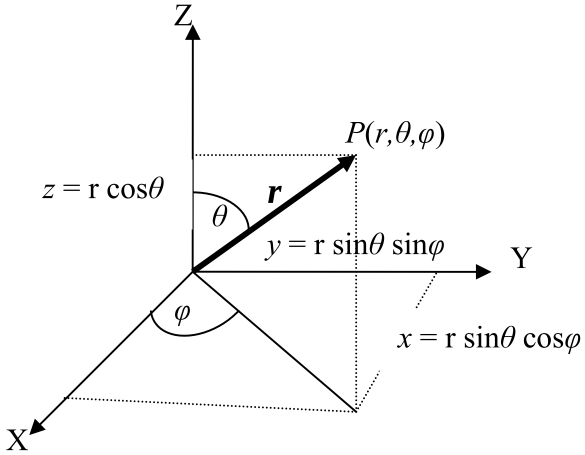
Laplace's Equation in Spherical Coordinates

When one is dealing with a problem having axial symmetry, it is generally convenient to use spherical polar coordinates (r,θ,ϕ) and my chose the axis of symmetry as the polar axis $\theta = 0$.



$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \\ z &= r\cos\theta, \qquad \varphi = \tan^{-1}\frac{y}{x} \end{aligned}\right\}, |\mathbf{r}| = \sqrt{x^2+y^2+z^2},$$

$$\theta \equiv \{0,\pi\}, \quad \varphi \equiv \{0,2\pi\}, \quad r \equiv \{0,\infty\},$$

$$d\tau = r^2\, dr\, d\Omega, \qquad d\Omega \equiv \text{Solid angle} = \sin\theta d\theta d\varphi,$$

H.W. for yourself: Prove the following:

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial r}{\partial x}}_{x/r}\frac{\partial}{\partial r} = \frac{x}{r}\frac{\partial}{\partial r}, \qquad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{x}{r}\frac{\partial}{\partial r}\right) = \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{\partial}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2},$$

$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\varphi$	$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\varphi}{r}$	$\frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\theta}$	$\frac{\partial x}{\partial\varphi} = -y$
$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi$	$\frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\varphi}{r}$	$\frac{\partial\varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta}$	$\frac{\partial y}{\partial\varphi} = x$
$\frac{\partial r}{\partial z} = \cos\theta$	$\frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}$	$\frac{\partial\varphi}{\partial z} = 0$	$\frac{\partial z}{\partial\varphi} = 0$

H.W. Prove that the Laplace's equation, $\nabla^2 V = 0$, in spherical coordinates is given by:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0 \quad (1)$$

H.W. Prove that: $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})=\frac{1}{r}\frac{\partial^2}{\partial r^2}(rV)$

Separation of Variables

Use $V(r,\theta,\phi)=R(r)\Psi(\theta,\phi)$, then equation (1) becomes:

$$\frac{\Psi}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})+\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{R}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=0 \quad (2)$$

Divided Eq. (2) by $R\Psi/r^2$, one obtains

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})+\frac{1}{\Psi\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{\Psi\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=0 \quad (3)$$

One can see that the first term is a function of r only while the remaining two terms are independent of r. The above equation is satisfied if we take:

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})=m \quad (4)$$

and

$$\frac{1}{\Psi\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{\Psi\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=-m \quad (5)$$

where m is a constant. The solutions of eqs. (4) and (5) take simpler forms, if one takes the constant m as $l(l+1)$ where the constant l is still arbitrary. We get

$$\frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})=l(l+1) \quad (4a)$$

And

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=-l(l+1)\Psi \quad (5a)$$

First we may find the solution of radial equation (4a). This may be expressed as

$$\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r})-l(l+1)R=0 \quad (4b)$$

Let us substitute $R(r)=U(r)/r$, Then eq. (4b) becomes

$$\frac{\partial^2 U(r)}{\partial r^2}-\frac{l(l+1)}{r^2}U(r)=0 \quad (4c)$$

From the form of (4c) it is apparent that a single power of r (rather than of power series) will satisfy it. One finds the solution to be

$$U(r)=Ar^{l+1}+\frac{B}{r^l} \quad \Rightarrow \quad R(r)=Ar^l+\frac{B}{r^{l+1}}$$

where l is yet undetermined and *A* and *B* are arbitrary constants.

