Chapter 10
Rotational motion

Important Terms

Angular Displacement “θ” in radians

$$\theta = \frac{s}{r},\ \text{where } s \text{ is the length of arc and } r \text{ is the radius.}$$

Angular Velocity “ω”

The rate at which θ changes.

Angular Acceleration; Constant Angular Acceleration “α”

The rate at which the angular velocity changes.

Instantaneous speed, or point’s linear speed (or, tangential speed)

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega .$$

Linear acceleration (or, tangential acceleration)

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha .$$

Centripetal acceleration

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha .$$

Equations and Symbols

$\theta = \frac{s}{r}$ $\omega = \lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$ $\alpha = \lim_{\Delta t \to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$ $v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$ $a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha$ $a_c = \frac{v_t^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$	s = length of arc r = radius v = velocity θ = angle in radians ω= Angular velocity α = angular acceleration v_t = Linear (tangential) velocity a_t = tangential acceleration a_c = Radial (Centripetal) acceleration

Basic Requirements:

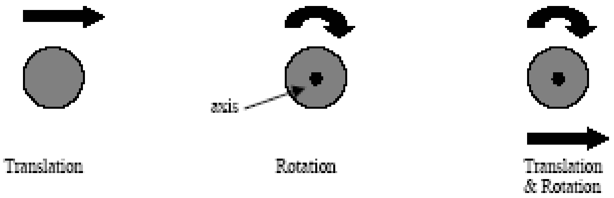
1. To be familiar with the angular terminologies, such as Angular: displacement, velocity and acceleration.
2. To relate the linear and angular variables.
3. Master the kinematic equations in case of rotation.
4. Differentiate between tangential and centripetal acceleration

10-1 ROTATIONAL VARIABLES

So far in our study of physics we have (with few exceptions) dealt with particles, objects whose spatial dimensions were unimportant for the questions we were asking. We now deal with the (elementary!) aspects of the motion of extended objects, objects whose dimensions are important.

The objects that we deal with are those which maintain a rigid shape (the mass points maintain their relative positions) but which can change their orientation in space. They can have translational motion, in which their center of mass moves but also rotational motion, in which we can observe the changes in direction of a set of axes that is “glued to” the object. Such an object is known as a rigid body. We need only a small set of angles to describe the rotation of a rigid body. Still, the general motion of such an object can be quite complicated.

Translation vs. Rotation

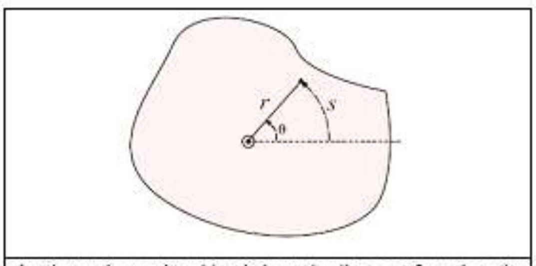


Since this is such a complicated subject, we specialize further to the case where a line of points of the object is fixed and the object spins about a rotation axis fixed in space. When this happens, every individual point of the object will have a circular path, although the radius of that circle will depend on which mass point we are talking about. And the orientation of the object is completely specified by one variable, an angle θ which we can take to be the angle between some reference line “painted” on the object and the x axis (measured counter-clockwise, as usual).

Because of the nice mathematical properties of expressing the measure of an angle in radians, we will usually express angles in radians all through our study of rotations; on occasion, though, we may have to convert to or from degrees or revolutions. Revolutions, degrees and radians are related by:

$$1 \text{ revolution} = 360^{o} = 2\pi \text{ radians}$$

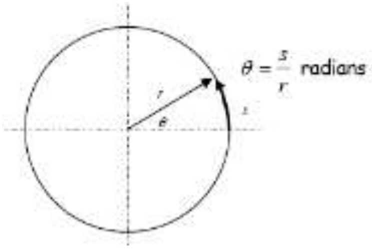
$$\Rightarrow \boxed{\frac{1 \text{ revolution}}{1 \,\text{radians}} = 2\pi}$$



A point on the rotating object is located a distance r from the axis; as the object rotates through an angle θ it moves a distance s.

[Later, because of its importance, we will deal with the motion of a (round) object which rolls along a surface without slipping. This motion involves rotation and translation, but it is not much more complicated than rotation about a fixed axis.]

Angular Displacement "θ"



As a rotating object moves through an angle θ from the starting position, a mass point on the object at radius r will move a distance s ; s length of arc of a circle of radius *r*, subtended by the angle θ. When θ is in radians, these are related by

$$\theta = \frac{s}{r}, \qquad \theta \text{ in radians} \qquad (1)$$

If we think about the consistency of the units in this equation, we see that since *s* and *r* both have units of length, θ is really *dimensionless*; but since we are assuming radian measure, we will often write "rad" next to our angles to keep this in mind.

Notes:

- ☞ We will assume that θ is + if it is counterclockwise from the + x axis.
- ☞ Although θ has both magnitude and direction it is not generally considered a vector quantity because addition of angular displacements is not communicative. Only in the limiting case of $\Delta\theta$ can an angular displacement be considered a vector.
- ☞ Normally we are interested in θ as a function of time or $\theta(t)$.
- ☞ 1 revolution = 360^0 = 2π radians
- ☞ 1 radian = 57.3^0 = 0.159 revolutions
- ☞ A complete revolution is some multiple integers of 2π radians, e.g. ($n \times 2\pi$): 2π, 4π, 6π, etc.
- ☞ If a body rotates about a fixed axis then all the particles will have same angular displacement (although linear displacement will differ from particle to particle in accordance with the distance of particles from the axis of rotation).

Simple Example:

a- What angle in radians is subtended by an arc that has length 1.80 m and is part of a circle of radius 1.20 m?

b- Express the same angle in degrees.

c- The angle between two radii of a circle is 0.620 rad. What arc length is subtended if the radius is 2.40 m?

Answer:

a- The equation $\theta = \frac{s}{r}$ relates arc-length, radius and subtended angle. We find:

$$\theta = \frac{s}{r} = \frac{1.80 \text{ m}}{1.20 \text{ m}} = 1.50 \text{ rad}$$

b- To express this angle in degrees use the relation: 360 deg = 2π rad (or, $180^\circ = \pi$ rad). Then we have:

$$1.50 \text{ rad} = (1.50 \text{ rad})\left(\frac{180^\circ}{\pi \text{ rad}}\right) = 85.9^\circ$$

c- We can find the arc length subtended by an angle θ by the relation: $s = r\theta$. Then for an angle of 0.620 rad and radius 2.40m, the arc-length is

$$s = r\theta = (2.40 \text{ m})(0.620) = 1.49 \text{ m}.$$

Angular Velocity "ω": is "the rate at which θ changes".
If in a time period Δt the object has rotated through an angular displacement $\Delta\theta$ then we define the average angular velocity for that period as:

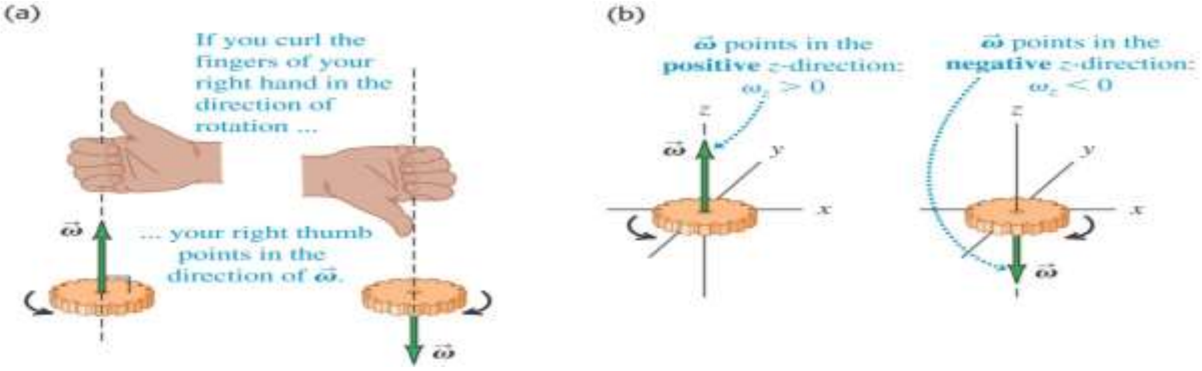
$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t} \tag{2}$$

A more interesting quantity is found as we let the time period Δt be vanishingly small. This gives us the instantaneous angular velocity, ω:

$$\omega = \lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{3}$$

Angular velocity has units of rad/s, or equivalently, 1/s or s^{-1}.

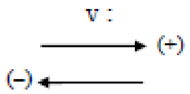
In more advanced studies of rotational motion, ω of a rotating object is defined in such a way that it is a **vector quantity**. For an object rotating counterclockwise about a fixed axis, this vector has magnitude ω and points outward along the axis of rotation. For our purposes, though, we will treat ω as a number which can be positive or negative, depending on the direction of rotation.

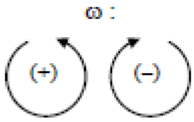


Notes:

- ☞ ω has the same value for all particles in a rotating system.
- ☞ Tangential velocity, which depends upon distance from the rotational axis, varies depending upon radius.
- ☞ $\omega_{\text{avg}} = \dfrac{\Delta\theta}{\Delta t}$ rad/s
- ☞ Angular velocity is a psuedovector. The direction is determined from the **right hand rule (RHR)**.
- ☞ If one curls their right hand around the axis of rotation with their fingers pointing in the direction of rotation, their thumb then gives the direction of the angular momentum vector.
- ☞ Note that the direction of the angular velocity vector is along the axis of rotation rather than in the direction of motion.

In 1D, velocity v has a sign (+ or –) depending on direction. Likewise, for fixed-axis rotation, ω has a sign convention, depending on the sense of rotation.





Simple Example: What is the angular speed in radians per second of

b- the Earth in its orbit about the Sun and

c- the Moon in its orbit about the Earth?

Answer:

b- The Earth goes around in a (nearly!) circular path with a period of one year. In seconds, this is:

$$1\,\text{yr} = (1\,\text{yr})\left(\frac{365.25\,\text{day}}{1\,\text{yr}}\right)\left(\frac{24\,\text{hr}}{1\,\text{day}}\right)\left(\frac{3600\,\text{s}}{1\,\text{hr}}\right) = 3.156\times10^7\,\text{s}$$

In one year its angular displacement is 2π radians (all the way around) so its angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{(3.156\times10^7\,\text{s})} = 1.99\times10^{-7}\,\frac{\text{rad}}{\text{s}}$$

c- How long does it take the moon to go around the earth? any good reference source will tell you that it is 27.3 days. Converting to seconds, we have:

$$P = 27.3\,\text{days} = (27.3\,\text{days})\left(\frac{24\,\text{hr}}{1\,\text{day}}\right)\left(\frac{3600\,\text{s}}{1\,\text{hr}}\right) = 2.36\times10^6\,\text{s}$$

In that length of time the angular displacement of the moon is 2π so its angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{(2.36\times10^6\,\text{s})} = 2.66\times10^{-6}\,\frac{\text{rad}}{\text{s}}$$

Angular Acceleration; Constant Angular Acceleration "α"

"The rate at which the angular velocity changes" is the angular acceleration of the object. If the object's (instantaneous) angular velocity changes by $\Delta\omega$ within a time period Δt , then the average angular acceleration for this period is

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t} \qquad (4)$$

But as you might expect, much more interesting is the instantaneous angular acceleration, defined as:

$$\alpha = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\,\omega}{dt} \qquad (5)$$

Notes:

- ☞ Angular acceleration has the same value for all particles in a rotating system.
- ☞ $\alpha_{\text{avg}} = \dfrac{\Delta\omega}{\Delta t}$ rad/s^2
- ☞ Angular acceleration is another psuedovector and its direction is also determined from the **RHR**.

10-2 ROTATION WITH CONSTANT ANGULAR ACCELERATION

We can derive simple equations for rotational motion if we know that α is constant. (Later we will see that this happens if the "torque" on the object is constant.) Then, if θ_o is the initial angular displacement, ω_o is the initial angular velocity and α is the constant angular acceleration, then we find:

$$\omega = \omega_o + \alpha t \tag{6}$$

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2 \tag{7}$$

$$\omega^2 = \omega_o^2 + 2\alpha\left(\theta - \theta_o\right) \tag{8}$$

$$\theta = \theta_o + \frac{1}{2}\left(\omega_o + \omega\right)t \tag{9}$$

where θ and ω are the angular displacements and velocity at time t . θ_o and ω_o are the values of the angle and angular velocity at $t = 0$.

These equations have exactly the same form as the kinematic equations for one–dimensional linear motion given in Chapter 2. The correspondences of the variables are:

$$x \rightarrow \theta, \quad v \rightarrow \omega, \quad a \rightarrow \alpha$$

It is almost always simplest to set $\theta_o = 0$ in these equations, so you will often see Eqs. 6—9 written with this substitution already made.

