r_A^{\ 3}} = \frac{T_B^{\ 2}}{r_B^{\ 3}}$. This means that planets further from the sun (larger r) have longer orbital periods (longer T).

Kepler's Laws were <u>empirical</u> rules, **based on observations** of the motions of the planets in the sky. Kepler had no theory to explain these rules.

Newton (1642-1727) started with Kepler's Laws and NII ($\mathbf{F}_{net} = m\mathbf{a}$) and <u>deduced</u> that $F_{grav\,(Sun-planet)} = G\frac{M_S m_P}{r_{SP}^{\ 2}}$. Newton applied similar reasoning to the motion of the Earth-Moon system (and to an Earth-apple system) and deduced that $F_{\substack{grav \\ (\text{Earth-mass m})}} = G\dfrac{M_E\, m}{r_{Em}^{\ 2}}$. Newton then made a mental leap, and realized that this law applied to <u>any</u> 2 masses, not just to the Sun-planet, the Earth-moon, and Earth-projectile systems. Starting with $\mathbf{F}_{net}$ = m**a** and F_{grav} = G Mm / r^2, Newton was able to <u>derive</u> Kepler's Laws (and much more!). Newton could explain the motion of everything!

Derivation of KIII (for special case of circular orbits). Consider a small mass m in circular orbit about a large mass M, with orbital radius r and period T. We aim to show that T^2 / r^3 = const. Start with NII: F_{net} = m a

The only force acting is gravity, and for circular motion a = v^2 / r $\Rightarrow$



$$G\frac{M\,\cancel{m}}{r^2} = \cancel{m}\frac{v^2}{\cancel{r}} \Rightarrow G\frac{M}{r} = v^2 = \left(\frac{2\pi r}{T}\right)^2$$

[recall the v = dist / time = $2\pi r / T$]

$$\Rightarrow \quad G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2} \quad \Rightarrow \quad \frac{T^2}{r^3} = \frac{4\pi^2}{G\,M} = \text{constant, independent of m}$$

(Deriving this result for elliptical orbits is much harder, but Newton did it.)

- Note that: The speed v of a satellite in circular orbit: $v = \sqrt{\dfrac{GM}{r}}$.

For low-earth orbit (few hundred miles up), this orbital speed is about 7.8 km/s ≅ 4.7 miles/second. The Space Shuttle must attain a speed of 4.7 mi/s when it reaches the top of the atmosphere (and it fuel has run out) or else it will fall back to Earth.

Example: The planet Mars has a satellite, Phobos, which travels in a circular orbit of radius 9.40×10^6 m, with a period of 2.754×10^4 s. Calculate the mass of Mars from this information.

Answer: Use KIII $T^2 = \left(\dfrac{4\pi^2}{GM}\right)r^3$, we have

$$\Rightarrow M = \left(\frac{4\pi^2}{G}\right)\frac{r^3}{T^2} = \left(\frac{4\pi^2}{6.65\times10^{-11}}\right)\frac{\left(9.4\times10^6\right)^3}{\left(2.754\times10^4\right)^2} = \underline{6.5\times10^{23}\text{kg}}.$$

Example: A satellite of Jupiter, has an orbital period of 1.77 days and an orbital radius of 4.22×10^5 km. Determine the mass of Jupiter.

Answer:

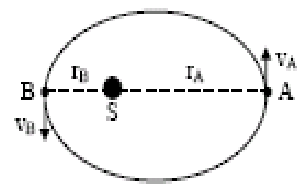
$$T^2 = \frac{4\pi^2 R^3}{GM_J} \quad \Rightarrow M_J = \frac{4\pi^2 R^3}{GM_J} \Rightarrow M_J = \frac{4\pi^2 R^3}{GT^2} = 1.90 \times 10^{27}\ \text{kg}$$

Example: A 20.0 kg satellite moves on a circular orbit around a planet of mass M = 4.06×10^{24} kg with a period of 2.40 h. What is the radius of the orbit of the satellite?
Answer:

$$T^2 = \frac{4\pi^2}{GM} \times r^3 \Rightarrow r = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$r = \left[\frac{6.67 \times 10^{-11} \times 4.06 \times 10^{24} \times (2.4 \times 3600)^2}{4\pi^2}\right]^{1/3} = 8.00 \times 10^6\ \text{m}$$

Example: The Fig shows a planet traveling in a counterclockwise direction on an elliptical path around a star S located at one focus of the ellipse. The speed of the planet at a point A is v_A and at B is v_B. The distance $A_S = r_A$ while the distance $B_S = r_B$. The ratio v_A/v_B is:



Answer: Conservation of angular momentum at points A and B requires that:

$$L_A = L_B \Rightarrow mv_A r_A = mv_B r_B \Rightarrow \frac{v_A}{v_B} = \frac{r_B}{r_A}.$$

Example: A planet makes a circular orbit with period T around a star. If the planet were to orbit, at the same distance, around a star with three times the mass of the original star, what would be the new period?