H.W. Prove the above solution.
Answer: Substituting this expression in the differential equation for $R(r)$ we obtain

$$\frac{1}{Ar^k}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(Ar^k\right)\right)=k(k+1)=\ell(\ell+1)$$

Therefore, the constant *k* must satisfy the following relation:

$$k^2+k=\ell(\ell+1)$$

This equation gives us the following expression for *k*

$$k=\frac{-1\pm\sqrt{1+4\ell(\ell+1)}}{2}=\frac{-1\pm(2\ell+1)}{2}\quad\Rightarrow\quad k=\begin{cases}\ell\\ \text{or}\\ -(\ell+1)\end{cases}$$

The general solution for $R(r)$ is thus given by

$$\boxed{R(r)=Ar^{\ell}+\frac{B}{r^{\ell+1}}}$$

where *A* and *B* are arbitrary constants.

Now, we may try for the solution of (5a). Any solution of (5a) Ψ is a function of θ and ϕ and is called a surface harmonic of degree 'l'. Adopting the same technique let the solution of (5a) be $\Psi=\Theta(\theta)\Phi(\phi)\equiv\Theta\Phi$, then (5a) will be:

$$\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}+l(l+1)\Theta\Phi=0 \tag{5b}$$

Dividing (5b) throughout by $\frac{\Theta\Phi}{\sin^2\theta}$, one obtains

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)\sin^2\theta+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0 \tag{5c}$$

One finds that the variables are again separable. The first two terms in (5c) are functions of θ only and the last term is a function of ϕ only. Let us take

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2\Phi \qquad \text{(Azimuthal)} \tag{6}$$

Where m is constant. The solution of (6) is

$$\Phi_m(\phi)=C\,e^{\pm im\phi} \tag{7}$$

where C is a constant. In order that potential *V* be single valued, it is essential that

$$e^{\pm im\phi}=e^{\pm im(\phi+2\pi)} \tag{8}$$

This is possible only if *m* is an integer. One can normalize the function Φ_m by choosing the constant in such way that $\int_0^{2\pi}\Phi_m\Phi_m^* d\phi=1$. For this to be satisfied, constant C must be equal to $\frac{1}{\sqrt{2\pi}}$. We must note that the functions Φ_m are also orthogonal, i.e.

$$\int_0^{2\pi}\Phi_m\Phi_n^* d\phi=\delta_{mn}$$

Note: Φ must be a periodic function whose period evenly divides 2π, i.e. $\Phi(\phi)=\Phi(\phi+2\pi)$, *m* is necessarily an integer and Φ is a linear combination of the complex exponentials $e^{\pm im\phi}$

Legendre's equation

Equation (5c) could be simplified using Equ. (6), with Azimuthal symmetry m = 0, and using the definition $\mu = \cos\theta$, to prove that:

$$\frac{d}{d\theta} = -\sin\theta\frac{d}{d\mu} = -(1-\mu^2)\frac{d}{d\mu},$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) = (1-\mu^2)\frac{d^2}{d\mu^2} - 2\mu\frac{d}{d\mu} = \frac{d}{d\mu}\left((1-\mu^2)\frac{d}{d\mu}\right)$$

This implies that the function *P* satisfies the equation

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_\ell(\mu)}{d\mu}\right) = -\ell(\ell+1)P_\ell(\mu)$$

(We now have P_ℓ since for every ℓ we will have a different function.). The last equation is the **Legendre** equation, and its solutions are the **Legendre polynomials** $P_m(\mu \equiv \cos\theta)$.