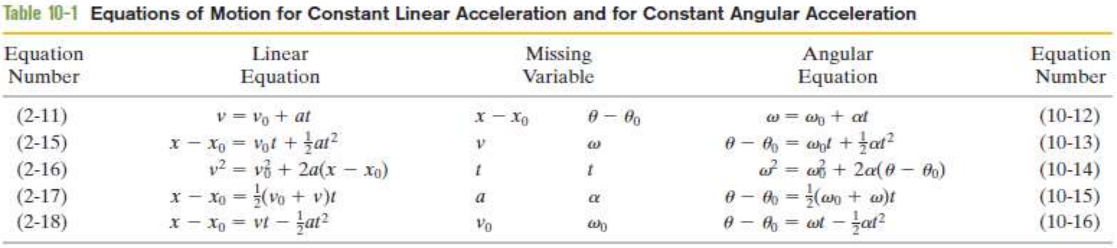
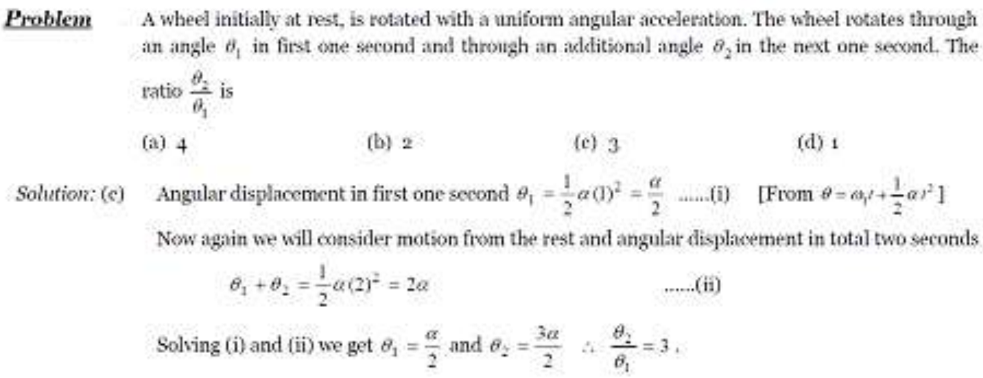


Table 10-1 Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable		Angular Equation	Equation Number
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(10-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(10-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(10-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(10-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(10-16)



Problem A wheel initially at rest, is rotated with a uniform angular acceleration. The wheel rotates through an angle θ_1 in first one second and through an additional angle θ_2 in the next one second. The ratio $\frac{\theta_2}{\theta_1}$ is

(a) 4 (b) 2 (c) 3 (d) 1

Solution: (c) Angular displacement in first one second $\theta_1 = \frac{1}{2}\alpha(1)^2 = \frac{\alpha}{2}$ ......(i) [From $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$]

Now again we will consider motion from the rest and angular displacement in total two seconds

$\theta_1 + \theta_2 = \frac{1}{2}\alpha(2)^2 = 2\alpha$ ......(ii)

Solving (i) and (ii) we get $\theta_1 = \frac{\alpha}{2}$ and $\theta_2 = \frac{3\alpha}{2}$ $\therefore \frac{\theta_2}{\theta_1} = 3$.

Example: Calculate the required time for a wheel, initially at rest, to turn through 10 full revolutions if it can accelerate at a rate of 1π rad/s^2.

Answer: given that $\alpha = 1\pi \text{ rad/s}^2$, $\theta - \theta_i = 10 \times 2\pi \text{ rad}$, and $\omega_i = 0 \text{ rad/s}$, then

$$\theta - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$$

$$20\,\pi \text{ rad} = 0\,t + \frac{1}{2}\left(1\pi \text{ rad/s}^2\right)\left(t\right)^2 \quad \Rightarrow \quad t = \underline{6.3 \text{ s}}.$$

Example: A wheel spins at a rate of 30 revs/sec $30\ \text{revs/s} = 60\,\pi\ \text{rad/s}$ comes to a complete stop in 10 seconds. Find:

a) the angular acceleration of the wheel

b) the number of revolutions the wheel undergoes before it comes to a stop

Answer: given that $\Delta t = 10\ \text{s}$, $\omega_i = 60\,\pi\ \text{rad/s}$, $\omega_f = 0$, and take $\theta_i = 0$, then

a)

$$\omega_f = \omega_i + \alpha t \quad \Rightarrow \quad \alpha = \frac{\omega_f - \omega_i}{\Delta t} = \frac{0 - 60\pi}{10} = -6\pi \frac{\text{rad}}{\text{s}^2}$$

b) use

$$\omega^2 = \omega_o^2 + 2\alpha\left(\theta - \theta_o\right) \quad \Rightarrow \quad \left(\theta - \theta_o\right) = \frac{\omega^2 - \omega_o^2}{2\alpha} = \frac{0 - \left(60\,\pi\right)^2}{2(-6\,\pi)} = \underline{300\,\pi\ \text{rad}}$$

Or, we can use

$$\theta - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2 = 60\,\pi\left(10\right) + \tfrac{1}{2}\left(-6\pi\right)\left(10\right)^2 = \underline{300\,\pi\ \text{rad}}.$$

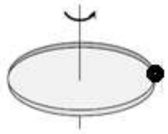
Since there are $2\,\pi$ radians per revolution, this yields 150 revolutions of the wheel.

Example: A car engine is idling at ω_0= 500 rev/min at a traffic light. When the light turns green, the crankshaft rotation speeds up at a constant rate to ω = 2500 rev/min over an interval of 3.0 s. The number of revolutions the crankshaft makes during these 3.0 s is:

Answer:

$$\Delta\theta = \left(\frac{\omega_i + \omega_f}{2}\right)\Delta t = \left(\frac{500 + 2500}{2}\right)\left(\frac{rev}{60}\right)3 = \underline{75\ \text{rev}}$$

Example: The angular position of a point on the rim of a rotating wheel is given by



$$\theta(t) = 4.0t - 3.0t^2 + t^3,$$

where θ is in radians if t is given in seconds.

a- What are the angular velocities at t = 2.0 s and t = 4.0 s?

b- What is the average angular acceleration for the time interval that begins at t = 2.0 s and ends at t = 4.0 s?

c- What are the instantaneous angular accelerations at the beginning and end of this time interval?

Answer:

a- In the problem we are given the angular position θ as a function of time. To find the (instantaneous) angular velocity at any time, use Eq. 3 and find:

$$\begin{aligned}\omega(t) &= \frac{d\theta}{dt} = \frac{d}{dt}\left(4.0t - 3.0t^2 + t^3\right) \\ &= 4.0 - 6.0t + 3.0t^2\end{aligned}$$

where, if t is given in seconds, ω is given in rad/s .

The angular velocities at the given times are then

$$\omega(2.0\,\text{s}) = 4.0 - 6.0(2.0) + 3.0(2.0)^2 = 4.0\,\tfrac{\text{rad}}{\text{s}}$$

$$\omega(4.0\text{s}) = 4.0 - 6.0(4.0) + 3.0(4.0)^2 = 28.0\,\tfrac{\text{rad}}{\text{s}}$$

b- Since we have the values of ω and t = 2.0 s and t = 4.0 s, Eq. 4 gives the average angular acceleration for the interval:

$$\begin{aligned}\overline{\alpha} &= \frac{\Delta\omega}{\Delta t} = \frac{(28.0\,\frac{\text{rad}}{\text{s}} - 4.0\,\frac{\text{rad}}{\text{s}})}{(4.0\text{s} - 2.0\text{s})} \\ &= 12.0\,\frac{\text{rad}}{\text{s}^2}\end{aligned}$$

The average angular acceleration is 12.0 rad/s^2.

c- We find the instantaneous angular acceleration from Eq. 5:

$$\begin{aligned}\alpha(t) &= \frac{d\omega}{dt} = \frac{d}{dt}\left(4.0 - 6.0t + 3.0t^2\right) \\ &= -6.0 + 6.0t\end{aligned}$$

where, if t is given in seconds, α is given in rad/s^2 .

Then at the beginning and end of our time interval the angular accelerations are:

$$\alpha(2.0\,\text{s}) = -6.0 + 6.0(2.0) = 6.0\,\frac{\text{rad}}{\text{s}^2}$$

$$\alpha(4.0\text{s}) = -6.0 + 6.0(4.0) = 18.0\,\frac{\text{rad}}{\text{s}^2}$$

H.W. Sample problem 10.01

The angular position of a point on the rim of a rotating wheel is given by

$$\theta(t) = -1.00 - 0.600t + 0.250t^2,$$

where θ is in radians if t is given in seconds.

H.W. At what time, $t_{\min}$, does $\theta(t)$ reach the minimum value? What is $\theta(t)$ at $t_{\min}$?

Answer: Calculate $\omega(t) = 0$ to find $t_{\min} = 1.20$ s, and $\theta(t_{\min}) = -1.36$ rad $\approx -77.9^{\circ}$

Example: An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative angular acceleration of magnitude 2.00 rad/s^2,

(a) How long does it take the wheel to stop?

(b) Through how many radians does it turn during the time found in (a)?

Answer:

(a) Convert the initial rotation rate to radians per second:

$$100\,\frac{\text{rev}}{\text{min}} = \left(100\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 10.5\,\frac{\text{rad}}{\text{s}}$$

When the wheel has stopped then of course its angular velocity is zero. Since we know ω_o, ω and α we can use Eq. 6 to get the elapsed time:

$$\omega = \omega_o + \alpha t \quad \Rightarrow \quad t = \frac{(\omega - \omega_o)}{\alpha}$$

and we get:

$$t = \frac{(0 - 10.5\,\frac{\text{rad}}{\text{s}})}{(-2.00\,\frac{\text{rad}}{\text{s}^2})} = 5.24\,\text{s}$$

The wheel takes 5.24 s to stop.

(b) We want to find the angular displacement θ during the time of stopping. Since we know that the angular acceleration is constant we can use Eq 9, and it might be simplest to do so. Then we have:

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t = \tfrac{1}{2}(10.5\,\tfrac{\text{rad}}{\text{s}} + 0)(5.24\,\text{s}) = 27.5\,\text{rad} \ .$$

The wheel turns through 27.5 radians in coming to stop.

10-3 RELATING THE LINEAR AND ANGULAR VARIABLES

As we wrote in Eq. 1, when a rotating object has an angular displacement $\Delta\theta$, then a point on the object at a radius r travels a distance $s = r\theta$. This is a relation between the angular motion of the point and the "linear" motion of the point (though here "linear" is a bit of a misnomer because the point has a circular path). The distance of the point from the axis does not change, so taking the time derivative of this relation give the instantaneous speed of the particle as:

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{10}$$

which we similarly call the point's linear speed (or, tangential speed) to distinguish it from the angular speed. Note, all points on the rotating object have the same angular speed but their linear speeds depend on their distances from the axis.

Similarly, the time derivative of the Eq. 10 gives the **linear acceleration** of the point:

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{11}$$

Here it is essential to distinguish the **tangential** acceleration from the **centripetal** acceleration that we recall from our study of uniform circular motion. It is still true that a point on the wheel at radius *r* will have a centripetal acceleration given by:

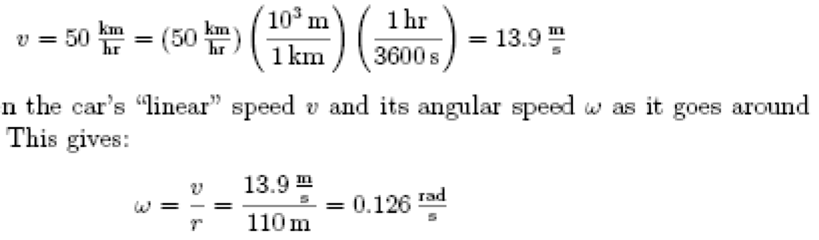
$$a_c = \frac{v_t^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{12}$$

These two components specify the acceleration vector of a point on a rotating object. (Of course, if α is zero, then $a_t = 0$ and there is only a centripetal component.)

Example: What is the angular speed of a car traveling at 50 km/h and rounding a circular turn of radius 110 m?

Answer:

To work consistently in SI units, convert the speed of the car:



$$v = 50\,\frac{\text{km}}{\text{hr}} = \left(50\,\frac{\text{km}}{\text{hr}}\right)\left(\frac{10^3\,\text{m}}{1\,\text{km}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right) = 13.9\,\frac{\text{m}}{\text{s}}$$

The relation between the car's "linear" speed v and its angular speed ω as it goes around the track is $v = r\omega$. This gives:

$$\omega = \frac{v}{r} = \frac{13.9\,\frac{\text{m}}{\text{s}}}{110\,\text{m}} = 0.126\,\frac{\text{rad}}{\text{s}}$$

Example: A disk, of radius 6.0 cm, is free to rotate at a constant rate of 1200 rpm about its axis. Find:

a- the radial acceleration
b- the tangential acceleration.

Answer:

a-

$$\omega = 1200 \text{ rpm } = 1200 \times \frac{2\pi}{60} = 125.7 \text{ rad/s}$$

$$\therefore a_r = \frac{v^2}{R} = R\omega^2 = (0.06)\times(125.7)^2 = \underline{948 \text{ m/s}^2}$$

b- $a_t = 0$; since ω is constant.