<u>Solution</u>: From Kepler's third law: $T^2 = \left(\frac{4\pi^2}{GM}\right)R^3$

If the distance (R) is the same, then $T \propto M^{-1/2}$

Let the old period be T and the new period be T_n, then:

$$T_n \propto 3^{-\frac{1}{2}} T = \frac{T}{\sqrt{3}} = 0.577\ T$$

Example: Both Venus and the Earth have approximately circular orbits around the Sun. The period of the orbital motion of Venus is 0.615 year, and the period of the Earth is 1 year. By what factor do the sizes of the two orbits differ (R_E/R_V)?
<u>Solution</u>:

$$\frac{r_E}{r_V} = \frac{T_E^{2/3}}{T_V^{2/3}} = \frac{(1\ \text{year})^{2/3}}{(0.615\ \text{year})^{2/3}} = 1.38$$

Summary of equations:

$$F = G\frac{m_1\ m_2}{r^2}, \quad g = a_g - \omega^2 R_E, \quad U(r) = -\frac{G\,M\,m}{r}, \quad v_{esc} = \sqrt{\frac{2\,G\,M}{R}}$$

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}, \quad L = mr^2\omega$$

$$\frac{T^2}{r^3} = \frac{4\pi^2}{G\,M} = \text{constant}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$E = -\frac{1}{2}\frac{GmM}{r}$$

SATELLITES: ORBITS AND ENERGY

From KIII , it was found $v = \sqrt{\frac{GM}{r}}$, then the total energy of an object in an orbit (circular or parabolic) is given by:

$$E = KE + PE = \frac{1}{2}mv^2 + U = \frac{1}{2}m\frac{GM}{r} - \frac{GM}{r}m = -\frac{1}{2}\frac{GmM}{r}$$

Example: A 1000 kg satellite is in a circular orbit of radius = $2R_e$ about the Earth. How much energy is required to transfer the satellite to an orbit of radius = $4R_e$? (R_e = radius of Earth = 6.37×10^6 m, mass of the Earth = 5.98×10^{24} kg).

Answer:

$$\Delta E = E_f - E_i = -\frac{GMm}{2r_f} - \left(-\frac{GMm}{2r_i}\right) = -\frac{GMm}{2(4R_e)} - \left(-\frac{GMm}{2(2R_e)}\right) = \frac{GMm}{8R_e}$$

$$= \frac{6.67\times10^{-11}\times5.98\times10^{24}\times1000}{8\times6.37\times10^6} = \underline{7.8\times10^9\ \text{J}}.$$

Extra Solved problems

Example: At what distance above the surface of Earth (radius = R_e) is the magnitude of the gravitational acceleration equal to g/16? (Where g = gravitational acceleration at the surface of Earth).

Answer: At a distance h above the surface of Earth, it is required that $a_g = \frac{g}{16}$, where $a_g = \frac{GM}{(h+R_e)^2}$ and $g = \frac{GM}{R_e^2}$, then:

$$\frac{GM}{(h+R_e)^2} = \frac{g}{16} = \frac{GM/R_e^2}{16} \quad \Rightarrow \quad (h+R_e)^2 = 16R_e^2$$

$$\Rightarrow \quad h = \underline{3R_e}.$$

Example: The magnitude of the acceleration due to gravity at the North Pole of planet Neptune is 10.7 m/s^2. Neptune has a radius of 2.5 x 10^4 km and rotates once around its axis in 16.0 hours. What is the magnitude of the acceleration due to gravity at the equator of Neptune? A: 10.4 m/s^2

Solution:

a_g = acceleration due to gravity at the pole = 10.7 m/s^2
T = period of revolution = (16) (3600) = 57600 s
ω = angular speed of the planet = $2\pi/T$ = 1.091 x 10^{-4} rad/s
g = acceleration due to gravity at the equator = $a_g - \omega^2 R$ = 10.4 m/s^2

Example: If the gravitational acceleration at the surface of Earth is 9.8 m/s^2, at what distance from the Earth's center (inside the Earth) will the gravitational acceleration be 4.0 m/s^2?

Solution: Let M be the mass of the earth, R be the radius of the Earth, r be the requested distance, a_{gs} be the gravitational acceleration at the surface, and a_g be the gravitational acceleration at the required location.

