Equilibrium and Elasticity

12-1 Static Equilibrium

In this chapter we study a special case of the dynamics of rigid objects covered in the last two chapters. It is the (very important!) special case where the center of mass of the object has no motion and the object is not rotating.

Conditions for Equilibrium of a Rigid Object

For a rigid object which is not moving at all we have the following conditions:

1. The (vector) sum of the external forces on the rigid object must equal zero:

$$\sum \vec{F} = 0 \qquad (1)$$

When this condition is satisfied we say that the object is in translational equilibrium.

☞ it really only tells us that a_{CM} is zero, but of course that includes the case where the object is motionless.

2. The sum of the external torques on the rigid object must equal zero.

$$\sum \vec{\tau} = 0 \qquad (2)$$

When this condition is satisfied we say that the object is in rotational equilibrium. (It really only tells us that τ about the given axis is zero, but —again— that includes the case where the object is motionless.)

When both 1 and 2 are satisfied we say that the object is in **static equilibrium**.
Nearly all of the problems we will solve in this chapter are two–dimensional problems (in the xy plane), and for these, Eqs. 1 and 2 reduce to

$$\sum F_x = 0\ ; \quad \sum F_y = 0\ ; \quad \sum \tau_z = 0 \qquad (3)$$

Two Important Facts for Working Statics Problems:

i. The force of gravity acts on all massive objects in our statics problems; its acts on all the individual mass points of the object. One can show that for the purposes of computing the forces and torques on rigid objects in statics problems we can treat the mass of the entire object as being concentrated at its center of mass; that is, for an object of mass *M* we can treat gravity as exerting a force *Mg* downward at the center of mass.

☞ This result depends on the fact that the acceleration of gravity, ***g*** is usually constant over the volume of the object. Otherwise it is not true.

ii. While there is only one way to write the conditions for the forces on a rigid object summing to zero, we have a choice in the way we write the equation for the total torque. Eq. 3 does not specify the choice of the axis for calculating the torque. In general it matters a great deal which axis we pick! But when the sum of torques about any one axis is zero and the sum of forces is zero (translational equilibrium) then the sum of torques about any axis will give zero; so for statics problems we are free to pick the most convenient axis for computing $\sum \vec{\tau} = 0$. Often this will be the point on the object where several unknown forces are acting, so that the resulting set of equations will be simpler to solve.

In Summary: In order to guarantee static equilibrium (it is not moving) for an object, we must have

1) it is not translating (not moving up, down, left, or right), i.e net force = 0 AND
2) it is not rotating, i.e net torque = 0

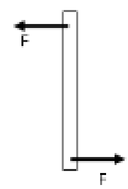
If a stationary mass is acted on by several forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, ..$, then in order to NOT translate, the net force must be zero.

$$\vec{F}_{net} = \vec{F}_{total} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = 0$$

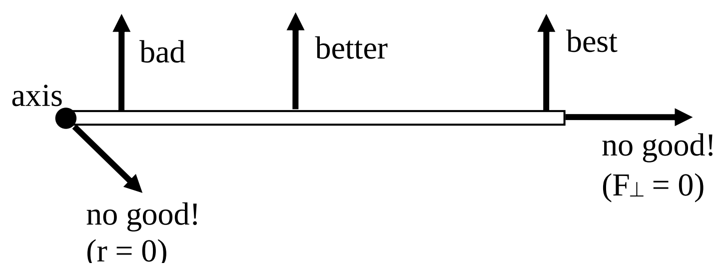
$$\Rightarrow F_{1x} + F_{2x} + F_{3x} + \ldots = 0 , \quad F_{1y} + F_{2y} + F_{3y} + \ldots = 0$$

$$\Rightarrow \boxed{\sum F_x = 0 , \quad \sum F_y = 0}$$ Equilibrium possible, but not guaranteed.

Example: Even though the net force is zero, the object might not be in static equilibrium. Here is a case (two forces acting on a bar) where the net force is zero, but the forces cause the object to *rotate*:



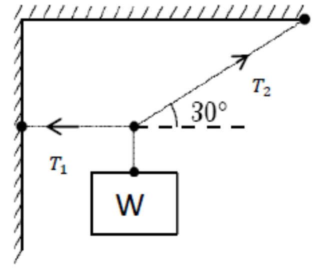
If you want to easily rotate an object about an axis, you want a large lever arm r and a large perpendicular force $F_\perp$:



---

i- Static equilibrium problem with $\vec{F}_{net} = 0$, but no torque.

Example: A weight W = 100 N is hung from two ropes as shown in the figure. Find the magnitude of the tension in the horizontal rope.



Answer:

$T_2 \sin 30 = 100 \Rightarrow T_2 = 200$ N

$T_2 \cos 30 - T_1 = 0$

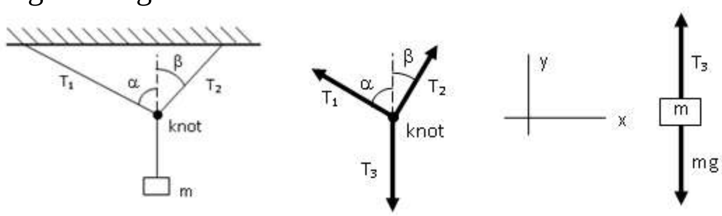
$(200)\cos 30 - T_1 = 0$

$\Rightarrow T_1 = 173$ N

Example: The figure shows a mass m hanging from two strings. Find T_1 and T_2.

Answer: Knowns: m, α , β , and unknowns: string tensions T_1 = ? T_2 = ?

Forces on knot: Notice that lengths of force arrows in diagram below have nothing to do with the lengths of the strings in diagram above.



Because the forces on the mass m must cancel, the tension in the bottom string = T_3 = mg.
The knot is a point object; there are no lever arms here, so no possibility of rotation, so we don't have to worry about torques. Apply equations of static equilibrium to the knot:

$$\sum F_x = 0 \quad \Rightarrow \quad -T_1 \sin\alpha + T_2 \sin\beta = 0 \quad \text{OR} \quad T_1 \sin\alpha = T_2 \sin\beta$$

$[|F_{left}| = |F_{right}|]$

$$\sum F_y = 0 \quad \Rightarrow \quad +T_1 \cos\alpha + T_2 \cos\beta - mg = 0 \text{ OR} \quad +T_1 \cos\alpha + T_2 \cos\beta = mg$$

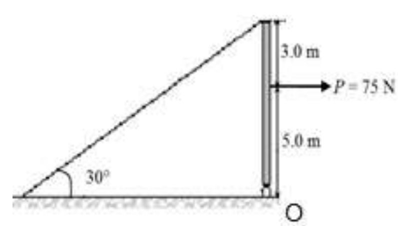
$[|F_{up}| = |F_{down}|]$

Now have 2 equations in 2 unknowns (T_1 and T_2), so we can solve. (I'll let you do that!)