Example: An astronaut is being tested in a centrifuge. The centrifuge has a radius of 10 m and, in starting, rotates according to $\theta(t) = 0.30\, t^2$, where t in seconds gives θ in radians. When t = 5.0 s, what are the astronaut's

(a) Angular velocity,
(b) Linear speed,
(c) Tangential acceleration (magnitude only) and
(d) Radial acceleration (magnitude only)?

Answer:

(a) We are given θ as a function of time. We get the angular velocity from its definition,

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(0.30t^2) = 0.60t$$

where we mean that when t is in seconds, ω is given in $\frac{\mathrm{rad}}{\mathrm{s}}$. When $t = 5.0\,\mathrm{s}$ this is

$$\omega = (0.60)(5.0)\,\frac{\mathrm{m}}{\mathrm{s}} = 3.0\,\frac{\mathrm{rad}}{\mathrm{s}}$$

(b) The linear speed of the astronaut is found from Eq. 10 (the linear or tangential speed of a mass point):

$$v = R\omega = (10\,\mathrm{m})(0.60t) = 6.0t$$

where we mean that when t is given in seconds, v is given in $\frac{\mathrm{m}}{\mathrm{s}}$. When $t = 5.0\,\mathrm{s}$ this is

$$v = (6.0)(5.0)\,\frac{\mathrm{m}}{\mathrm{s}} = 30.0\,\frac{\mathrm{m}}{\mathrm{s}}$$

(c) The (magnitude of the) tangential acceleration of a mass point is given by Eq. .11. We will need the angular acceleration α, which is

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.60t) = 0.60$$

so that

$$a_T = R\alpha = (10\,\mathrm{m})(0.60) = 6.0\,\frac{\mathrm{m}}{\mathrm{s}^2}\ .$$

Here, since a_T is constant we have written in the appropriate units, which are $\frac{\mathrm{m}}{\mathrm{s}^2}$. (Since a_T is constant, the answer is the same at $t = 5.0\,\mathrm{s}$ as at any other time.)

(d) The radial acceleration of the astronaut is the centripetal acceleration. From Eq. .12 we can get the magnitude of a_c from:

$$a_c = R\omega^2 = (10\,\mathrm{m})(0.60t)^2 = 3.6t^2$$

where we mean that if t is given in seconds, a_c is given in $\frac{\mathrm{m}}{\mathrm{s}^2}$. When $t = 5.0\,\mathrm{s}$ this is

$$a_c = (3.6)(5.0)^2\,\frac{\mathrm{m}}{\mathrm{s}^2} = 90\,\frac{\mathrm{m}}{\mathrm{s}^2}$$

Extra Problems

Q1: A disk—a horizontal rotating platform—of radius *r* is initially at rest, and then begins to accelerate constantly until it has reached an angular velocity ω after 2 complete revolutions. What is the angular acceleration of the disk during this time? Ans: $\omega^2/8\pi$

Answer: Given the quantities: $\omega_o = 0, \theta_o = 0, \theta_f = 2\times 2\pi,$ then, the angular acceleration of the disk can be determined by using rotational kinematics:

$$\omega^2 = \omega_0{}^2 + 2\alpha\theta$$

$$\alpha = \frac{\omega^2 - \omega_0{}^2}{2\theta} = \frac{\omega^2}{2(2 \bullet 2\pi)} = \frac{\omega^2}{8\pi}$$

Q2: A rotating wheel moves uniformly from rest to an angular speed of 0.16 rev/s in 33 s.

a) Find its angular acceleration in rad/s^2.

b) Would doubling the angular acceleration during the given period have doubled final angular speed?

Answer: Given the quantities: $\omega_o = 0, \theta_o = 0, \omega_f = 0.16\times 2\pi$ rad/s, and t = 33 s, then, the angular acceleration of the disk can be determined by

a) Using the kinematic equation:

$$\omega = \omega_0 + \alpha t$$

From rest to an angular speed of 0.16 rev/s in 33s we should have:

$$\alpha = \frac{\Delta\omega}{t} = \frac{\omega - \omega_0}{t} = \frac{0.16 \times 2\pi}{33} = 0.030 \text{ rad/s}^2$$

a) For double the angular acceleration we should have:

$$2\alpha = 2\frac{\Delta\omega}{t} = \frac{2\Delta\omega}{t}$$

The angular speed will be doubled as well

Q3: A racing car travels on a circular track of radius 275 m. Suppose the car moves with a constant linear speed of 51.5 m/s.
a) Find its angular speed.
b) Find the magnitude and direction of its acceleration.

Answer: Given that $r = 275$ m, $v_t = v =$ 51.5 m/s

a) Angular and linear (tangential) speed are always related through : $v = v_t = r\omega$

$$\omega = \frac{v}{r} = \frac{51.5}{275} = 0.19 \text{ rad/s}$$

b) With a constant linear speed the acceleration is radial $(a = a_r = \frac{v^2}{r} = v_t = r\omega$ as $a_t = \frac{dv}{dr} = 0)$

$$a = \frac{v^2}{r} = \frac{51.5^2}{275} = 9.645 \text{ m/s}^2$$

Q4: A wheel 1.65 m in diameter lies in a vertical plane and rotates about its central axis with a constant angular acceleration of 3.70 rad/s^2. The wheel starts at rest at t = 0, and the radius vector of a certain point P on the rim makes an angle of 57.3^o with the horizontal at this time. At $t = 2.00$ s, find the following:
a) the angular speed of the wheel.
b) the tangential speed of the point P.
c) the total acceleration of the point P.
d) the angular position of the point P.

Answer: Given the quantities: $\omega_o = 0$, $\theta_o = 57.3^\circ \times \frac{\pi}{180}$ rad , $\alpha = 3.70$ rad/s^2, at t = 2 s, then:

a) The wheel started at rest., therefore:

$$\omega = \omega_0 + \alpha t = \alpha t = 3.70 \times 2 = 7.40 \text{ rad/s}$$

b) The tangential speed of point P located on the rim:

$$v = r\omega = \frac{1.65}{2} \times 7.40 = 6.11 \text{ m/s}$$

c) To calculate the total acceleration of the point P, we need to calculate both the radial and tangential components

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha = \frac{1.65}{2} \times 3.70 = 3.05 \text{ m/s}^2$$

$$a_r = \frac{v^2}{r} = \frac{6.11^2}{\frac{1.65}{2}} = 45.18 \text{ m/s}^2$$

And finally:

$$a = \sqrt{a_t^2 + a_n^2} = 45.28 \text{ m/s}^2$$

Its direction β with respect to the radius to P can be evaluated from

$$\tan\beta = \frac{a_t}{a_n} = \frac{3.05}{45.18}$$

i.e. $\beta = 3.86^{\circ}$

d)

$$\begin{aligned}\theta &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 = 57.3 \times \frac{\pi}{180} + 0 + \frac{1}{2} \times 3.70 \times 2^2 \\ &= 8.40 \text{ rad}\end{aligned}$$

Q: The angular position of a point on the rim of a rotating wheel of radius R is given by:

$$\theta\,(t) = 6.0\,t + 3.0\,t^2 - 2.0\,t^3,$$

where θ is in radians and t is in seconds. What is the average angular acceleration for a point at R/2 for the time interval between t = 0 and t = 5 s? – 24 rad/s^2

Answer

$\theta\,(t) = 6.0\,t + 3.0\,t^2 - 2.0\,t^3 \quad \Rightarrow \quad \omega\,(t) = 6.0 + 6.0\,t - 6.0\,t^2$

$\omega\,(0) = 6.0$, $\omega\,(5) = -114$

$$\Rightarrow \quad \bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{-114-6}{5-0} = -24$$

Q: A uniform disk starts from rest and rotates, about fixed central axis, with a constant angular acceleration. It reaches an angular velocity of 13.7 rad/s when it has completed 5.00 revolutions. What is the angular velocity when it has completed 9.00 revolutions? 18.4 rad/s

Answer: First calculate the acceleration

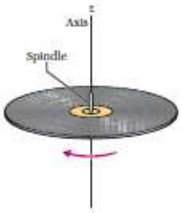
$$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta\theta} = \frac{(13.7)^2 - 0}{2\times 5(2\pi)} = 2.987 \quad \text{rad/s}^2$$

Second $\omega(9 \text{ revolutions}) = \sqrt{\omega_i^2 + 2\alpha\Delta\theta} = \sqrt{0 + 2\times 2.987\times 9\times 2\pi} = 18.38 \text{ rad/s}$

Q: A phonograph turntable rotating at 33.3 rev/min slows down and stops in 30 s after the motor is turned off.

(a) Find its (uniform) angular acceleration in units of rev/min^2.

(b) How many revolutions did it make in this time?



Answer:

(a) Here we are given the initial angular velocity of the turntable and its final angular velocity (namely zero, when it stops) and the time interval between them. We can use Eq. 6 to find α, which we are told is constant. We have:

$$\omega = \omega_o + \alpha t \quad \Rightarrow \quad \alpha = \frac{(\omega - \omega_o)}{t}$$

We don’t need to convert the units of the data to radians and seconds; if we watch our units, we can use revolutions and minutes. Noting that the time for the turntable to stop is t = 30 s = 0.50min, and with ω_o = 33.3 rev/min and ω = 0 we find:

$$\alpha = \frac{0 - 33.3\,\frac{\text{rev}}{\text{min}}}{0.50\,\text{min}} = -66.7\,\frac{\text{rev}}{\text{min}^2}$$

The angular acceleration of the turntable during the time of stopping was −66.7 rev/min^2. (The minus sign indicates a deceleration, that is, an angular acceleration opposite to the sense of the angular velocity.)

(b) Here we want to find the value of θ at t = 0.50 min. To get this, we can use either Eq. 7 or Eq. 9. With $\theta_o = 0$, Eq. 9 gives us:

$$\begin{aligned} \theta &= \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \\ &= \tfrac{1}{2}(33.3\,\tfrac{\text{rev}}{\text{min}} + 0)(0.50\,\text{min}) \\ &= 8.33\,\text{rev} \end{aligned}$$

The turntable makes 8.33 revolutions as it slows to stop.

Q: A disk, initially rotating at 120 rad/s, is slowed down with a constant angular acceleration of magnitude 4.0 rad/s^2.

(a) How much time elapses before the disk stops?

(b) Through what angle does the disk rotate in coming to rest?

Answer:

(a) We are given the initial angular velocity of the disk, ω_o = 120 rad/s. (We let the positive sense of rotation be the same as that of the initial motion.) We are given the magnitude of the disk’s angular acceleration as it slows, but then we must write

$$\alpha = -4.0\,\frac{\text{rad}}{\text{s}^2}\ .$$

The final angular velocity (when the disk has stopped!) is ω = 0. Then from Eq. 6 we can solve for the time t:

$$\omega = \omega_o + \alpha t \quad \Rightarrow \quad t = \frac{(\omega - \omega_o)}{\alpha}$$

and we get:

$$t = \frac{(0 - 120\,\frac{\text{rad}}{\text{s}})}{(-4.0\,\frac{\text{rad}}{\text{s}^2})} = 30.0\,\text{s}$$

(b) We’ll let the initial angle be $\omega_o = 0$. We can now use any of the constant–α equations containing θ to solve for it; let’s choose Eq. 8, which gives us:

$$\omega^2 = \omega_o^2 + 2\alpha(\theta) \quad \Rightarrow \quad \theta = \frac{(\omega^2 - \omega_o^2)}{2\alpha}$$

and we get:

$$\theta = \frac{(0^2 - (120\,\frac{\text{rad}}{\text{s}})^2)}{2(-4.0\,\frac{\text{rad}}{\text{s}^2})} = 1800\,\text{rad}$$

The disk turns through an angle of 1800 radians before coming to rest.

Q: A wheel, starting from rest, rotates with a constant angular acceleration of 2.00 rad/s^2. During a certain 3.00 s interval, it turns through 90.0 rad.

(a) How long had the wheel been turning before the start of the 3.00 s interval?
(b) What was the angular velocity of the wheel at the start of the 3.00 s interval?

Answer:

(a)We are told that sometime after the wheel starts from rest we measure the angular displacement for some 3.00 s interval and it is 9.00 rad. Suppose that we start measuring time at the beginning of this interval; since this time measurement isn't from the beginning of the wheel's motion, we'll call it t_o. Now, with the usual choice $\theta_o = 0$ we know that at $t_o = 3.00$ s we have θ = 90.0 rad. Also α = 2.00 rad/s^2. Using Eq. 7 to get:

$$90.0\,\mathrm{rad} = \omega_0(3.00\,\mathrm{s}) + \tfrac{1}{2}(2.00\,\tfrac{\mathrm{rad}}{\mathrm{s}^2})(3.00\,\mathrm{s})^2$$

which we can use to solve for ω_o:

$$\omega_0(3.00\,\mathrm{s}) = 90.0\,\mathrm{rad} - \tfrac{1}{2}(2.00\,\tfrac{\mathrm{rad}}{\mathrm{s}^2})(3.00\,\mathrm{s})^2 = 81.0\,\mathrm{rad}$$

so that

$$\omega_0 = 27.0\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

(Looking ahead, we can see that we've already answered part (b)!)