Combining the solutions for $R(r)$ and $P_\ell(\mu)$ we obtain the most general solution of Laplace's equation in a spherical symmetric system with azimuthal symmetry:

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(Ar^\ell + \frac{B}{r^{\ell+1}}\right)P_\ell(\mu)}$$

Applications

1- r- dependent of V

Example: Find the general solution to Laplace's equation in spherical coordinates, for the case where $V(r)$ depends only on *r*.

Answer: Start with the Laplace's equation in spherical coordinates and use the condition *V* is only a function of *r* then:

$$\frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial\phi} = 0$$

Therefore, Laplace's equation can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

The solution *V* of this second-order differential equation must satisfy the following first-order differential equation:

$$r^2\frac{\partial V}{\partial r} = a = \text{ constant}$$

This differential equation can be rewritten as

$$\frac{\partial V}{\partial r} = \frac{a}{r^2} \quad\Rightarrow\quad V = -\frac{a}{r} + b$$

where *b* is a constant. If *V* = 0 at infinity, such as Columbic potential, then *b* must be equal to zero, and consequently

$$V = -\frac{a}{r}$$

2- r and θ dependent of V (Similar to Polar coordinates)

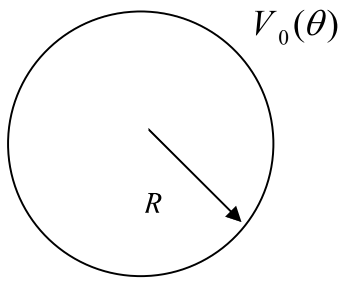
Consider a spherical symmetric system. Assuming that the system has ***azimuthal symmetry*** ($\Rightarrow \frac{\partial V}{\partial \varphi} = 0$), then Laplace's equation reduces to:

$$\frac{1}{R(r)}\frac{\partial}{\partial r}(r^2\frac{\partial R(r)}{\partial r}) = \ell(\ell+1) = \text{ constant} \qquad \text{(Radial equation)}$$

and

$$\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right) = -\ell(\ell+1) \qquad \text{(Angular equation)}$$

Example 1: The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius *R*. Find the electrostatic potential inside the sphere.



Answer: The system has spherical symmetry and we can therefore use the most general solution of Laplace's equation in spherical coordinates:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Consider the region inside the sphere ($r < R$). In this region $B_\ell = 0$, otherwise $V(r,\theta)$ would blow up at $r = 0$. Thus:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

The potential at $r = R$ is therefore equal to

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$$

To calculate the constant A_ℓ we are going to use the Fourier's trick, with the orthogonality relation of the Legendre polynomials, i.e.

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_0^\pi P_{\ell'}(\cos\theta)P_\ell(\cos\theta)\sin\theta d\theta}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}} = \int_0^\pi P_{\ell'}(\cos\theta)V_0(\theta)\sin\theta d\theta$$

This implies:

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi V_0(\theta)P_\ell(\cos\theta)\sin\theta d\theta$$

Then

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty}\left[\frac{2\ell+1}{2R^\ell}\int_0^\pi V_0(\theta)P_\ell(\cos\theta)\sin\theta d\theta\right] r^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{2}\left[\int_0^\pi V_0(\theta)P_\ell(\cos\theta)\sin\theta d\theta\right]\left(\frac{r}{R}\right)^\ell P_\ell(\cos\theta)$$

H.W. Find the electrostatic potential outside the sphere**.** Here you will find $A_\ell = 0$ and

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta d\theta$$

Example: The potential at the surface of a sphere, of radius *R*, is given by:

$$V_o(\theta) = K \cos(3\theta)$$

where K is a constant. Find the potential inside and outside the sphere. (Assume that there is no charge inside or outside the sphere.)