----

ii- Static equilibrium problem with $\vec{\tau}_{net} = 0$, but no Forces.

Example: A uniform 100 kg beam is held in a vertical position by a pin at its lower end, a cable at its upper end, and by applying a horizontal force P = 75 N as shown in **Figure**. Find the tension in the cable.



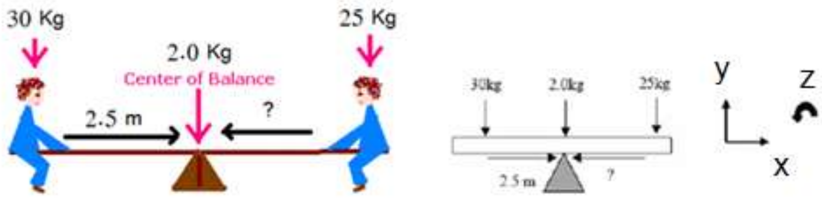
Answer: Take the torque about the pin

$$\tau_o = T \times 8 \times \cos 30^\circ - 75 \times 5 = 0 \quad \Rightarrow \quad T = \frac{75 \times 5}{8 \times \cos 30^\circ} = 54.13$$

----

iii- **Static equilibrium problem with** $\vec{F}_{\text{net}} = 0$ and $\vec{\tau}_{\text{net}} = 0$

Example: Consider a playground seesaw. The mass of the plank is 2.0 kg, the masses of two children on it are 25 kg and 30 kg with the 30 kg child sitting 2.5 meters from the center of the plank (the fulcrum) as shown below. Where must the second child sit in order for this system to be in equilibrium?



Answer:
Noting that a normal force directed upwards acts at the point of the fulcrum, the FBD's for the **first condition** yield:

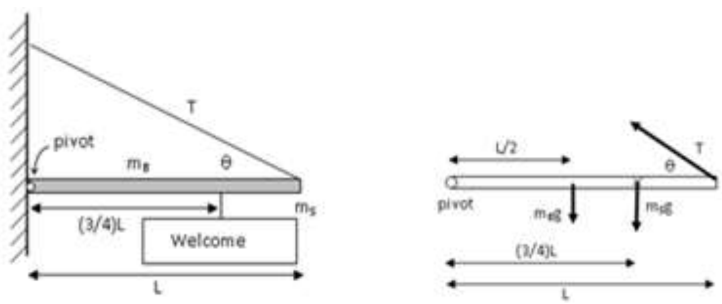
$$\sum F_y = 0 \Rightarrow N - (30 + 35 + 2)g = 0$$

Note that while this is all true it is not, by itself, particularly useful.
To apply the **second condition** we must first choose an axis about which to compute torques. The axis that makes the most physical sense would be one directly through the board over the fulcrum, but we could choose any axis that made computations easier. In this case choosing the axis associated with the fulcrum eliminates the forces created by the mass of the board itself since these act on the center of mass of the board which is located directly over the fulcrum.

$$\sum \vec{\tau} = 0 \Rightarrow (30\text{g}) \times (2.5 \text{ m}) - (25\text{g}) \times (x \text{ m}) = 0$$

Solving this equation for x yields a distance of 3 meters.

Example: A store welcome's sign with mass m_S is hung from a uniform bar of mass m_B and length L. The sign is suspended from a point ¾ of the way from the wall. The bar is held up with a cable at an angle θ as shown.



a- Calculate the tension T in the cable.

b- If the wall is exerting some force $\vec{F}_W$ on the left end of the bar. What are the components F_{wx} and F_{wy} of this force?

Answer:

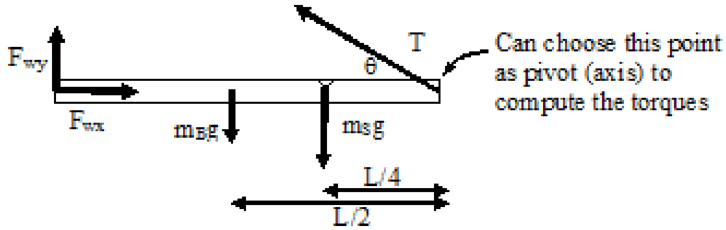
a- Knowns: m_B , m_S , L, θ , Unknown: tension T = ?

Torque diagram, showing forces and lever arms about the pivot:
Choose pivot as the axis:

$\sum \tau = 0 \quad \Rightarrow \quad +L\,(T\sin\theta) - (3/4)L(m_S g) - (L/2)(m_B g) = 0$ L's cancel, so

$$T\sin\theta = \tfrac{3}{4}m_S g + \tfrac{1}{2}m_B g \Rightarrow T = \frac{\left(\tfrac{3}{4}m_S + \tfrac{1}{2}m_B\right)g}{\sin\theta}$$

b- All forces on bar:



Method I: Assume we know tension T (from previous problem). Then can use

$$\sum F_x = 0, \quad \sum F_y = 0 \quad \Rightarrow \quad F_{wx} = T\cos\theta, \quad F_{wy} = m_B g + m_S g - T\sin\theta$$

Method II: Assume that we do not know tension T.

Torque is always computed with respect to some axis or pivot point. If the object is not moving at all, we can pick <u>any</u> point as the axis. We can always pretend that the object is about to rotate about that point. Let us choose the right end of the bar as our pivot point. Then the tension force does not produce any torque (since the lever arm is zero), and the (unknown) variable T does not appear in our torque equation.

$$\sum \tau = 0 \Rightarrow \frac{L}{4}m_S g + \frac{L}{2}m_B g - LF_{wy} = 0 \qquad \text{(L's cancel)}$$

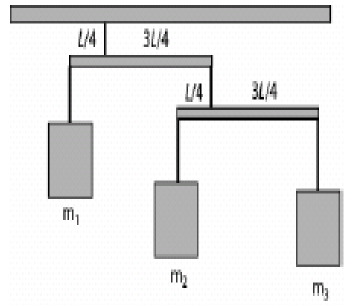
$$F_{wy} = \frac{m_S g}{4} + \frac{m_B g}{2}$$

(Still have to get F_{wx} using method I above.)
What is the magnitude of the total force on the bar from the wall?