Now suppose we measure time from the beginning of the wheel's motion with the variable t. We want to find the length of time required for ω to get up to the value 27.0 rad/s. For this period the initial angular velocity is ω_o = 0 and the final angular velocity is 27.0 rad/s. Since we have α we can use Eq. 6 to get t:

$$\omega = \omega_o + \alpha t \quad \Rightarrow \quad t = \frac{(\omega - \omega_o)}{\alpha}$$

which gives

$$t = \frac{(27.0\,\frac{\mathrm{rad}}{\mathrm{s}} - 0\,\frac{\mathrm{rad}}{\mathrm{s}})}{2.00\,\frac{\mathrm{rad}}{\mathrm{s}^2}} = 13.5\,\mathrm{s}$$

This tells us that the wheel had been turning for 13.5 s before the start of the 3.00 s interval.
(b) In part (a) we found that at the beginning of the 3.00 s interval the angular velocity was 27.0 rad/s.

Chapter 10
Rotational motion

Summary of last lecture:

motion

Classification	displacemet	velocity	acceleration
Linear	$\vec{s}$	$\vec{v}=\frac{d\,\vec{s}}{dt}$	$\vec{a}=\frac{d\,\vec{v}}{dt}$
Rotation	θ	ω	α

Important Terms

- **Moment of inertia (rotational inertia)** "I" in kg.m^2: a quantity expressing a body's tendency to resist angular acceleration. For a point mass the moment of inertia is just the mass times the square of perpendicular distance to the rotation axis, I = m r^2. That point mass relationship becomes the basis for all other moments of inertia. Total moment of inertial is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation
- **Rotational kinetic energy** "$K_{rot}=\frac{1}{2}I\omega^2$"
- **Parallel axis theorem:** can be used to determine the mass moment of inertia of a rigid body about any axis, given the body's moment of inertia about a parallel axis through the object's center of gravity and the perpendicular distance between the axes.

Equations and Symbols

$\theta=\frac{s}{r}$ $\omega=\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt}$, $\alpha=\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}=\frac{d\omega}{dt}$ $v_t=\frac{ds}{dt}=r\frac{d\theta}{dt}=r\omega$ $a_t=\frac{dv_t}{dt}=r\frac{d\omega}{dt}=r\alpha$ $a_c=\frac{v_t^2}{r}=\frac{(r\omega)^2}{r}=r\omega^2$ $I=\sum_i m_i r_i^2$ $K_{rot}=\frac{1}{2}I\,\omega^2$ $I=I_{CM}+MD^2$	s = length of arc r = radius v = velocity θ = angle in radians ω = Angular velocity α = angular acceleration v_t = Linear (tangential) velocity a_t = tangential acceleration a_c = Radial (Centripetal) acceleration I = moment of inertia (rotational inertia) m_i = mass of particles *i*. M = total mass D = distance

Basic Requirements:

1. Master the calculation of the moment of inertia of a system of particles.
2. Use the **Parallel axis theorem** to calculate the moment of inertia of a system of particles.

10-4 KINETIC ENERGY OF ROTATION

Because a rotating object is made of many mass points in motion, it has kinetic energy; but since each mass point has a different linear speed $\left(v = r\omega\right)$ our formula from translational particle motion, $K = \frac{1}{2} m\, v^2$ no longer applies. If we label the mass points of the rotating object as m_i , having individual (different!) linear speeds v_i , then the total kinetic energy of the rotating object is

$$K_{rot} = \sum_i \frac{1}{2} m_i v_i^2$$

If r_i is the distance of the ith mass point form the axis, then $v_i = r_i \omega$ and we then have:

$$K_{rot} = \sum_i \frac{1}{2} m_i v_i^{\,2} = \frac{1}{2} \sum_i m_i \left(r_i \omega\right)^2 = \frac{1}{2}\left(\sum_i m_i r_i^{\,2}\right)\omega^2$$

The sum $\sum_i m_i r_i^{\,2}$ is called the **moment of inertia** for the rotating object (which we discuss further in the next section), and usually denoted I . (It is also called the **rotational inertia** in some books.) It has units of kg · m^2 in the SI system. I of a body is a measure of the rotational inertia of the body. With this simplification, our last equation becomes

$$K_{rot} = \frac{1}{2} I\, \omega^2 \tag{13}$$

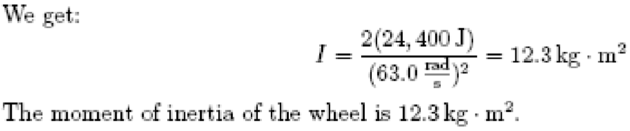
Example: Calculate the rotational inertia of a wheel that has a kinetic energy of 24, 400 J when rotating at $\omega = 602\ \text{revs/min}$.

Answer:

First, find the angular speed of the wheel in rad/s: $\omega = 602\ \text{revs/min} = 63.0\ \text{rad/s}$.

Finally, we have

$$K_{rot} = \frac{1}{2} I \omega^2 \quad \Rightarrow \quad I = \frac{2K_{rot}}{\omega^2}$$



We get:

$$I = \frac{2(24,400\ \text{J})}{(63.0\ \frac{\text{rad}}{\text{s}})^2} = 12.3\ \text{kg} \cdot \text{m}^2$$

The moment of inertia of the wheel is 12.3 kg · m^2.

10-5 CALCULATING THE ROTATIONAL INERTIA

For a rotating object composed of many mass points, the moment of inertia I is given by

$$I = \sum_i m_i r_i^2 \qquad (14)$$

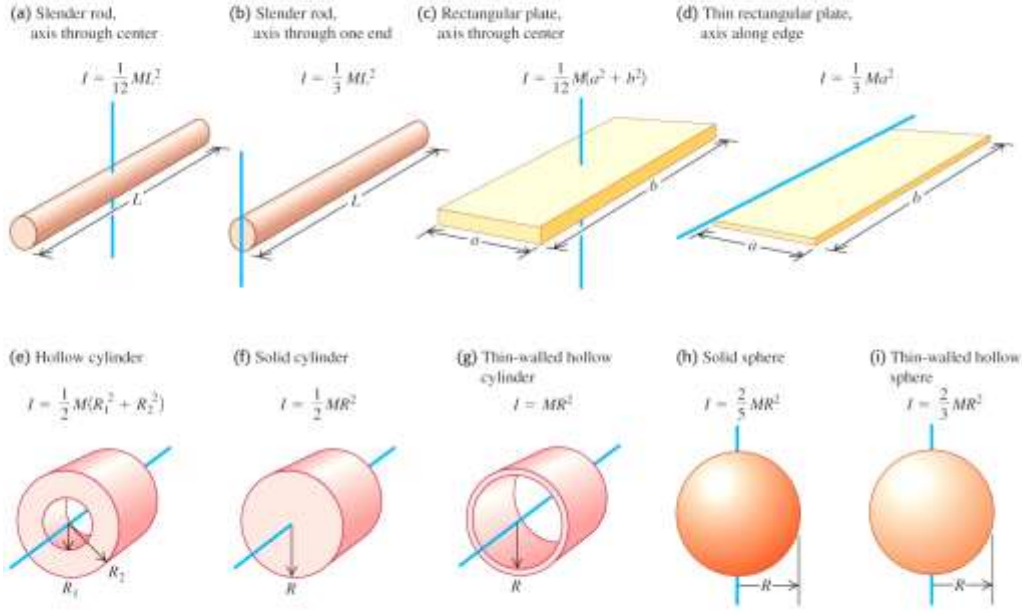
I has units of kg $\cdot$m^2 in the SI system, and as we use it in elementary physics, it is a scalar (i.e. a single number which in fact is always positive). More frequently we deal with a rotating object which is a continuous distribution of mass, and for this case we have the more general expression (not required in our course)

$$I = \int r^2 dm \qquad (15)$$

Here, the integral is performed over the volume of the object and at each point we evaluate r^2, where r is the distance measured perpendicularly from the rotation axis.
The evaluation of this integral for several cases of interest is a common exercise in multi-variable calculus. In most of our problems we will only be using a few basic geometrical shapes, and the moments of inertia for these are given in the Appendix.

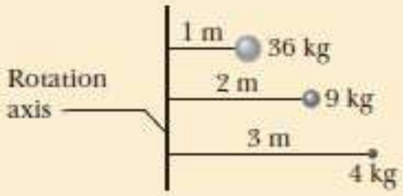
Appendix:

Moments of Inertia for some shapes *M* is the total mass, *a*, *b*, and *L* are lengths, and *R* is the radius.

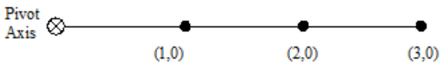


Checkpoint 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



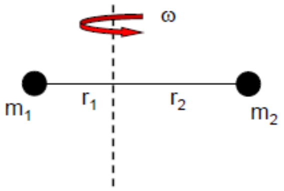
Example: As shown in the figure, three masses, of 1.5 kg each, are fastened at fixed position to a very light rod pivoted at one end. Find the moment of inertia for the rotation axes shown



Answer: Apply the equation

$$I=\sum_i^3 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = m(r_1^2+r_2^2+r_3^2) = 1.5(1^2+2^2+3^2) = 21 \text{ kg.m}^2$$

Example: The figure shows a rigid body consists of two particles attached to a rod of negligible mass. The masses are m$_1$ = 2.00 kg and m$_2$ = 1.00 kg and they are separated by a distance r = r$_1$ + r$_2$.



(a) Find the moment of inertia of the body. Assume r$_1$ = 0.33 m and r$_2$ = 0.67 m are the distance between m$_1$ and the rotation axis and m$_2$ and the rotation axis (the dashed, vertical line) respectively.

(b) What is the moment of inertia if the axis is moved so that is passes through m$_1$?

(c) What is your comment on the two calculated values? Which one will be easy to rotate?

Answer:

(a) Apply the formula (14) of the moment of inertia, we can have

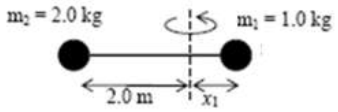
$$I=\sum_i^2 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 = 2.0(0.33^2)+1.0(0.67^2) = 0.67 \text{ kg.m}^2$$

(b)

$$I=\sum_i^2 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 = 2.0(0.00^2)+1.0(1.00^2) = 1.00 \text{ kg.m}^2$$

(c)???

Example: A rigid body consists of two particles attached to a rod of negligible mass. The rotational inertia of the system about the axis shown in Figure is 10 kg m^2. What is x_1?



Answer:

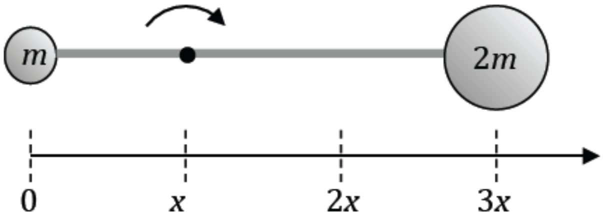
$$I=\sum_i^2 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 \quad \Rightarrow \quad 10 = 1\times x_1^2 + 2\times 2^2 \quad \Rightarrow x_1 = \underline{1.41 \text{ m}}$$

Example: A hoop rolls without sliding on a horizontal floor. The ratio of its translational kinetic energy to its rotational kinetic energy (about its central axis) is

Answer: The ratio is

$$\frac{K_{edge}}{K_{center}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}I\omega^2} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}\left(mR^2\right)\left(v/R\right)^2} = 1$$

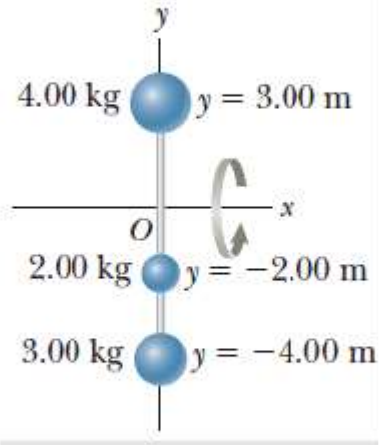
Example: A solid sphere of mass *m* is fastened to another sphere of mass 2*m* by a thin rod with a length of 3*x*. The spheres have negligible size, and the rod has negligible mass. What is the moment of inertia of the system of spheres as the rod is rotated about the point located at position *x*, as shown?



Answer: Moment of inertia for a system of discrete masses is calculated as follows:

$$I=\sum_{i}^{2} m_i r_i^2 = m(x)^2 + 2m(2x)^2 = 9mx^2$$

Q3: Rigid rods of negligible mass lying along the y axis connect three particles. The system rotates about the x axis with an angular speed of 2.10 rad/s.



a) Find the moment of inertia about the x axis.

b) Find the total rotational kinetic energy evaluated from $K_{rot}=\frac{1}{2}I\omega^2$

c) Find the tangential speed of each particle.

d) Find the total kinetic energy evaluated from $K_{rot}=\sum_i \frac{1}{2}m_i v_i^2$

e) Your comment for b and d.