Answer: The most general solution of Laplace's equation in spherical coordinates is:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\mu)$$

Part A: Consider the region inside the sphere ($r < R$). In this region $B_\ell = 0$, otherwise $V(r,\theta)$ would blow up at $r = 0$. Thus:

$$V_{\text{ins}}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\mu)$$

The potential at $r = R$ is therefore equal to

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\mu) = K \cos(3\theta)$$

Using trigonometric relations we can rewrite $\cos(3\theta)$ as:

$$\cos(3\theta) = 4\mu^3 - 3\mu = 4\left[\tfrac{1}{5}\{2P_3(\mu) + 3P_1(\mu)\}\right] - 3[P_1] = \frac{8}{5}P_3(\mu) - \frac{3}{5}P_1(\mu)$$

Substituting this expression in the equation for $V(R,\theta)$ we obtain

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\mu) = K\left[\frac{8}{5}P_3(\mu) - \frac{3}{5}P_1(\mu)\right]$$

This equation immediately shows that $A_\ell = 0$ unless $\ell = 1$ or $\ell = 3$. If $\ell = 1$ or $\ell = 3$ then

$$A_1 = -\frac{3}{5}\frac{K}{R}, \quad A_3 = \frac{8}{5}\frac{K}{R^3}$$

The electrostatic potential inside the sphere is therefore equal to

$$\boxed{V_{\text{ins}}(r,\theta) = -\frac{3}{5}\frac{K}{R} r P_1(\mu) + \frac{8}{5}\frac{K}{R^3} r^3 P_3(\mu)}$$

Part B: Now consider the region outsider the sphere (*r* > *R*). In this region $A_\ell = 0$, otherwise $V(r,\theta)$ would blow up at infinity. The solution of Laplace's equation in this region is therefore equal to:

$$V_{\text{out}}(r,\theta) = \sum_{\ell=0}^{\infty} \left[\frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\mu)$$

The potential at $r = R$ is therefore equal to (from part A)

$$V(R,\theta) = \sum_{\ell=0}^{\infty} \left[\frac{B_\ell}{R^{\ell+1}} \right] P_\ell(\mu) = K\left[\frac{8}{5}P_3(\mu) - \frac{3}{5}P_1(\mu)\right]$$

The equation immediately shows that $B_\ell = 0$ except when $\ell = 1$ or $\ell = 3$. If $\ell = 1$ or $\ell = 3$ then:

$$B_1 = -\frac{3}{5}KR^2, \quad B_3 = \frac{8}{5}KR^4$$

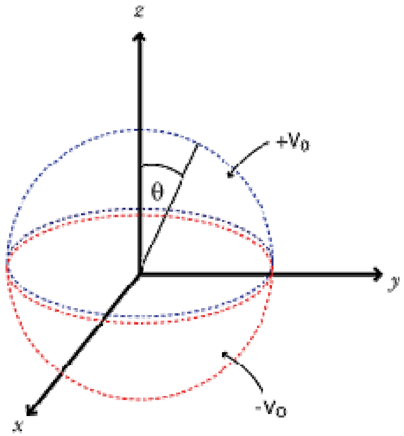
The electrostatic potential outside the sphere is thus equal to

$$\boxed{V_{\text{out}}(r,\theta) = -\frac{3K}{5}\frac{R^2}{r^2}P_1(\mu) + \frac{8K}{5}\frac{R^4}{r^4}P_3(\mu)}$$

H.W. Check if the $V_{\text{out}}(r,\theta) = V_{\text{in}}(r,\theta)$ at $r = R$.

Example: Find the potential outside two insulated conducting spheres, each of radius *R*, with the given boundary condition:

$$V(R,\theta) = \begin{cases} +V_o & 0 \le \theta \le \dfrac{\pi}{2} \\ -V_o & \dfrac{\pi}{2} \le \theta \le \pi \end{cases}$$



where V_o is a constant. Assume that there is no charge inside or outside the spheres.