$$F_w = \left|\vec{F}_w\right| = \sqrt{F_{wx}^{\,2} + F_{wy}^{\,2}}$$

12-2 SOME EXAMPLES OF STATIC EQUILIBRIUM

Example: Fig. 1 shows a three boxes of masses m_1, m_2 and m_3 hanging from a ceiling. The crossbars are horizontal and have negligible mass and same length L. If m_3 = 1.0 kg, then m_1 is equal to:



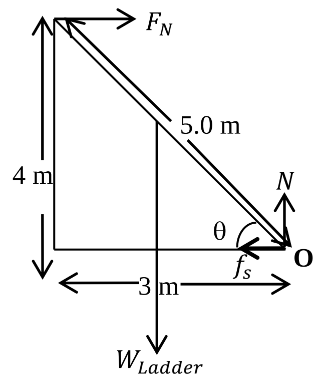
Answer:

$$\sum \vec{\tau}_1 = 0 \Rightarrow (m_3)\times\left(\frac{3L}{4}\right) = (m_2)\times\left(\frac{L}{4}\right) \Rightarrow m_2 = 3m_3$$

$$\sum \vec{\tau}_1 = 0 \Rightarrow \left(\overbrace{m_3 + m_2}^{4}\right)\times\left(\frac{3L}{4}\right) = (m_1)\times\left(\frac{L}{4}\right)$$

$$\Rightarrow m_3 = 12 m_1 = \underline{12\text{ kg}}$$

Example: A uniform ladder whose length is 5.0 m and whose weight is 4.0×10^2 N leans against a frictionless vertical wall. The foot of the ladder can be placed at a maximum distance of 3.0 m from the base of the wall on the floor without the ladder slipping. Determine the coefficient of static friction between the foot of the ladder and the floor.



Answer:

$$\theta = tan^{-1}\left(\frac{4}{3}\right) = 53.1°$$

Applying $\sum F_x = 0 = F_N - f_s = 0 \Rightarrow f_s = F_N = \mu_s N$

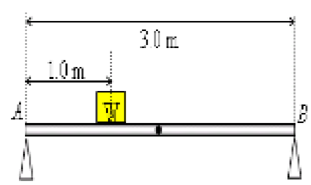
Also $\sum F_y = 0 = N - W_{ladder} \Rightarrow N = W_{ladder} = 4 \times 10^2\ N$

About O, $\sum \tau = 0 = W_{ladder} \times 2.5 \times cos\ 53.1 - F_N \times 4 = 0$

$$F_N = \frac{W_{ladder} \times 2.5 \times \cos 53.1}{4} = \frac{400 \times 2.5 \times \cos 53.1}{4} = 150.1\ N$$

$$\mu_s = \frac{f_s}{N} = \frac{F_N}{N} = \frac{150.1}{400} = 0.38$$

Example: A uniform steel bar of length 3.0 m and weight 20 N rests on two supports (A and B) at its ends. A block of weight W = 30 N is placed at a distance 1.0 m from A (see Figure). The forces on the supports A and B respectively are:



Answer:

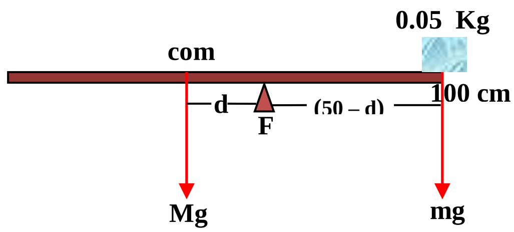
$$\sum \vec{\tau}_B = 0 \Rightarrow (30)\times(2.0\text{ m}) + (20)\times(1.5\text{ m}) - N_B \times (3.0\text{ m}) = 0$$

$$\therefore N_A = \underline{30\text{ N}}$$

$$\sum \overrightarrow{\tau_A} = 0 \Rightarrow (30)\times(1.0\text{ m}) + (20)\times(1.5\text{ m}) - N_A \times (3.0\text{ m}) = 0$$

$$\therefore N_B = \underline{20\text{ N}}$$

Example: A uniform 0.20 kg meter stick can be balanced horizontally on a knife edge if 0.050 kg point mass is placed at the 100-cm mark. Find the position of the knife edge measured from the zero-cm mark.



Answer:

Taking torque about knife edge F

$$0.2 \times d \times \cancel{g} = 0.05 \times (50 - d) \times \cancel{g}$$

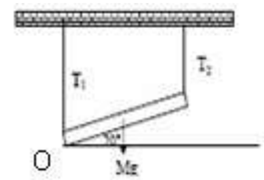
$$(0.2 + 0.05)d = 0.050 \times 50$$

$$d = 10 \text{ cm}$$

Position of knife edge from zero-cm mark = 50+10 = 60 cm

Extra Problems

Q: A uniform meter stick has mass M = 1.25 kg. As shown in the figure, this meter stick is supported by two vertical strings, one at each end, in such a manner that it makes an angle of 20° with the horizontal. Find the tension in each string.



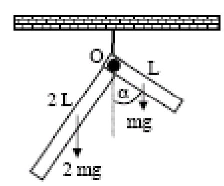
Answer:

$$\sum \overrightarrow{F_y} = 0 \;\Rightarrow T_1 + T_2 = 1.25g \text{ N}$$

$$\sum \overrightarrow{\tau_O} = 0 \;\Rightarrow (T_2)\times\left(h\cos 20^\circ\right) = (Mg)\times\left(\frac{h}{2}\cos 20^\circ\right) \Rightarrow T_2 = \frac{Mg}{2} = \underline{6.1 \text{ N}}$$

$$\therefore T_1 = 1.25g - \frac{1.25g}{2} = \underline{6.1 \text{ N}}$$

Q: A thin right angled rod is made of a uniform material. The shorter end is half the length of the longer end. It is hanging by a string attached at point O (see figure). At equilibrium, calculate the angle α between the shorter rod and the vertical.

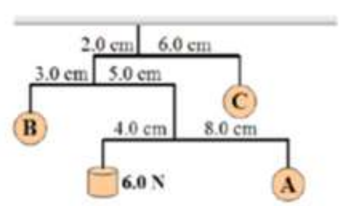


Answer:

$$\sum \overrightarrow{\tau_O} = 0 \;\Rightarrow \left(\frac{m}{2}g\right)\times\left(\frac{L}{4}\sin\alpha\right) = (mg)\times\left(\frac{L}{2}\underbrace{\sin(90^\circ-\alpha)}_{\cos\alpha}\right)$$

$$\Rightarrow \frac{\sin\alpha}{\cos\alpha} = 4 \quad \Rightarrow \alpha = \underline{76^\circ}$$

Q: Consider the assembly shown in **Figure**, where four objects are held in equilibrium by horizontal massless rods. What is the weight of ball C?