Answer:

a) $$I = \sum m_i r_i^2 = 4\times 3^2 + 2\times 2^2 + 3\times 4^2 = 92 \text{ kg.m}^2$$

b) $$Ki = \frac{1}{2}I\omega^2 = 0.5\times 92\times 2.10^2 = 202.86 \text{ J}$$

c) Different linear speeds for different radius. However, all particles are rotating at same angular speed: $v_i = r_i\omega$

Mass 1: $v_1 = r_1\omega = 3 \times 2.10 = 6.30 \text{ m/s}$

Mass 2: $v_2 = r_2\omega = 2 \times 2.10 = 4.20 \text{ m/s}$

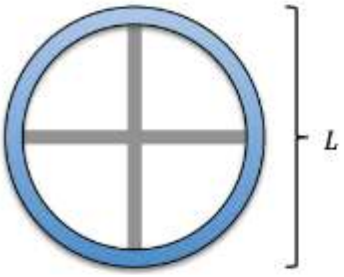
Mass 3: $v_3 = r_3\omega = 4 \times 2.10 = 8.40 \text{ m/s}$

d) The total kinetic energy is:

$$\begin{aligned} Ki &= \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2}(4 \times 6.30^2 + 2 \times 4.20^2 + 3 \times 8.40^2) \\ &= 202.86 \text{ J} \end{aligned}$$

e) Both expressions lead to the same value.

Q4: A pair of long, thin, rods, each of length *L* and mass *M*, are connected to a hoop of mass *M* and radius *L*/2 to form a 4-spoked wheel as shown in the figure. Express all answers in terms of the given variables and fundamental constants. Calculate the moment of inertia for the entire spoked-wheel assembly for an axis of rotation through the center of the assembly and perpendicular to the plane of the wheel.



Answer:

The moment of inertia for the spoked wheel is simply the sum of the individual moments of inertia of its three components: the two long thin rods and the hoop around the outside:

$$I_{total} = I_{rod} + I_{rod} + I_{hoop}$$

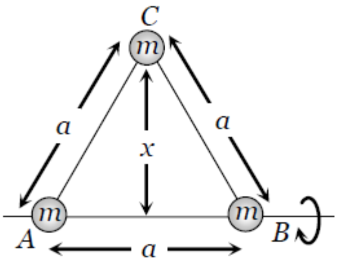
$$I_{total} = \frac{1}{12}ML^2 + \frac{1}{12}ML^2 + MR^2$$

$$I_{total} = \frac{1}{12}ML^2 + \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{5}{12}ML^2$$

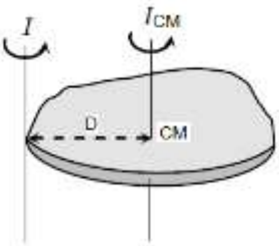
Example: Three point masses, i.e. they have no moment of inertia, each of mass *m* are placed at the corners of an equilateral triangle of side *a*. Calculate the moment of inertia of this system about an axis passing along one side of the triangle. Choose the side *AB*.

Answer: The moment of inertia of system about *AB* side of triangle

$$\therefore I_{\text{system}} = I_A + I_B + I_C = 0 + 0 + mx^2 = mx^2 = m\left(\frac{a\sqrt{3}}{2}\right)^2 = \frac{3}{4}ma^2$$



Parallel Axis Theorem



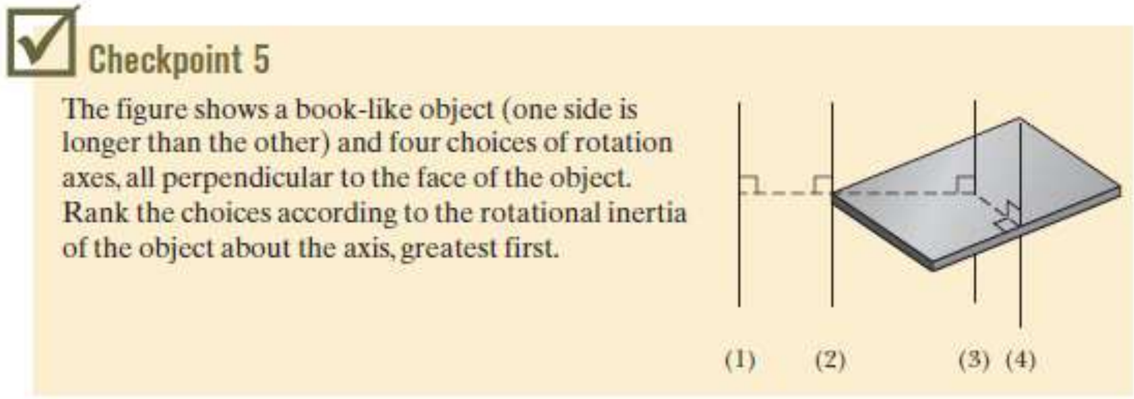
Moment of inertia of a body about a given axis *I* is equal to the sum of moment of inertia of the body about an axis parallel to given axis and passing through center of mass of the body is I_{CM} and MD^2, where M is the mass of the body and D is the perpendicular distance between the two axes. This situation is shown in the figure. In symbol, the new moment of inertia of the object about the new axis will have a new value *I*, given by

$$I = I_{CM} + MD^2 \qquad (16)$$

Eq. 16 is known as the **Parallel Axis Theorem** and is sometimes handy for computing moments of inertia if we already have a listing for a moment of inertia through the object's center of mass.

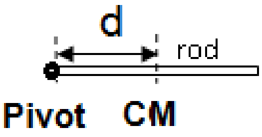
Relates I_{CM} (axis through center-of-mass) to I w.r.t. some other axis: $I = I_{CM} + MD^2$

(See proof in text.)



Checkpoint 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.

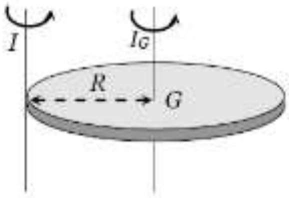


Example: Calculate the moment of inertia for a rod about its end point.

Answer: Rod of length L, mass M

$$\mathrm{I_{CM}} = \tfrac{1}{12}\mathrm{M\,R^2} \;,\; \mathrm{d = L/2} \Rightarrow$$

$$\mathrm{I_{Pivot}} = \mathrm{I_{CM}} + \mathrm{M\,d^2} = \tfrac{1}{12}\mathrm{M\,L^2} + \tfrac{1}{4}\mathrm{M\,L^2} = \tfrac{1}{3}\mathrm{M\,L^2}$$

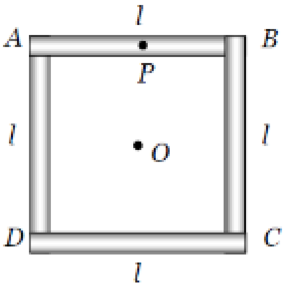


Example: Moment of inertia of a disc about an axis through its center of mass and perpendicular to its plane is $I_G = \frac{1}{2}MR^2$. Calculate the moment of inertia about an axis through its tangent perpendicular to the plane

Answer: The moment of inertia about an axis through its tangent perpendicular to the plane is given by:

$$\mathrm{I} = \mathrm{I_G} + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Example: Four thin rods of same mass *M* and same length *l*, form a square as shown in figure. Calculate the moment of inertia of this system about an axis through center *O* and perpendicular to its plane.



Answer: $\mathrm{I_{CM}(rod)} = \frac{1}{22}\mathrm{M}\,\ell^2$

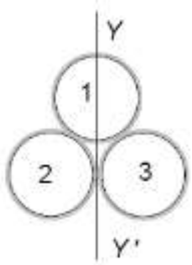
M.I. of rod *AB* about point $P = \frac{1}{12}\mathrm{M}\,\ell^2$

M.I. of rod *AB* about point $O = \frac{1}{12}\mathrm{M}\,\ell^2 + \mathrm{M}\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\mathrm{M}\,\ell^2$

[by the theorem of parallel axis]

and the system consists of 4 rods of similar type so by the symmetry $I_{system} = \frac{4}{3}\mathrm{M}\,\ell^2$

Example: Three rings each of mass m and radius R are arranged as shown in the figure. Calculate the moment of inertia of the system about YY'.

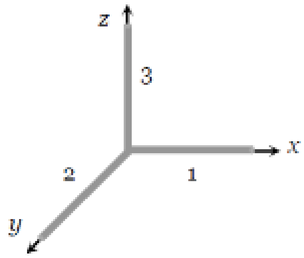


Answer: $\mathrm{I_{CM}(ring)} = \frac{1}{2}\mathrm{mR}^2$

M.I of system about YY' $\therefore I_{\text{system}} = I_1 + I_2 + I_3$, where I_1 = moment of inertia of ring about CM, $I_2 = I_3 =$ M.I. of inertia of ring about a tangent in a plane

$$\therefore I_{\text{system}} = \frac{1}{2}\mathrm{mR}^2 + \left(\frac{1}{2}\mathrm{mR}^2 + \mathrm{mR}^2\right) + \left(\frac{1}{2}\mathrm{mR}^2 + \mathrm{mR}^2\right) = \frac{7}{2}\mathrm{mR}^2$$

Example: Three identical thin rods, each of length L and mass M, are welded perpendicular to one another as shown in the figure. They are placed along *X*, *Y* and *Z*-axes in such a way that one end of each of the rod is at the origin. The moment of inertia of this system about *Z* axis is



Answer: $\mathrm{I_{CM}(rod)} = \frac{1}{12}ML^2$

Moment of inertia of the system about *z*-axis can be find out by calculating the moment of inertia of individual rod about *z*-axis

$I_1 = I_2 = \frac{1}{3}ML^2$ because *z*-axis is the edge of rod 1 and 2 and $I_3 = 0$ because rod in lying on *z*-axis

$$\therefore I_{\text{system}} = I_1 + I_2 + I_3 = \frac{1}{3}ML^2 + \frac{1}{3}ML^2 + 0 = \frac{2}{3}ML^2$$

Example: Find the moment of inertia of a uniform ring of radius R and mass M about an axis 2R from the center of the ring as shown in the Figure.

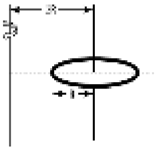
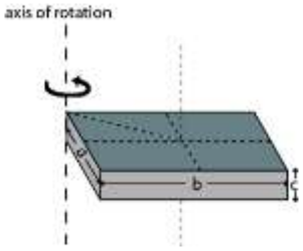


Figure:

Answer:

$$I_o = I_{CM} + Md^2 = MR^2 + M(2R)^2 = 5MR^2$$

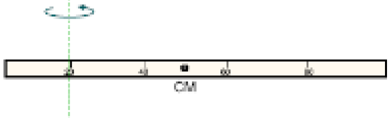
Extra Problems



Q: A uniform **slab** of dimensions: a = 60 cm, b = 80 cm, and c = 2.0 cm (see Figure) has a mass of 6.0 kg. Its rotational inertia about an axis perpendicular to the larger face and passing through one corner of the slab is:

Answer: Use the equation: $I = I_{CM} + MD^2$

$$\Rightarrow \quad I = \frac{M}{12}(a^2+b^2) + M\left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right] = \frac{M}{3}\underbrace{(a^2+b^2)}_{1} = \underline{2.0\ \mathrm{kg.m^2}}$$



Q: Calculate the rotational inertia of a meter stick with mass 0.56 kg, about an axis perpendicular to the stick and located at the 20 cm mark.

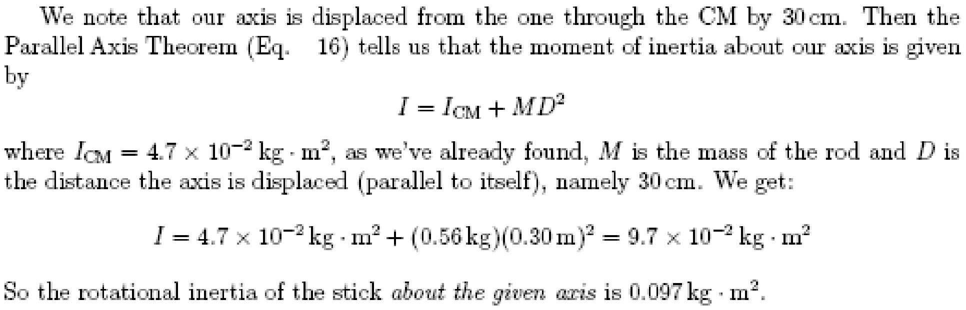
Answer:

A picture of this rotating system is given in the figure. The stick is one meter long (being a meter stick and all that) and we take it to be uniform so that its center of mass is at the 50 cm mark. But the axis of rotation goes through the 20 cm mark.

Now if the axis did pass through the center of mass (perpendicular to the stick), we would know how to find the rotational inertia; from Figure we see that it would be

$$I_{CM} = \frac{1}{12} M L^2 = \frac{1}{12}(0.56)(1.00)^2 = 4.7 \times 10^{-2}\ \mathrm{kg.m^2}$$

The rotational inertia about our axis will not be the same.