Answer: The most general solution of Laplace's equation in spherical coordinates is:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\mu)$$

Consider the region outside the sphere (*r* > *R*). In this region $A_\ell = 0$, otherwise $V(r,\theta)$ would blow up at infinity. The solution of Laplace's equation in this region is:

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty}\left[\frac{B_\ell}{r^{\ell+1}}\right]P_\ell(\mu)$$

The potential at $r = R$ is therefore equal to

$$V_{out}(r = R,\theta) = \sum_{\ell=0}^{\infty}\left[\frac{B_\ell}{R^{\ell+1}}\right]P_\ell(\mu)$$

Where

$$B_\ell = \frac{2\ell+1}{2}R^{\ell+1}\int_0^{\pi} V(R,\theta)P_\ell(\cos\theta)\sin\theta d\theta$$

$$= \frac{2\ell+1}{2}R^{\ell+1}V_o\left\{\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta d\theta - \int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta d\theta\right\}$$

Change the variable $x = \cos\theta \quad \Rightarrow \quad dx = -\sin\theta d\theta$, we have:

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} V_o \left\{ -\int_1^0 P_\ell(x)dx + \underbrace{\int_0^{-1} P_\ell(x)dx}_{x\to -x} \right\}$$

In the second term, change the variable $y = -x \quad \Rightarrow \quad dy = -dx$, then change back $y = x$. Then, we can have:

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} V_o \left\{ \int_0^1 P_\ell(x)dx - \int_0^1 \underbrace{P_\ell(-x)}_{P_\ell(-x)=(-1)^\ell P_\ell(x)} dx \right\}$$

and is reduced to:

$$B_\ell = \begin{cases} 0 & \ell \text{ is even} \\ (2\ell+1) R^{\ell+1} V_o \int_0^1 P_\ell(x)dx & \ell \text{ is odd} \end{cases}$$

$$V_{out}(r,\theta) = V_o \sum_{\ell=1,3,5,\cdots}^{\infty} (2\ell+1)\left(\frac{R}{r}\right)^{\ell+1} \left(\int_0^1 P_\ell(x)dx \right) P_\ell(\mu)$$

$$= V_o \left\{ 3\left(\frac{R}{r}\right)^2 \left(\int_0^1 \underbrace{P_1(x)}_{x} dx \right) P_1(\cos\theta) + 7\left(\frac{R}{r}\right)^4 \left(\int_0^1 \underbrace{P_3(x)}_{\frac{1}{2}(5x^3-3x)} dx \right) P_3(\cos\theta) + \cdots \right\}$$

$$= V_o \left\{ \frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) + \cdots \right\}$$

For a point on z-axis $r = z, \theta = 0$, we have

$$V(z,0) = V_o \left\{ \frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(1) - \frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(1) + \cdots \right\}, \qquad P_\ell(1) = 1$$

H.W. Check the above results with the exact one:

$$V(z,0) = V_o \left\{ 1 - \frac{r^2 - R^2}{r\sqrt{r^2+R^2}} \right\}, \qquad P_\ell(1) = 1$$

H.W. Find the potential inside the sphere, and check the result:

$$V_{in}(r,\theta) = V_o \left\{ \frac{3}{2}\left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \cdots \right\}$$

It is simply changing $\left(\frac{R}{r}\right)^{\ell+1}$ by $\left(\frac{r}{R}\right)^{\ell}$

Associated Legendre Polynomials

When Helmholtz's equation is separated in spherical polar coordinates, one of the separated ODE's is the associated Legendre equation

Differential equation

$$\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}+\left\{n(n+1)-\frac{m^2}{(1-x^2)}\right\}\right]P_n^m(x)=0$$