Answer:

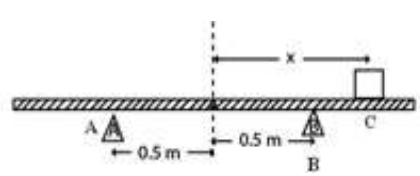
$$6 \times 4 = 8\,A \;\rightarrow A = \frac{24}{8} = 3 \text{ N}$$

$$5 \times 9 = 3\,B \;\rightarrow B = \frac{45}{3} = 15 \text{ N}$$

$$2 \times 24 = 6\,C \;\rightarrow C = \frac{48}{6} = 8 \text{ N}$$

--

Q: A uniform rigid rod having a mass of 50 kg and a length of 2.0 m rests on two supports A and B as shown in the Figure. When a block of mass 60 kg is kept at point C at a distance of *x* from the center, the rod is about to be lifted from A The value of *x* is:



Answer:

$$\sum \overrightarrow{\tau_B} = 0 \;\Rightarrow (60g)\times(x-0.5) = (50g)\times(0.5) \quad \Rightarrow \quad x = 0.5 + \frac{0.5\times 50}{60} \approx \underline{0.9 \text{ m}}$$

Q: To know the location of the CM of a person, one has to use the arrangement shown in figure #. A plank of weight 40 N is placed on two scales separated by 2.0 m. A person lies on the plank and the right scale reads 314 N and the left scale reads 216 N. What is the distance from the right scale to the person's CM?

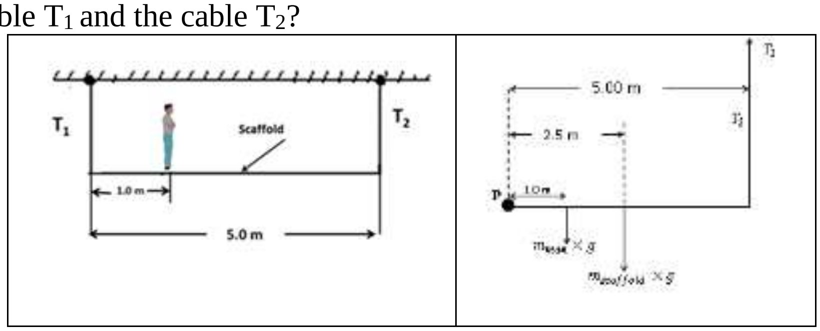
Answer

Calculate the torque about the CM of the person:

$$(216\text{N})(2 - x) - (40\text{N})(1 - x) - (314\text{N})(x) = 0$$

and x = 0.80 m. Note that the person's weight (which we could find using Sum $F_y = 0$) does not enter into the calculation because I chose the "pivot" at the CM.

Q: A scaffold of mass 50.0 kg and length 5.00 m is supported in a horizontal position by a vertical cable at each end. A person of mass 80.0 kg stands at a point 1.00 m from one end of the scaffold as shown in. In the equilibrium position of the system (scaffold + person), what is the tension in the cable T_1 and the cable T_2?



Answer:

Taking moments about P

$$\sum \tau = 0 = T_2 \times 5 - m_{man} \times g \times 1 - m_{scaffold} \times g \times 2.5$$

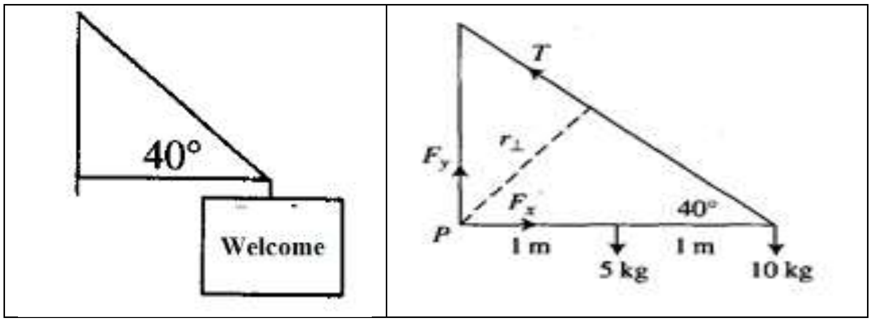
$$T_2 = \frac{(m_{man} \times 1 + m_{scaffold} \times 2.5)}{5} g = \left(\frac{80 \times 1 + 50 \times 2.5}{5}\right) \times 4.8 = 402\ N$$

$$For\ T_1\ calculation\ apply \sum F_y = 0$$

$$T_2 + T_1 - m_{man} \times g - m_{scaffold} \times g = 0$$

$$T_1 = (m_{man} + m_{scaffold})g - T_2 = 130 \times 9.8 - 401.8 = 872.2 \approx 872\text{ N}$$

Example: A 5-kg beam 2 m long is used to support a 10-kg sign by means of a cable attached to a building, as shown in the figure. What is the tension in the cable and compressive force exerted by the beam?



Answer:

Calculate the torque about a pivot at point P.

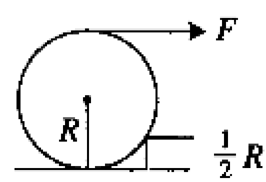
$$r_\perp = 2\sin 40^\circ = 1.3\,\text{m}$$

$$T(1.3\,\text{m}) - 5\,\text{kg}\,(1\,\text{m}) - 10\,\text{kg}\,(2\,\text{m}) = 0$$

$$T = 19\,\text{kg} = (19)(9.8) = 188\ \text{N}$$

$$\sum F_x = 0 \qquad F_x - T\cos 40^\circ = 0 \qquad F_x = 144\ \text{N}$$

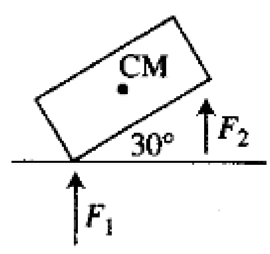
Example: What horizontal force applied as shown here is required to pull a wheel of weight Wand radius R over a curb of height h = R/2?



Answer:
The torque due to F about the contact point must balance the torque due to gravity acting at the center of the sphere.

$$w\left(\frac{\sqrt{3}}{2}R\right) - F\left(\frac{3R}{4}\right) = 0 \quad F = \frac{2\sqrt{3}}{3}W$$

Example: Two people carry a refrigerator of weight 800 N up a ramp inclined at 30° above horizontal. Each exerts a vertical force at a corner. The CM of the refrigerator is at its center. Its dimensions in the drawing are 0.72 m x 1.8 m. What force does each person exert?