The effective mass of the earth (m*) at the required location is given by the ratio of the volumes.

$$a_g = \frac{G\, m^*}{r^2} = \frac{G}{r^2}\frac{r^3}{R^3} M = \frac{G\, M}{R^2}\frac{r}{R} = a_{gs}\frac{r}{R}$$

Thus:

$$r = \frac{a_g}{a_{gs}} R = \frac{4.0}{9.8} \times 6370 = 2600 \text{ km}$$

Example: A spherical asteroid has a radius of 500 km. The acceleration due to gravity at the surface of the asteroid is 3.00 m/s^2. With what speed will an object hit the surface of the asteroid if it is dropped from rest from 300 km above the surface?

Solution: **with**

$$a_g = \frac{GM}{R^2} \rightarrow GM = a_g R^2$$

$$K_i + U_i = K_f + U_f \rightarrow K_f = U_i - U_f$$

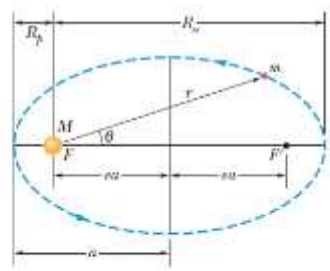
$$\frac{1}{2} m v^2 = -\frac{GmM}{R+h} + \frac{GmM}{R}$$

$$v^2 = \frac{2GM}{R} - \frac{2GM}{R+h} = 2 a_g R^2 \left(\frac{1}{R} - \frac{1}{R+h}\right)$$

$$v^2 = 2 \times 3.0 \times 2.5 \times 10^{11} \left(\frac{1}{5 \times 10^5} - \frac{1}{8 \times 10^5}\right) = 1125000$$

Thus: v = 1.06 km/s

Example: The semimajor axis of planet Pluto is 5.92 x 10^{12} m and the eccentricity of its orbit around the Sun is e = 0.248. Find Pluto's closest distance from the Sun.



Solution:

a = semimajor axis, e = eccentricity, R_p = closest distance

a = ae + R_p

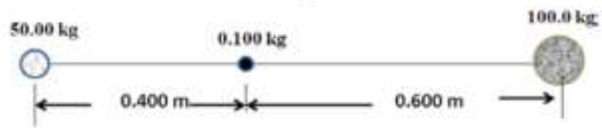
R_p = a – ae = a (1 – e) = 5.92 x 10^{12} x (1 – 0.248) = 4.45 x 10^{12} m

Phys101	Final	Zero Version
Term: 131	Thursday, January 02, 2014	Page: 9

Q15.

Three solid uniform spheres are located in space, as shown in **Figure 7**. The 50.0 kg and 100 kg spheres are fixed and the 0.100 kg sphere is released from its initial position with its center 0.400 m from the center of the 50.0 kg sphere. Find the kinetic energy of the 0.100 kg sphere when it has moved 0.400 m to the right from its initial position.

Figure 7



A) +1.81 nJ
B) −1.81 nJ
C) − 5.34 nJ
D) +5.34 nJ
E) +7. 45 nJ

Ans:

$$\Delta K = -\Delta U = U_i - U_f;\ K_i = 0$$

$$K_f = \frac{1}{2} m v_f^2 = U_i - U_f$$

$$U_i = -G m_{0.1}\left(\frac{m_{50}}{0.4} + \frac{m_{100}}{0.6}\right) - \frac{G m_{50} m_{100}}{1}$$

$$U_f = -G m_{0.1}\left(\frac{m_{50}}{0.8} \pm \frac{m_{100}}{0.2}\right) - \frac{G m_{50} m_{100}}{1}$$

$$K_f = \frac{1}{2}\cancel{m}_{0.1} v_f^2 = U_i - U_f = G\cancel{m}_{0.1}\left(\frac{m_{50}}{0.8} + \frac{m_{100}}{0.2} - \frac{m_{50}}{0.4} - \frac{m_{100}}{0.6}\right)$$

$$K_f = 0.1 \times 6.67 \times 10^{-11}\left(\frac{50}{0.8} + \frac{100}{0.2} - \frac{50}{0.4} - \frac{100}{0.6}\right) = 1.80 \times 10^{-9} J$$

Phys101	Final	Zero Version
Term: 131	Thursday, January 02, 2014	Page: 10

Q16.

The potential energy of a satellite of mass 1.00×10^2kg on a surface of a planet is -1.00×10^6 J. Find the escape speed of the satellite from the surface of the planet.

A) 1.41×10^2 m/s
B) 2.00×10^2 m/s
C) 3.54×10^4m/s
D) 9.80×10^6 m/s
E) 9.80×10^3m/s

Ans:

$$K_i + U_i = 0 \Rightarrow K_i = \frac{1}{2}mv_{esc}^2 = -U_i \Rightarrow v_{esc} = \sqrt{\frac{-2U_i}{m}}$$

$$v_{esc} = \sqrt{\frac{2\times10^6}{100}} = 1.41\times10^2\ m/s$$