Definition

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x); \qquad n=0,1,2,3,\cdots$$

$$m=0,1,2,\cdots,n$$

$$P_n^0(x)=P_n(x)$$

$$P_n^{-m}(x)=(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x)$$

$$P_n^m(-x)=(-1)^{n+m}P_n^m(x)$$

$$P_n^m(x)=0 \qquad \text{if } m>n$$

Generating function

$$g(x,h)=(2m-1)!!\frac{(1-x^2)^{m/2}h^m}{\left(1-2xh+h^2\right)^{m+1/2}}=\sum_{n=0}^{\infty}h^nP_n^m(x), \qquad |h|<1, \quad |x|\le +1$$

Recurrence relations

$$(2n+1)xP_n^m(x)=(n+m)P_{n-1}^m(x)+\left(n-m+1\right)P_{n+1}^m(x);$$

$$(2n+1)\sqrt{1-x^2}P_n^m(x)=P_{n-1}^{m+1}(x)-P_{n-1}^{m+1}(x)(x);$$

Orthogonality relation

$$\int_{-1}^{1}P_n^m(x)P_\ell^m(x)dx=\frac{2}{2n+1}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\delta_{n\ell}$$

H.W. Check the following table

m	n	$P_n^m(x)$
1	**1**	$\sqrt{1-x^2}=\sin\theta$
1	**2**	$3x\sqrt{1-x^2}=3\cos\theta\sin\theta$
2	**2**	$3\left(1-x^2\right)=3\sin^2\theta$
1	**3**	$\frac{3}{2}\left(5x^2-1\right)\sqrt{1-x^2}=\frac{3}{2}\left(5\cos^2\theta-1\right)\sin\theta$

Spherical Harmonic Function $Y_{\ell,m}(\theta,\varphi)$

Definition

$$Y_{\ell,m}(\theta,\varphi)=\Theta(\theta)\frac{e^{im\varphi}}{\sqrt{2\pi}}$$

$$=(-1)^m\left[\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}P_\ell^m(\cos\theta)e^{im\varphi};\qquad m\geq 0$$

$$Y_{\ell,-m}(\theta,\varphi)=(-1)^m Y^*_{\ell,m}(\theta,\varphi);$$

where $\ell=0,1,2,\cdots;\quad m=-\ell,-\ell+1,\cdots,+\ell$. $\Theta(\theta)=\left[\frac{(2\ell+1)}{2}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}P_\ell^m(\cos\theta)$ is the normalized angular function. An asterisk * indicates complex conjugation.

Differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\ell(\ell+1)\right]Y_{\ell,m}(\theta,\varphi)=0$$

Orthogonality relation

$$\langle \ell m\,|\,\ell' m\rangle=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\,Y^*_{\ell,m}(\theta,\varphi)Y_{\ell',m'}(\theta,\varphi)=\delta_{\ell\ell'}\delta_{mm'}$$

The statement of completeness is that any function $f(\theta,\varphi)$ can be represented as a sum over spherical harmonics:

$$f(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}f_{\ell m}Y_{\ell m}(\theta,\phi) \tag{2.5.8}$$

for some coefficients $f_{\ell m}$. By virtue of Eq. (2.5.7), these can in fact be calculated as

$$f_{\ell m}=\int f(\theta,\phi)Y^*_{\ell m}(\theta,\phi)\,d\Omega. \tag{2.5.9}$$

Equation (2.5.8) means that the spherical harmonics form a complete set of *basis functions* on the sphere.

It is interesting to see what happens when Eq. (2.5.9) is substituted into Eq. (2.5.8). To avoid confusion we change the variables of integration to θ' and ϕ':

$$\begin{aligned}f(\theta,\phi) &= \sum_\ell\sum_m Y_{\ell m}(\theta,\phi)\int f(\theta',\phi')Y^*_{\ell m}(\theta',\phi')\,d\Omega' \\ &= \int f(\theta',\phi')\left[\sum_\ell\sum_m Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)\right]d\Omega'.\end{aligned}$$