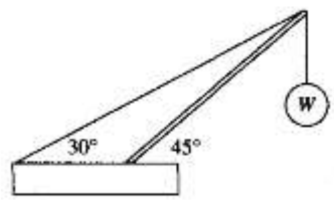


Answer:
F1 + F2 = 800 N, and the torque about the point of application of F1 is zero.

$$(F_2)(1.8\cos 30^\circ) - (800\,\text{N})(0.9\cos 30^\circ - 0.36\sin 30^\circ) = 0$$

$$F_2 = 308\,\text{N} \quad F_1 = 492\,\text{N}$$

Example: In the crane here the boom is 3.2 m long and weighs 1200 N. The cable can support a tension of 10,000 N. The weight is attached 0.5 m from the end of the boom. What maximum weight can be lifted?



Answer:

Calculate the torque about the base of the boom. The cable is one side of a triangle with angles 30°, 135° and 15°

$$(10{,}000\ \mathrm{N})(3.2 \sin 15^\circ) - w(2.7 \cos 45^\circ) = 0 \quad w = 4340\ \mathrm{N}$$

Equilibrium and Elasticity

12-3 ELASTICITY

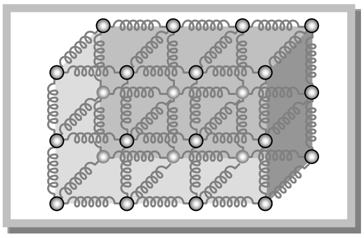
I- Elastic Property of Matter (Reading only)

(1) **Elasticity:** The property of matter by virtue of which a body tends to regain its original shape and size after the removal of deforming force is called elasticity.

(2) **Plasticity:** The property of matter by virtue of which it does not regain its original shape and size after the removal of deforming force is called plasticity.

(3) **Perfectly elastic body:** If on the removal of deforming forces the body regain its original configuration completely it is said to be perfectly elastic. A quartz fiber and phosphor bronze (an alloy of copper containing 4% to 10% tin, 0.05% to 1% phosphorus) is the nearest approach to the perfectly elastic body.

(4) **Perfectly plastic body:** If the body does not have any tendency to recover its original configuration, on the removal of deforming force, it is said to be perfectly plastic. Paraffin wax, wet clay are the nearest approach to the perfectly plastic body. Practically there is no material which is either perfectly elastic or perfectly plastic and the behavior of actual bodies lies between the two extremes.



(5) **Reason of elasticity:** In a solids, atoms and molecules are arranged in such a way that each molecule is acted upon by the forces due to neighboring molecules. These forces are known as intermolecular forces. For simplicity, the two molecules in their equilibrium positions (at inter-molecular distance $r = r_0$) (see figure) are shown by connecting them with a spring.

In fact, the spring connecting the two molecules represents the inter-molecular force between them. On applying the deforming forces, the molecules either come closer or go far apart from each other and restoring forces are developed. When the deforming force is removed, these restoring forces bring the molecules of the solid to their respective equilibrium position ($r = r_0$) and hence the body regains its original form.

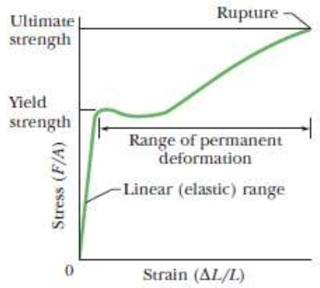


Figure 12-13 A stress–strain curve for a steel test specimen such as that of Fig. 12-12. The specimen deforms permanently when the stress is equal to the *yield strength* of the specimen's material. It ruptures when the stress is equal to the *ultimate strength* of the material.

(6) **Elastic limit:** Elastic bodies show their property of elasticity upto a certain value of deforming force. If we go on increasing the deforming force then a stage is reached when on removing the force, the body will not return to its original state. The maximum deforming force upto which a body retains its property of elasticity is called elastic limit of the material of body. Elastic limit is the property of a body whereas elasticity is the property of material of the body.

(7) **Elastic fatigue:** The temporary loss of elastic properties because of the action of repeated alternating deforming force is called elastic fatigue. It is due to this reason

(i) Bridges are declared unsafe after a long time of their use.

(ii) Spring balances show wrong readings after they have been used for a long time.

(iii) We are able to break the wire by repeated bending.

(8) **Elastic after effect:** The time delay in which the substance regains its original condition after the removal of deforming force is called elastic after effect. It is the time for which restoring forces are present after the removal of the deforming forces it is negligible for perfectly elastic substance, like quartz, phosphor bronze and large for glass fiber.

===

A- Stress

When a force is applied on a body there will be relative displacement of the particles and due to property of elasticity an internal restoring force is developed which tends to restore the body to its original state. *The internal restoring force acting per unit area of cross section of the deformed body is called* ***stress***.

At equilibrium, restoring force is equal in magnitude to external force, stress can therefore also be defined as external force per unit area on a body that tends to cause it to deform.

If external force *F* is applied on the area *A* of a body then,

$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$$

Unit of stress: N/m^2 (S.I.), Dimension: $[\text{ML}^{-1}\text{T}^{-2}]$

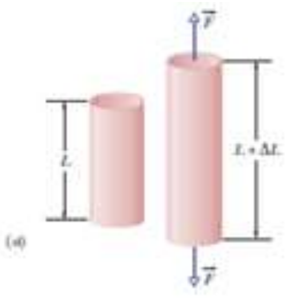
Stress developed in a body depends upon how the external forces are applied over it. On this basis there are two types of stresses: **Normal** and **Shear** or **Tangential** stress.

(1) **Normal stress:** Here the force is applied normal to the surface.

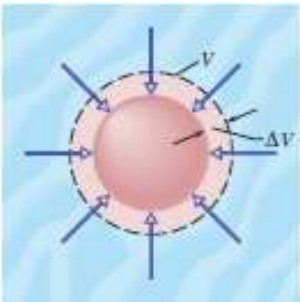
It is again of two types: **Longitudinal** and **Bulk or volume stress**

(i) ***Longitudinal stress***

(ii) ***Bulk (Hydraulic) or Volume stress***



Longitudinal stress



Bulk (Hydraulic) or Volume stress

i- Longitudinal stress **(Reading)**

a) It occurs only in solids and comes in picture when one of the three dimensions *viz*. lengths, breadth, height is much greater than other two.

b) Deforming force is applied parallel to the length and causes increase in length.

c) Area taken for calculation of stress is area of cross section.