We note that our axis is displaced from the one through the CM by 30 cm. Then the Parallel Axis Theorem (Eq. 16) tells us that the moment of inertia about our axis is given by

$$I = I_{CM} + MD^2$$

where $I_{CM} = 4.7 \times 10^{-2}\ \mathrm{kg \cdot m^2}$, as we've already found, M is the mass of the rod and D is the distance the axis is displaced (parallel to itself), namely 30 cm. We get:

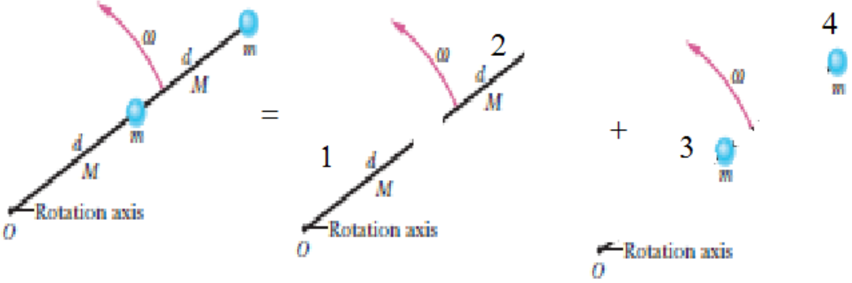
$$I = 4.7 \times 10^{-2}\ \mathrm{kg \cdot m^2} + (0.56\ \mathrm{kg})(0.30\ \mathrm{m})^2 = 9.7 \times 10^{-2}\ \mathrm{kg \cdot m^2}$$

So the rotational inertia of the stick *about the given axis* is $0.097\ \mathrm{kg \cdot m^2}$.

Q: In the Figure, two particles, each with mass m 0.85 kg, are fastened to each other, and to a rotation axis at O, by two thin rods, each with length d = 5.6 cm and mass M = 1.2 kg. The combination rotates around the rotation axis with the angular speed ω = 0.30 rad/s. Measured about O, what are the combination's

(a) rotational inertia and

(b) kinetic energy?



Answer:

The particles are treated "point-like" in the sense that Eq. 10-33 yields their rotational inertia, and the rotational inertia for the rods is figured using Table 10-2(e) and the parallel-axis theorem (Eq. 10-36).

(a) With subscript 1 standing for the rod nearest the axis and 4 for the particle farthest from it, we have

$$I_O = I_{1,O} + I_{2,O} + I_{3,O} + I_{4,O}$$
$$I_{1,O} = \left(\frac{1}{12}Md^2 + M\left(\frac{1}{2}d\right)^2\right),$$
$$I_{2,O} = \left(\frac{1}{12}Md^2 + M\left(\frac{3}{2}d\right)^2\right),$$
$$I_{3,O} = md^2$$
$$I_{4,O} = m(2d)^2.$$

$$I_O = \frac{8}{3}Md^2 + 5md^2 = \frac{8}{3}(1.2\text{ kg})(0.056\text{ m})^2 + 5(0.85\text{ kg})(0.056\text{ m})^2$$
$$= 0.023\text{ kg}\cdot\text{m}^2.$$

(b) Using Eq. 10-34, we have

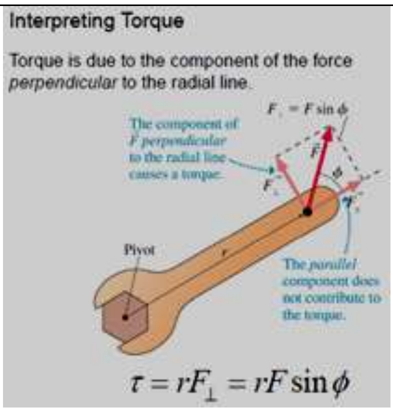
$$K_{rot} = \frac{1}{2}I_O\omega^2 = \frac{1}{2}\left(\frac{8}{3}M + 5m\right)d^2\omega^2 = \left[\frac{4}{3}(1.2\text{ kg}) + \frac{5}{2}(0.85\text{ kg})\right](0.056\text{ m})^2(0.30\text{ rad/s})^2$$
$$= 1.1\times10^{-3}\text{ J}.$$

Rotational motion

Basic Requirements:

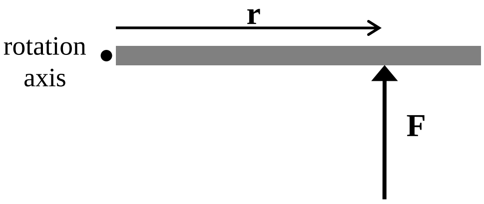
1. Master the calculation of the rotational torque.
2. Calculate the power in case of rotational.

Basic Principles: Why is the handle on a door located far away from the hinge? Why is it easier to loosen a nut using a long wrench? Why are long wheel-base cars more stable than short wheel-base cars?

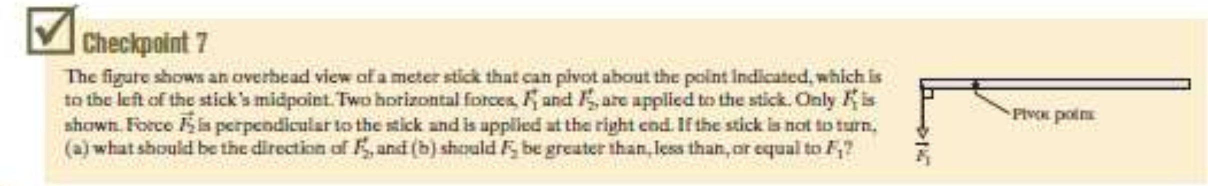
 handle on a door is located far away from the hinge	 loosen a nut using a long wrench

The “ability” of a force to rotate an object about an axis depends on two variables:

1. The magnitude of the *force F*.
2. The *distance r* between the axis of rotation and the point where the force is applied.



Try opening a door by applying the *same* force F at *different* points: r = outer edge, middle, near hinge. You will quickly realize that the resulting motion of the door – the acceleration α – depends on *F and r*. It turns out that the “*turning ability*” of a force is simply the product of F and r. The technical name for this turning ability is ***torque***. **Torque** comes from Latin and means “**to twist**”.



Checkpoint 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than, or equal to F_1?

10-6 Torque "Is a force that causes a rotational acceleration of a rigid body about an axis or motion of a single particle relative to some fixed point".

1- Torque is a vector.
2- Torque is positive when the body rotate counterclockwise (convention)
3- Torque is negative when the body rotate clockwise (convention)

Suppose the force F (whose direction lies in the plane of rotation) is applied at a point r (relative to the rotation axis which is the pivot). Suppose that the (smallest) angle between r and F is ϕ. Then the magnitude of the torque exerted on the object by this force is

$$\left|\vec{\tau}\right| = r\,(F\sin\phi) \qquad (17)$$

By some very simple regrouping, this equation can be written as

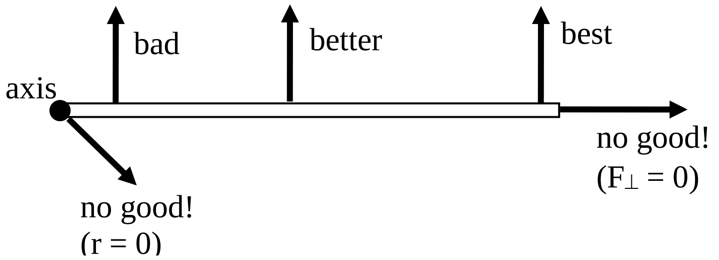
$$\vec{\tau} = r(F\sin\phi) = r\,F_\perp = \underbrace{r_\perp}_{\text{moment arm of } \vec{F}} F = \vec{r}\times\vec{F}$$

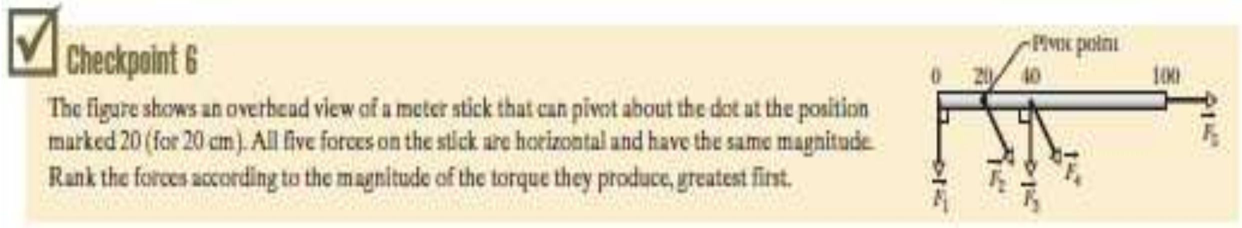
SI unit of torque is **N.m (same as the work); but** Never use Joules as a unit of torque, because Joules is a unit of work.

In Summary:

☞ *Force* causes *linear* acceleration.
☞ *Torque* causes *angular* acceleration.

If you want to easily rotate an object about an axis, you want a large lever arm r and a large perpendicular force F⊥:

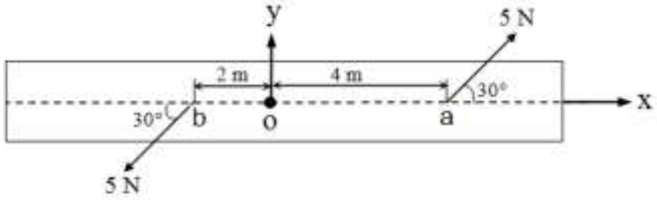




Checkpoint 6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.

Example: Calculate the net torque (magnitude, in N.m, and direction) on a uniform beam shown in the Figure about a point O passing through its center.



Answer:

1) $\tau_{tot} = \tau_a + \tau_b$

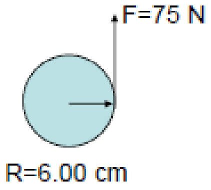
2) $\tau_a = r_a \times F_a = + F_a r_a \sin(30°)\, \hat{k} = (5N)(4m)(0.5)\, \hat{k}$

3) $\tau_b = r_b \times F_b = + F_b r_b \sin(30°)\, \hat{k} = +(5N)(2m)\ (0.5)\, \hat{k}$

4) $\tau_{tot} = (10 + 5)\, \hat{k}\ N \cdot m = 15\, \hat{k}\ N \cdot m$

Example: The pull cord of an engine is wound around a drum of radius 6.00 cm. The cord is pulled with a force of 75.0 N by the engine. What magnitude torque does the cord apply to the drum?

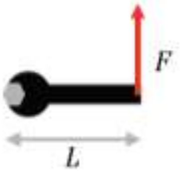
Answer:



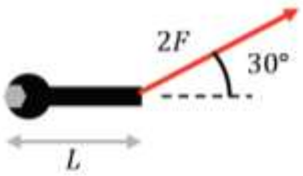
$$\begin{aligned}\tau &= r_{\perp}F \\ &= rF \\ &= (0.06\,\text{m})(75.0\,\text{N}) = 4.5\,\text{Nm}\end{aligned}$$

Q: A series of wrenches of different lengths is used on a hexagonal bolt, as shown below. Which combination of wrench length and Force applies the greatest torque to the bolt?

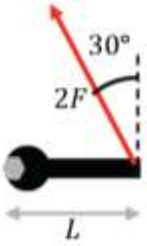
a.



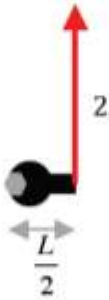
b.



c.



d.

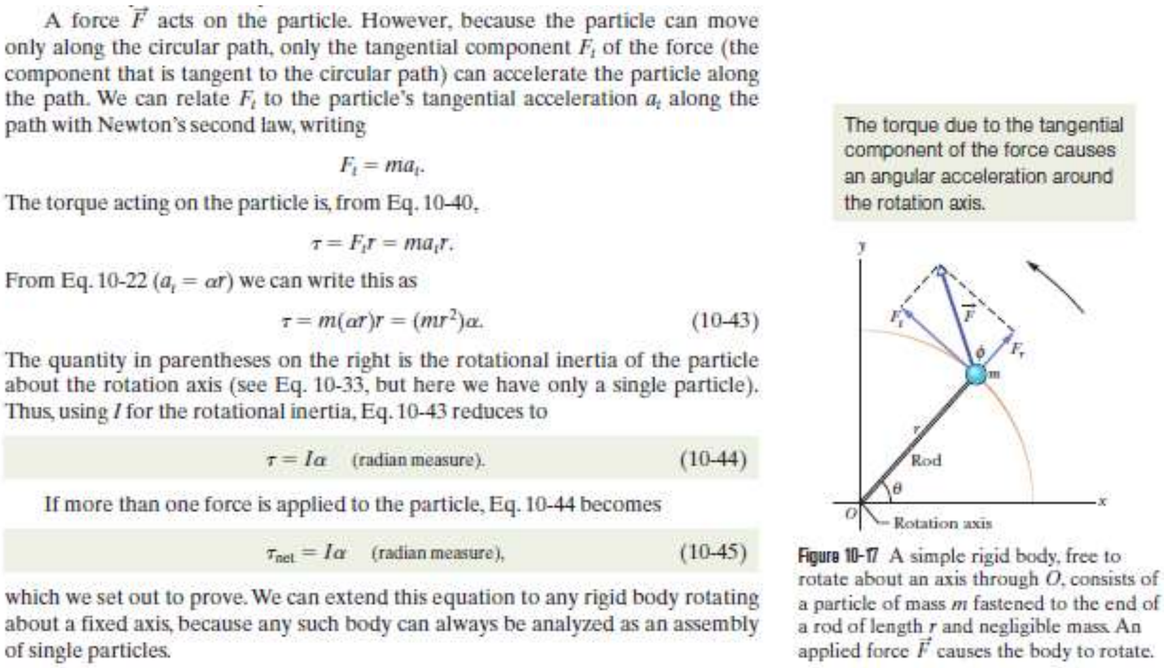


e.