The quantity within the large square brackets is such that when it is multiplied by $f(\theta', \phi')$ and integrated over the primed angles, it returns $f(\theta, \phi)$. This must therefore be a product of two δ-functions, one for θ and the other for ϕ. More precisely stated,

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta}, \tag{2.5.10}$$

where the factor of $1/\sin\theta$ was inserted to compensate for the factor of $\sin\theta'$ in $d\Omega'$ (the δ-function is enforcing the condition $\theta' = \theta$). Equation (2.5.10) is known as the *completeness relation* for the spherical harmonics. This is analogous to a well-known identity,

$$\int \left(\frac{1}{\sqrt{2\pi}} e^{ikx'}\right)^* \left(\frac{1}{\sqrt{2\pi}} e^{ikx}\right) dk = \delta(x - x'),$$

in which the integral over dk replaces the discrete summation over ℓ and m; the basis functions $(2\pi)^{-1/2} e^{ikx}$ are then analogous to the spherical harmonics.

l	m	$Y_{lm}(\theta,\varphi)$
0	0	$\frac{1}{\sqrt{4\pi}}$
1	0	$\sqrt{\frac{3}{4\pi}}\cos\theta$
1	±1	$\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$
2	0	$\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)$
2	±1	$\mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{\pm i\varphi}$
2	±2	$\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$

Recurrence relations

$$\cos\theta Y_{\ell,m}(\theta,\varphi) = \left[\frac{(\ell+1+m)(\ell+1-m)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1,m}(\theta,\varphi)$$
$$+\left[\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1,m}(\theta,\varphi);$$

$$\sin\theta Y_{\ell,m}(\theta,\varphi)=\left[\frac{(\ell+1-m)(\ell+2-m)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y_{\ell+1,m-1}(\theta,\varphi)$$
$$+\left\{\left[\frac{(\ell+m)(\ell+m-1)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y_{\ell-1,m-1}(\theta,\varphi)\right\}e^{i\varphi}$$

Example: $Y_{l,0}=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$

Example:

$$Y_{3,0}=\sqrt{\frac{7}{4\pi}}P_3(\cos\theta)=\sqrt{\frac{7}{4\pi}}\frac{1}{2}(5\cos^2\theta-3\cos\theta)=\frac{1}{4}\sqrt{\frac{7}{\pi}}\cos\theta(5\cos\theta-3)$$
$$=\frac{1}{4}\sqrt{\frac{7}{\pi}}\frac{z}{r}(5\frac{z^2}{r}-3)=\frac{1}{4}\sqrt{\frac{7}{\pi}}\frac{z}{r^3}\left(5z^2-3r^2\right)$$

Example:

$$\psi=x+iy=r\sin\theta(\cos\varphi+i\sin\varphi)=r\sin\theta\, e^{i\varphi}$$
$$=-\sqrt{\frac{8\pi}{3}}\,r\,Y_{1,1}$$

Example:

$$\sin\theta\left(\sin\varphi+\cos\theta\cos\varphi\right)=-\frac{1}{2i}\sqrt{\frac{8\pi}{3}}Y_{1,1}-\frac{1}{2i}\sqrt{\frac{8\pi}{3}}Y_{1,-1}-\frac{1}{2}\sqrt{\frac{8\pi}{15}}Y_{2,1}+\frac{1}{2}\sqrt{\frac{8\pi}{15}}Y_{2,-1}$$

Example: $\sin\theta\left(1-\cos\theta\right)e^{i\varphi}=-\sqrt{\frac{8\pi}{3}}Y_{1,1}-\sqrt{\frac{8\pi}{15}}Y_{2,1}$

Example: $\sqrt{\pi}-\sqrt{3\pi}\cos^2\theta=-\pi\sqrt{\frac{16}{5}}Y_{2,0}$

Example: $\sin\theta\cos\varphi=-\sqrt{\frac{2\pi}{3}}Y_{1,1}+\sqrt{\frac{2\pi}{3}}Y_{1,-1}$