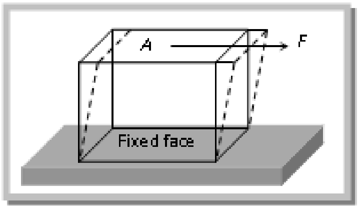
d) Longitudinal stress produced due to increase in length of a body under a deforming force is called tensile stress.

e) Longitudinal stress produced due to decrease in length of a body under a deforming force is called compressional stress.

ii- Bulk (Hydraulic) or Volume stress **(Reading)**

a) It occurs in solids, liquids or gases.
b) In case of fluids only bulk stress can be found.
c) It produces change in volume and density, shape remaining same.
d) Deforming force is applied normal to surface at all points.
e) Area for calculation of stress is the complete surface area perpendicular to the applied forces.
f) It is equal to change in pressure because change in pressure is responsible for change in volume.

(2) **Shear or tangential stress (Reading only):** It comes in picture when successive layers of solid move on each other *i.e.* when there is a relative displacement between various layers of solid.



a) Here deforming force is applied tangential to one of the faces.

b) Area for calculation is the area of the face on which force is applied.

c) It produces change in shape, volume remaining the same.

Difference between Pressure and Stress	
Pressure	**Stress**
Pressure is always normal to the area.	Stress can be normal or tangential.
Always compressive in nature.	May be compressive or tensile in nature.

***Example1*.** A and B are two wires. The radius of A is twice that of B. they are stretched by the same load. Calculate the value of the factor C that satisfy the relation between the stress on B, $(\text{Stress})_B$, and stress on A, $(\text{Stress})_A$, as :

$$(\text{Stress})_B = C \times (\text{stress})_A .$$

Solution : Given that $r_a = 2r_b$ and $\text{Stress} = \dfrac{\text{Force}}{\text{Area}} = \dfrac{F}{\pi r^2}$

$$\therefore \text{Stress} \propto \frac{1}{r^2} \Rightarrow \frac{(\text{Stress})_B}{(\text{Stress})_A} = \left(\frac{r_A}{r_B}\right)^2 = (2)^2 \Rightarrow (\text{Stress})_B = 4 \times (\text{stress})_A \qquad [\text{As } F = \text{constant}]$$

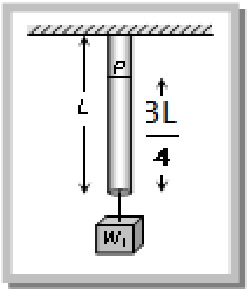
The value of C = 4.

***Example2*.** On suspending a weight Mg, the length L of elastic wire and area of cross-section A its length becomes double the initial length. Calculate the stress on the wire.

Solution: When the length of wire becomes double, its area of cross section will become half because volume of wire is constant $(V = AL)$.

So the stress $= \frac{\text{Force}}{\text{Area}} = \frac{Mg}{A/2} = \frac{2Mg}{A}$.

***Example3*.** One end of a uniform wire of length L and of weight W is attached rigidly to a point in the roof and a weight W_1 is suspended from its lower end. If S is the area of cross-section of the wire, the stress in the wire at a height $3L/4$ from its lower end is



Solution: As the wire is uniform so the weight of wire below point P is $\frac{3W}{4}$

$\therefore$ Total force at point $P = W_1 + \frac{3W}{4}$ and area of cross-section = S

$\therefore$ Stress at point $P = \frac{\text{Force}}{\text{Area}} = \frac{W_1 + \frac{3W}{4}}{S}$

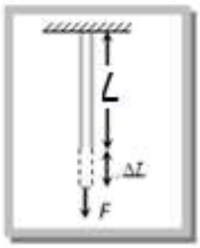
==

B- Strain

The ratio of change in configuration to the original configuration is called strain. Being the ratio of two like quantities, it has no dimensions and units.

Strains are of three types:

(1) **Linear strain:** If the deforming force produces a change in length alone, the strain produced in the body is called **linear** strain or **tensile strain**.



$$\text{Linear strain} = \frac{\text{Change in length}(\Delta L)}{\text{Original length}(L)}$$

Linear strain in the direction of deforming force is called **longitudinal strain** and in a direction perpendicular to force is called **lateral strain**.

***Example*.** A wire is stretched to double its length. Calculate the linear strain.

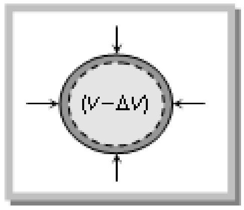
Solution: Strain $= \frac{\text{Change in length}}{\text{Original length}} = \frac{2L - L}{L} = 1$

***Example*.** The length of a wire increases by 1% by a load of 2.0 N. Calculate the linear strain produced in the wire.

Solution: Strain $= \dfrac{\text{Change in length}}{\text{Original length}} = \dfrac{1\% \text{ of } L}{\text{L}} = \dfrac{L/100}{L} = 0.01$

----

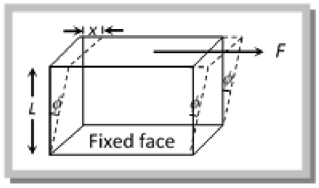
(2) **Volumetric strain:** If the deforming force produces a change in volume alone the strain produced in the body is called volumetric strain.



$$\text{Volumetric strain} = \frac{\text{Change in volume}(\Delta V)}{\text{Original volume}(V)}$$

----

(3) **Shearing strain:** If the deforming force produces a change in the shape of the body without changing its volume, strain produced is called shearing strain. It is defined as angle in radians through which a plane perpendicular to the fixed surface of the cubical body gets turned under the effect of tangential force.



$$\phi = \frac{x}{L}$$

***Example*.** A cube of aluminum of sides 0.1 *m* is subjected to a shearing force of 100 *N*. The top face of the cube is displaced through 0.02 *cm* with respect to the bottom face. The shearing strain would be

Solution : Shearing strain $\phi = \dfrac{x}{L} = \dfrac{0.02cm}{0.1m} = 0.002$

Note: ❑ When a beam is bent both compression strain as well as an extension strain is produced.

Three elastic moduli are used to describe the elastic behavior (deformations) of objects as they respond to forces that act on them. The strain (fractional change in length) is linearly related to the applied stress (force per unit area) by the proper modulus, according to the general stress–strain relation

$$\text{Stress} = \text{modulus} \times \text{Strain}.$$

When an object is under tension or compression, the stress–strain relation is written as

----

A- Young's Modulus (*E*). (Tension and Compression)

It is defined as the ratio of normal stress to longitudinal strain within limit of proportionality.

$$E = \frac{\text{Normal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L} \quad \Rightarrow \boxed{\Delta L = \frac{FL}{AE}}$$

Unit of Young's modulus: N/m^2 (S.I.).