Answer: The correct answer is *c*. Torque, the "turning effect" produced by a force applied to a moment-arm, is calculated according to $\vec{\tau} = \vec{r} \times \vec{F} = r(F \sin\theta)$, where θ is the angle between the vectors **r** and **F**. Here, each combination of wrench length and force produces a net torque of LF except for answer *c*:

$$\vec{\tau} = \vec{r} \times \vec{F} = r(F\cos\theta) = L(2F\cos 30^{\circ}) = \sqrt{3}LF$$

10-7 NEWTON'S SECOND LAW FOR ROTATION



A force $\vec{F}$ acts on the particle. However, because the particle can move only along the circular path, only the tangential component F_t of the force (the component that is tangent to the circular path) can accelerate the particle along the path. We can relate F_t to the particle's tangential acceleration a_t along the path with Newton's second law, writing

$$F_t = ma_t.$$

The torque acting on the particle is, from Eq. 10-40,

$$\tau = F_t r = ma_t r.$$

From Eq. 10-22 ($a_t = \alpha r$) we can write this as

$$\tau = m(\alpha r)r = (mr^2)\alpha. \qquad (10\text{-}43)$$

The quantity in parentheses on the right is the rotational inertia of the particle about the rotation axis (see Eq. 10-33, but here we have only a single particle). Thus, using I for the rotational inertia, Eq. 10-43 reduces to

$$\tau = I\alpha \quad \text{(radian measure).} \qquad (10\text{-}44)$$

If more than one force is applied to the particle, Eq. 10-44 becomes

$$\tau_{net} = I\alpha \quad \text{(radian measure),} \qquad (10\text{-}45)$$

which we set out to prove. We can extend this equation to any rigid body rotating about a fixed axis, because any such body can always be analyzed as an assembly of single particles.

The torque due to the tangential component of the force causes an angular acceleration around the rotation axis.

Figure 10-17 A simple rigid body, free to rotate about an axis through O, consists of a particle of mass m fastened to the end of a rod of length r and negligible mass. An applied force $\vec{F}$ causes the body to rotate.

$\tau = I\alpha$ *is the rotational analogue of* $F = ma$.

- **Newton's Second Law** "Linear Acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass (inertia)." $a \propto F, a \propto 1/m \Rightarrow a = F/m$

- **Newton's Second Law for Rotation** "Angular Acceleration of an object is directly proportional to the net ~~force~~ torque acting on it and inversely proportional to its ~~mass~~ rotational inertia." $\alpha \propto \tau, \alpha \propto 1/I \Rightarrow \alpha = \tau/I$

In Summary: we are going to use the following expressions for the torque

$$\overbrace{I\alpha \qquad (1) \qquad = \qquad \vec{r}\times\vec{F} \qquad (2)}^{\tau}$$

and the equation

$$\alpha = \frac{a}{R} \qquad (3)$$

Examples

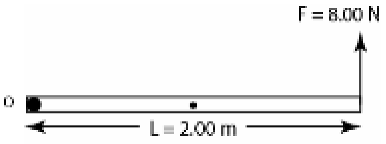
Example: The flywheel of a stationary engine has a moment of inertia 0f 30 kg.m^2. What constant torque is required to accelerate the flywheel to an angular velocity of 900 rpm in 10 seconds, starting from rest?

Answer:

$$\omega_f = 900\,\text{rpm} = 900\times\left(\frac{2\pi}{60}\right) = 94.2\,\text{rad/s}; \qquad \omega_i = 0$$

$$\left|\vec{\tau}\right| = I\,\alpha = I\,\frac{d\omega}{dt} = 30\left(\frac{94.2-0}{10}\right) = \underline{282.7\ \text{N.m}};$$

Example: A uniform thin rod of mass M = 3.0 kg and length L = 2.0 m is pivoted at one end O and acted upon by a force F = 8.0 N at the other end as shown in Figure. Calculate the angular acceleration of the rod at the moment the rod is in the horizontal position as shown in this figure.



Answer: First calculate the M.I. about the pivot O using the PAT:

$$I_0 = I_{CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2; \qquad (1)$$

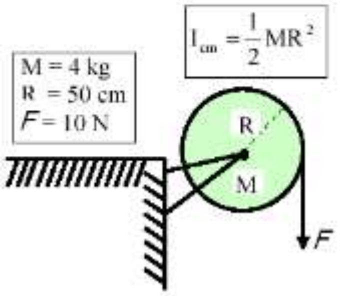
Then, use the torque equations as follows:

$$\overbrace{I_o\alpha = \left(\tfrac{1}{3}ML^2\right)\alpha \quad (2) \qquad\qquad \boldsymbol{r}\times\boldsymbol{F} = Fr_\perp = \text{FL} \quad (3)}^{\tau}$$

Equating (2) = (3) implies

$$\alpha = \frac{\text{FL}}{I_0} = \frac{8\times 2}{\frac{1}{3}(3)2^2} = \underline{4\frac{\text{rad}}{\text{s}}} \quad \underline{\text{Counterclockwise}}$$

Example: A uniform disk of radius 50 cm and mass 4 kg is mounted on a frictionless axle, as shown in Figure. A light cord is wrapped around the rim of the disk and a steady downward pull of 10 N is exerted on the cord. Find the tangential acceleration of a point on the rim of the disk.



Answer: Apply Newton's second law gives:

$$ma = F - T = 0 \quad \Rightarrow \quad F = T \quad (1)$$

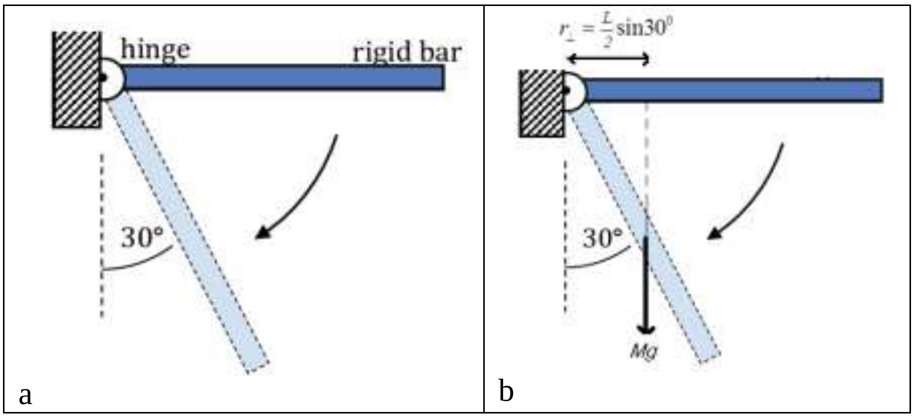
Then, use the torque equations as follows:

$$\overbrace{I_{CM}\alpha = \left(\tfrac{1}{2}MR^2\right)\alpha \quad (2) \qquad\qquad \boldsymbol{r}\times\boldsymbol{F} = Fr_\perp = \text{TR} \quad (3)}^{\tau}$$

Equating (2) = (3) and using $\alpha = \frac{a}{\text{R}}$ implies

$$\tfrac{1}{2}Ma = T \quad \Rightarrow \quad a = \frac{2}{M}T = \frac{2}{4}10 = \underline{5\ \text{m/s}^2} \quad \underline{\text{Clockwise}}$$

Example: A rigid bar with a mass *M* and length *L* is free to rotate about a frictionless hinge at a wall, see figure a. The bar has a moment of inertia $I = 1/3\ ML^2$ about the hinge, and is released from rest when it is in a horizontal position as shown. What is the instantaneous angular acceleration when the bar has swung down so that it makes an angle of 30° to the vertical?



Answer: The bar is accelerating angularly in response to the torque due to the force of gravity acting on the center of mass. Its angular acceleration due to this torque τ at the final position, see figure (b), can be calculate as follows:

$$\overbrace{I\alpha=\left(\frac{1}{3}ML^2\right)\alpha \quad (1) \qquad \boldsymbol{r}\times\boldsymbol{F} = Fr_\perp = Mg\left(\frac{L}{2}\sin30^0\right) \quad (2)}^{\tau}$$

(1) = (2) implies

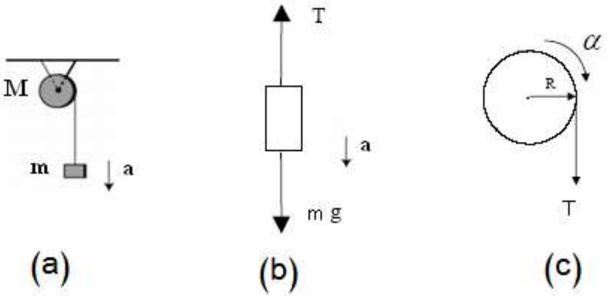
$$\alpha=\frac{3g}{4L} \quad \underline{\text{Clockwise}}$$

Note that:

1- At the horizontal position, we have the maximum torque: $Mg\left(\frac{L}{2}\sin90^0\right)=MgL/2\cdot$

2- At the vertical position, we have the minimum torque: $Mg\left(\frac{L}{2}\sin0^0\right)=0$, i.e. no rotation motion.

Example: For the system in figure (a) with $M = 2.5$ kg, $m = 1.2$ kg, and $R = 0.2$ m, find a, T and α.



Answer: Draw the FBD for the masses, Figures (b) and (c), and then apply Newton's 2nd law as follows:

1- For m: Figure (b), consider the motion is going down:

$$ma = mg - T \qquad (1),$$

2- For M: Figure (c), consider the rotation clockwise is positive:

$$\overbrace{I_{CM}\alpha = \frac{1}{2}MR^2\alpha \quad (2) \qquad\qquad TR \quad (3)}^{\tau}$$

Equating (2) = (3) and using $\alpha = \dfrac{a}{\mathrm{R}}$ implies

$$T = \frac{1}{2}Ma \qquad (4)$$

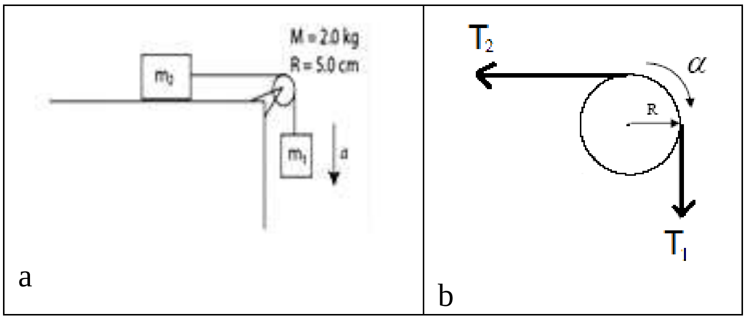
Substitute (4) in (1), we can have

$$ma = mg - \frac{1}{2}Ma \quad \Rightarrow a = \left(\frac{2m}{M+2m}\right)g = \frac{2\times 1.2}{2.5 + 2\times 1.2} = \underline{4.8 \text{ m/s}^2}$$

$$\Rightarrow T = \frac{1}{2}Ma = \underline{6.0 \text{ N}}$$

$$\alpha = \frac{a}{R} = \frac{4.8}{0.2} = \underline{24 \text{ rad/s}^2} \quad \underline{\text{Clockwise}}$$

Example: A mass, m_1 = 5.0 kg, hangs from a string and descends with a linear acceleration "a". The other end is attached to a mass m_2 = 4.0 kg which slides on a frictionless horizontal table. The string goes over a pulley (a uniform disk) of mass M = 2.0 kg and radius R = 5.0 cm (see Figure a). Find the value of a.



Answer: The equations of motion for the two masses are given by:

$$\left.\begin{aligned} m_1 a &= m_1 g - T_1 \quad (1), \\ m_2 a &= T_2 \qquad\quad\; (2) \end{aligned}\right\} \Rightarrow (m_1 + m_2)a = m_1 g + (T_2 - T_1) \quad (3)$$

We are taking the clockwise direction is positive (see figure b, then

$$\overbrace{I_{CM}\alpha = \frac{1}{2}MR^2\alpha \qquad (4) \qquad\qquad\qquad (\mathrm{T}_1\text{-}\mathrm{T}_2)\mathrm{R} \qquad (5)}^{\tau}$$

Equating (4) = (5) and using $\alpha = \frac{a}{\mathrm{R}}$ implies

$$\tfrac{1}{2}Ma = (T_1 - T_2) \qquad (6)$$

Use Eqs. (3) and (6) to solve for a, one finds:

$$a = \frac{m_1 g}{(m_1 + m_2) + M/2} = \underline{4.9\frac{\mathrm{m}}{\mathrm{s}^2}}$$

Example: In Fig. 10-41, block 1 has mass $m_1 = 0.460$ kg, block 2 has mass $m_2 = 0.500$ kg, and the pulley, which is mounted on a horizontal axle with negligible friction, has radius R = 5.00 cm. When released from rest, block 2 falls 75.0 cm in 5.00 s without the cord slipping on the pulley. (a) What is the magnitude of the acceleration of the blocks? What are (b) tension T_2 and (c) tension T_1? (d) What is the magnitude of the pulley's angular acceleration? (e) What is its rotational inertia?