If force is applied on a wire of radius *r* by hanging a weight of mass *M*, then

$$E = \frac{MgL}{\pi r^2 \Delta L}$$

Important points

i. If the length of a wire is doubled,

Then longitudinal strain = $\frac{\text{change in length}(\Delta L)}{\text{initial length}(L)} = \frac{\text{final length} - \text{initial length}}{\text{Initial length}} = \frac{2L-L}{L} = 1$

$\therefore$ Young's modulus = $\frac{\text{stress}}{\text{strain}}$ $\Rightarrow$ E = stress [As strain = 1]

So young's modulus is numerically equal to the stress which will double the length of a wire.

ii. Increment in the length of wire $\Delta L = \dfrac{FL}{\pi r^2 E}$ $\left[\text{As } E = \dfrac{FL}{A\Delta L}\right]$

So if same stretching force is applied to different wires of same material, $\Delta L \propto \dfrac{L}{r^2}$ [As *F* and *E* are constant]

i.e., greater the ratio $\dfrac{L}{r^2}$, greater will be the elongation in the wire.

***Example1*.** The diameter of a brass rod is 4 *mm* and Young's modulus of brass is $9\times10^{10}\ N/m^2$. The force required to stretch by 0.1% of its length is

Solution: $r = 2\times10^{-3} m,\ E = 9\times10^{10} N/m^2,\ \Delta L = 0.1\% L \quad \Rightarrow \frac{\Delta L}{L} = 0.001$

As $E = \frac{F}{A}\frac{L}{\Delta L}$ $\quad \therefore F = EA\frac{\Delta L}{L} = 9\times10^{10}\times\pi\ (2\times10^{-3})^2\times0.001 = 360\pi$ N

***Example2*.** A wire of length 2.0 m is made from 10 cm^3 of copper. A force *F* is applied so that its length increases by 2.0 mm. Another wire of length 8.0 m is made from the same volume of copper. If the force *F* is applied to it, its length will increase by:

Solution: $\Delta L = \frac{FL}{AE} = \frac{FL^2}{VE}$ $\quad \therefore\ \Delta L \propto L^2$ [As *V*, *E* and *F* are constant]

$$\frac{\Delta L_2}{\Delta L_1}=\left[\frac{L_2}{L_1}\right]^2=\left(\frac{8}{2}\right)^2=16$$

$$\Rightarrow \quad \Delta L_2 = 16\times\Delta L_1 = 16\times 2 \text{ mm} = 32 \text{ mm} = 3.2 \text{ cm}$$

Example3. A wire of length L and radius r is rigidly fixed at one end. On stretching the other end of the wire with a force F, the increase in its length is ΔL_1. If another wire of same material but of length $2L$ and radius $2r$ is stretched with a force of $2F$, the increase in its length will be

Solution: $$\Delta L = \frac{FL}{\pi r^2 E} \Rightarrow \frac{\Delta L_2}{\Delta L_1} = \frac{F_2}{F_1}\frac{L_2}{L_1}\left(\frac{r_1}{r_2}\right)^2 = 2\times 2\times\left(\frac{1}{2}\right)^2 = 1$$

$\Delta L_1 = \Delta L_2$ *i.e.* the increment in length will be same.

Shearing When an object is under a shearing stress, the Equation

$$\text{Stress} = \text{modulus} \times \text{Strain}$$

is written as

$$\frac{F}{A} = G\frac{\Delta x}{L}$$

where $\Delta x / L$ is the shearing strain of the object, Δx is the displacement of one end of the object in the direction of the applied force $\vec{F}$, and G is the **shear modulus** of the object. The stress is F/A.

Hydraulic Stress When an object undergoes *hydraulic compression* due to a stress exerted by a surrounding fluid, the Equation

$$\text{Stress} = \text{modulus} \times \text{Strain}$$

is written as

$$\underbrace{\frac{F}{A}}_{\text{Hydraulic Stress}} = p = B\underbrace{\frac{\Delta V}{V}}_{\text{Strain}}, \qquad B \equiv \text{bulk modulus}$$

where p is the pressure (*hydraulic stress*) on the object due to the fluid, $\Delta V/V$ (the strain) is the absolute value of the fractional change in the object's volume due to that pressure, and B is the **bulk modulus** of the object.

Example. A solid copper cube has an edge length of 85.5 cm. How much hydraulic stress must be applied to the cube to reduce the edge length to 85.0 cm? The bulk modulus of copper is 1.40×10^{11} N/m^2.

Answer:

$$|p| = B\left|\frac{\Delta V}{V}\right| = 1.40\times10^{11}\times\left[\frac{(85)^3-(85.5)^3}{(85.5)^3\times10^{-6}}\right]\times10^{-6}$$

$$= 2.44\times10^9 \text{N/m}^2$$

Extra problems

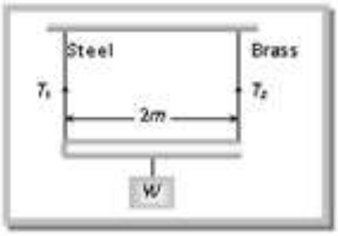
***Problem*.** The force required to stretch a steel wire of 1 cm^2 cross-section to 1.1 times its length would be $(Y = 2\times10^{11}\ Nm^{-2})$

(a) $2\times10^{6}\ N$ (b) $2\times10^{3}\ N$ (c) $2\times10^{-6}\ N$ (d) $2\times10^{-7}\ N$

Solution : (a) $L_2 = 1.1 L_1$ $\therefore \text{Strain} = \frac{l}{L_1} = \frac{L_2 - L_1}{L_1} = \frac{1.1L_1 - L_1}{L_1} = 0.1.$

$$F = EA\frac{\Delta L}{L} = 2\times10^{11}\times1\times10^{-4}\times0.1 = 2\times10^{6}\ N.$$

***Problem*.** A 2-*meter* long rod is suspended with the help of two wires of equal length. One wire is of steel and its cross-sectional area is 0.1 cm² and another wire is of brass and its cross-sectional area is 0.2 cm². If a load *W* is suspended from the rod and stress produced in both the wires is same then the ratio of tensions in them will be



(a) Will depend on the position of *W*
(b) $T_1 / T_2 = 2$
(c) $T_1 / T_2 = 1$
(d) $T_1 / T_2 = 0.5$