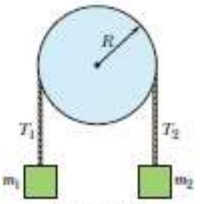


Figure 10-41
Problems 51 and 83.

Answer:

(a) We use constant acceleration kinematics. If down is taken to be positive and a is the acceleration of the heavier block m_2, then its coordinate is given by $y = \frac{1}{2}at^2$, so

$$a = \frac{2y}{t^2} = \frac{2(0.750\ \mathrm{m})}{(5.00\ \mathrm{s})^2} = 6.00 \times 10^{-2}\ \mathrm{m/s^2}.$$

Block 1 has an acceleration of 6.00×10^{-2} m/s^2 upward.

(b) Newton's second law for block 2 is $m_2 g - T_2 = m_2 a$, where m_2 is its mass and T_2 is the tension force on the block. Thus,

$$T_2 = m_2(g - a) = (0.500\ \mathrm{kg})\left(9.8\ \mathrm{m/s^2} - 6.00\times10^{-2}\ \mathrm{m/s^2}\right) = 4.87\ \mathrm{N}.$$

(c) Newton's second law for block 1 is $m_1 g - T_1 = -m_1 a,$ where T_1 is the tension force on the block. Thus,

$$T_1 = m_1(g + a) = (0.460\ \mathrm{kg})\left(9.8\ \mathrm{m/s^2} + 6.00\times10^{-2}\ \mathrm{m/s^2}\right) = 4.54\ \mathrm{N}.$$

(d) Since the cord does not slip on the pulley, the tangential acceleration of a point on the rim of the pulley must be the same as the acceleration of the blocks, so

$$\alpha = \frac{a}{R} = \frac{6.00\times10^{-2}\ \mathrm{m/s^2}}{5.00\times10^{-2}\ \mathrm{m}} = 1.20\ \mathrm{rad/s^2}. \qquad \underline{\text{Clockwise}}$$

(e) The net torque acting on the pulley is $\tau = (T_2 - T_1)R$. Equating this to $I\alpha$ we solve for the rotational inertia:

$$I = \frac{(T_2 - T_1)R}{\alpha} = \frac{(4.87\ \mathrm{N} - 4.54\ \mathrm{N})(5.00\times10^{-2}\ \mathrm{m})}{1.20\ \mathrm{rad/s^2}} = 1.38\times10^{-2}\ \mathrm{kg\cdot m^2}.$$

10-8 WORK AND ROTATIONAL KINETIC ENERGY

In **linear motion**, we knew that

$$\Delta W = \Delta K = K_f - K_i = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2; \qquad W = \int_{x_i}^{x_f} F dx, \quad \text{if } F \text{ is constant}$$

$$\Rightarrow P = \frac{\Delta W}{\Delta t} = Fv$$

Similarly, for **rotational motion**, we can have:

$$W = \Delta K = K_f - K_i = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2;$$

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta = \tau(\theta_f - \theta_i) \quad \text{if } \tau \text{ is constant}$$

$$P = \frac{dW}{dt} = \tau\,\omega \qquad \text{(power, rotation about fixed axis)}$$

Example: A horizontally-mounted disk with moment of inertia I spins about a frictionless axle. At time $t = 0$, the initial angular speed of the disk is ω . A constant torque τ is applied to the disk, causing it to come to stop in time t . How much Power is required to dissipate the wheel's energy during this time?

Answer: Given that: $\omega_f = 0,\ \omega_i = \omega$. The Power required to dissipate the wheel's initial energy is calculated using $P=W/t$, where W is the Work required to change the wheel's kinetic energy from its initial value to 0:

$$P = \frac{W}{\Delta t}, \quad W = \Delta K = K_f - K_i = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 = 0 - \tfrac{1}{2}I\omega^2$$

$$P = \frac{\Delta W}{\Delta t} = \frac{I\omega^2}{2t}$$

Example: The engine delivers 1.20 ×10^5 W to a plane fan at $\omega = 2400$ rev/min $= 251$ rad/s . How much work does the engine do in one revolution?

Answer: Since $s = vt \Rightarrow \theta = \omega t; \quad t = \frac{\theta}{\omega}$, then the periodic time will be: $T = \frac{2\pi}{\omega}$ s $= 0.025$ s .

Consequently,

$$P = \tau\omega = \frac{\Delta W}{\Delta t} \quad \Rightarrow \quad \Delta W = P\Delta T = 1.2\times10^5 \times 0.025 = \underline{3000\text{ J}}$$

Example: A grinding wheel of moment of inertia 0f 0.01 kg.m^2 is brought to rest, in 10 revolutions, from an initial angular velocity of $\omega = 3000$ rpm $= 314.2$ rad/s. What is the power dissipated?

Answer:

$$W = \Delta K = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.01)(314.2)^2 = 493.5\text{J}, \quad \Delta t = \frac{10}{(3000/60)} = 0.2 \text{ s};$$

$$P = \frac{\Delta K}{\Delta t} = \underline{2.47\times10^3\text{W}}$$

Sample Problem 10.11 Work, rotational kinetic energy, torque, disk

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Analogous Linear and Angular Quantities

Linear			Angular		Relation
Linear displacement	s	↔	Angular displacement	θ	$s = r\theta$
Linear speed	v	↔	Angular speed	ω	$v = r\omega$
Linear acceleration	a	↔	Angular acceleration	α	$a = r\alpha$
Mass (Inertia)	m	↔	Moment of inertia	I	$I = mr^2$
Force	F	↔	Torque	τ	$\vec{\tau} = \vec{r} \times \vec{F}$
Linear momentum	mv	↔	Angular momentum	$mvr = I\omega$	
Linear impulse	$F\,t$	↔	Angular impulse	τt	

Linear $F = ma$ $K.E. = \frac{1}{2}mv^2$ work = $F\ s$ power = $F\,v$

Angular $\tau = I\,\alpha$ $K.E. = \frac{1}{2}I\,\omega^2$ work = $\tau\ \theta$ power = $\tau\ \omega$

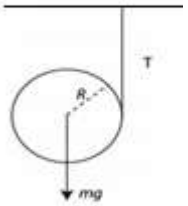
Section Summary We now have some understanding of why objects rotate the way they do. We built the laws of rotational motion in analogy to Newton's Laws of Motion for translation,

- **Newton's First Law for Rotation**
  "Every object will move with a constant angular velocity unless a torque acts on it."

- **Newton's Second Law for Rotation**
  "Angular acceleration of an object is directly proportional to the net torque acting on it and inversely proportional to its rotational inertia."

Extra problems

Q. A string is wrapped around a solid disk of mass m, radius R. The string is stretched in the vertical direction and the disk is released as shown in the Figure. Find the tension (T) in the string.



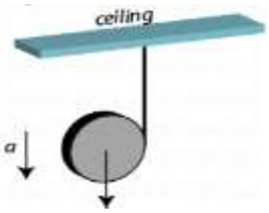
Answer:

$$ma = mg - T \quad (1),$$

$$\because I_{cm}\alpha = TR; \quad \alpha = \frac{a}{\mathrm{R}} \quad \Rightarrow \quad \tfrac{1}{2}mR^2\left(\frac{a}{R}\right) = TR \quad \Rightarrow \quad \tfrac{1}{2}ma = T \quad (2)$$

$$\therefore (2) \rightarrow (1) \quad \Rightarrow \quad a = \frac{2}{3}\mathrm{g} \Rightarrow \ T = \ \underline{\frac{1}{3}\mathrm{g}}$$

Q: A string (one end attached to the ceiling) is wound around a uniform solid cylinder of mass M = 2.0 kg and radius R = 10 cm (see Figure). The cylinder starts falling from rest as the string unwinds. The linear acceleration of the cylinder is:



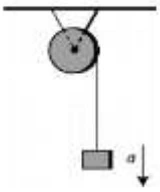
Answer:

$$ma = mg - T \quad (1),$$

$$\because I_{cm}\alpha = TR; \quad \alpha = \frac{a}{\mathrm{R}} \quad \Rightarrow \quad \tfrac{1}{2}mR^2\left(\frac{a}{R}\right) = TR \quad \Rightarrow \quad \tfrac{1}{2}ma = T \quad (2)$$

$$\therefore (1) \Rightarrow a = \frac{2}{3}\mathrm{g} = \ \underline{6.53\ \mathrm{m/s}^2}$$

Q: A 16 kg block is attached to a cord that is wound around the rim of a flywheel of radius 0.20 m and hangs vertically, as shown in Fig 4. The rotational inertia of the flywheel is 0.50 kg·m^2. When the block is released and the cord unwinds, the acceleration of the block is:



Answer:

$$ma = mg - T \quad (1),$$

$$\because I\alpha = TR; \quad \alpha = \frac{a}{\mathrm{R}} \quad \Rightarrow \quad 0.5\left(\frac{a}{0.2}\right) = T \times 0.2 \quad \Rightarrow \quad T = 12.5a \quad (2)$$

$$\therefore (1) \Rightarrow 16a = 16g - 12.5a \Rightarrow \ a = \frac{16}{28.5}g \ = \ \underline{5.5\ \mathrm{m/s}^2}$$

H.W.: A torque of 0.80 N.m applied to a pulley increases its angular speed from 45.0 rpm to 180 rpm in 3 seconds. Find the moment of inertia of the pulley.

Answer: 0.17 kg.m^2

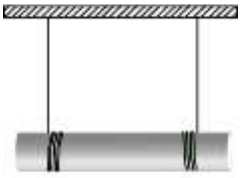


Figure 4

Q: A uniform 2.0 kg cylinder of radius 0.15 m is suspended by two strings wrapped around it, as shown in Figure 4. The cylinder remains horizontal while descending. The acceleration of the center of mass of the cylinder is:

Answer: Start with the equations of motion:

$$ma = mg - 2T \quad (1),$$

$$I\alpha = 2tr \quad (2),$$

$$\alpha = \frac{a}{r} \quad (3),$$

With the given information $I = \frac{1}{2}mr^2, \quad m = 2\text{ kg}, \quad r = 0.15\text{ m}$, one has:

$$(3) \Rightarrow T = \frac{1}{4}ma$$

$$(1) \Rightarrow \frac{3}{2}ma = mg \Rightarrow a = \frac{2}{3}g = \underline{6.53\frac{\text{m}}{\text{s}^2}}$$

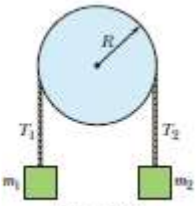


Figure 10-41
Problems 51 and 83.

Q: In Fig. 10-41, two blocks, of mass m_1 = 400 g and m_2 = 600 g, are connected by a massless cord that is wrapped around a uniform disk of mass *M*= 500 g and radius *R* = 12.0 cm. The disk can rotate without friction about a fixed horizontal axis through its center; the cord cannot slip on the disk. The system is released from rest. Find (a) the magnitude of the acceleration of the blocks, (b) the tension T_1 in the cord at the left, and (c) the tension T_2 in the cord at the right.

Answer:

We choose positive coordinate directions (different choices for each item) so that each is accelerating positively, which will allow us to set $a_1 = a_2 = R\alpha$ (for simplicity, we denote this as *a*). Thus, we choose upward positive for m_1, downward positive for m_2 and (somewhat unconventionally) clockwise for positive sense of disk rotation. Applying Newton's second law to m_1m_2 and (in the form of Eq. 10-45) to *M*, respectively, we arrive at the following three equations.

$$T_1 - m_1g = m_1a_1$$

$$m_2g - T_2 = m_2a_2$$

$$T_2R - T_1R = I\alpha$$

(a) The rotational inertia of the disk is $I = \frac{1}{2}MR^2$ (Table 10-2(c)), so we divide the third equation (above) by *R*, add them all, and use the earlier equality among accelerations — to obtain:

$$m_2g - m_1g = \left(m_1 + m_2 + \frac{1}{2}M\right)a$$

which yields $a = \frac{4}{25}g = 1.57\text{ m/s}^2$.

(b) Plugging back in to the first equation, we find

$$T_1 = \frac{29}{25}m_1g = 4.55\text{ N}$$

where it is important in this step to have the mass in SI units: m_1 = 0.40 kg.

(c) Similarly, with m_2 = 0.60 kg, we find

$$T_2 = \frac{5}{6}m_2g = 4.94\text{ N}.$$