Solution : (d) $\text{Stress} = \frac{\text{Tension}}{\text{Area of cross-section}} = \text{constant}$

$$\therefore \frac{T_1}{A_1} = \frac{T_2}{A_2} \Rightarrow \frac{T_1}{T_2} = \frac{A_1}{A_2} = \frac{0.1}{0.2} = \frac{1}{2} = 0.5.$$

***Problem*.** A certain wire, hanging from a ceiling, stretches 0.9 cm when outward force with magnitude F is applied to the free end. The same force is applied to a wire of the same material but with three times the diameter and three times the length. The second wire stretches:

Answer:

Calculate the ratio:

$$\frac{\Delta L_1}{\Delta L_2} = \frac{F_1 L_1 / A_1 E_1}{F_2 L_2 / A_2 E_2} = \frac{L_1 A_2}{L_2 A_1} = \frac{1L_1\times\pi(3d/2)^2}{3L_1\times\pi(d/2)^2} = 3$$

$$\Delta L_2 = \frac{\Delta L_1}{3} = \frac{0.9}{3} = 0.3$$

***Problem*.** A 10.2 m long steel beam with a cross-sectional area of 0.120 m² is mounted between two concrete walls with no room for expansion. When the temperature rises, such a beam will expand in length by 1.20 mm if it is free to do so. What force must be exerted by the concrete walls to prevent the beam from expanding? Young's modulus for steel is 2.00×10^{11} N/m².

Answer:

$$F = E\times\frac{\Delta L}{L}\times A$$

$$= 2\times10^{11}\times\frac{1.20\times10^{-3}}{10.2}\times0.120 = 2.82\times10^{6}\ \text{N}$$

***Example*.** Two wires *A* and *B* are of same materials. Their lengths are in the ratio 1:2 and diameters are in the ratio 2:1 when stretched by force F_A and F_B respectively they get equal increase in their lengths. Calculate the ratio F_A/F_B.

Solution: $E=\frac{FL}{\pi r^2 \Delta L} \quad \therefore \; F=E\pi r^2\frac{\Delta L}{L}$

$$\frac{F_A}{F_B}=\frac{E_A}{E_B}\left(\frac{r_A}{r_B}\right)^2\left(\frac{\Delta L_A}{\Delta L_B}\right)\left(\frac{L_B}{L_A}\right)=1\times\left(\frac{2}{1}\right)^2\times(1)\times\left(\frac{2}{1}\right)=8$$

***Example*.** A uniform plank of Young's modulus *E* is moved over a smooth horizontal surface by a constant horizontal force *F*. The area of cross-section of the plank is *A*. the compressive strain on the plank in the direction of the force is

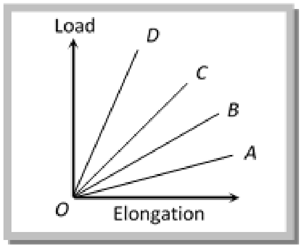
Solution: Compressive strain $=\frac{\text{Stress}}{\text{Young's modulus}}=\frac{F/A}{E}=\frac{F}{AE}$

***Example*.** A wire is stretched by 0.01 *m* by a certain force *F*. Another wire of same material whose diameter and length are double to the original wire is stretched by the same force. Then its elongation will be

Solution: $\Delta L=\frac{FL}{\pi r^2 E} \; \therefore \Delta L \propto \frac{L}{r^2}$ [As *F* and *E* are constants]

$$\frac{\Delta L_2}{\Delta L_1}=\left(\frac{L_2}{L_1}\right)\left(\frac{r_1}{r_2}\right)^2=(2)\times\left(\frac{1}{2}\right)^2=\frac{1}{2}\Rightarrow \Delta L_2=\frac{\Delta L_1}{2}=\frac{0.01}{2}=0.005m.$$

***Example*.** The load versus elongation graph for four wires of the same material is shown in the figure. The thickest wire is represented by the line



(a) *OD*
(b) *OC*
(c) *OB*
(d) *OA*

Solution: (a) Young's modulus $E=\frac{FL}{A\Delta L} \; \therefore \Delta L \propto \frac{1}{A}$ (As *E*, *L* and *F* are constant)

From the graph it is clear that for same load elongation is minimum for graph *OD*. As elongation (ΔL) is minimum therefore area of cross-section (*A*) is maximum. So thickest wire is represented by *OD*.

***Example*.** A 5 m long aluminum wire $(E = 7\times10^{10}\,\text{N/m}^2)$ of diameter 3 mm supports a 40 kg mass. In order to have the same elongation in a copper wire $(E = 12\times10^{10}\,\text{N/m}^2)$ of the same length under the same weight, the diameter should now be, in *mm*

Solution: $\Delta L = \dfrac{FL}{\pi r^2 E} = \dfrac{4FL}{\pi d^2 E}$ [As $r = d/2$]

If the elongation in both wires (of same length) is same under the same weight then $d^2 E$ = constant

$$\left(\frac{d_{Cu}}{d_{Al}}\right)^2 = \frac{E_{Al}}{E_{Cu}} \Rightarrow d_{Cu} = d_{Al}\times\sqrt{\frac{E_{Al}}{E_{Cu}}} = 3\times\sqrt{\frac{7\times10^{10}}{12\times10^{10}}} = 2.29\ \text{mm}$$

***Example*.** On applying a stress of $20\times10^{8}\,\text{N/m}^2$ the length of a perfectly elastic wire is doubled. Its Young's modulus will be

Solution : When strain is unity then Young's modulus is equal to stress.

***Example*.** The dimensions of four wires of the same material are given below. In which wire the increase in length will be maximum when the same tension is applied

Solution : If same force is applied on four wires of same material then elongation in each wire depends on the length and diameter of the wire and given by $\Delta L \propto \dfrac{L}{d^2}$ and the ratio of $\dfrac{L}{d^2}$ is maximum for (d) option.

***Example*.** The Young's modulus of a wire of length *L* and radius *r* is $Y\ N/m^2$. If the length and radius are reduced to *L*/2 and *r*/2, then its Young's modulus will be

Solution : Young's modulus do not depend upon the dimensions of wire. It is constant for a given material of wire.

***Example*.** A fixed volume of iron is drawn into a wire of length *L*. The extension *x* produced in this wire by a constant force *F* is proportional to

Solution : $\Delta L = \dfrac{FL}{AE} = \dfrac{FL^2}{ALE} = \dfrac{FL^2}{VE}$ for a fixed volume $\Delta L \propto L^2$