Few Special Functions

INTRODUCTION

Equations in the form:

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \qquad \text{(Legendre's equation)}$$

(n is a real constant),

$$x^2y''+xy'+\left\{x^2-n^2\right\}=0 \qquad \text{(Bessel's equation)}$$

(n is a positive constant or zero),

$$y''-2ty'+2ky=0 \qquad \text{(Hermit equation)}$$

where k is usually a non-negative integer,

$$xy''-(1-x)y'+ny=0 \qquad \text{(Laguerre equation)}$$

(n is a positive constant or zero), and

$$y''-ty=0 \qquad \text{(Airy's equation)}$$

(t could be positive or negative constant) occur in many physical problems, such as QM, EM,CM, SM, etc. The solutions of these functions, with others, are called special functions. In these lectures, we discuss the methods of solving these differential equations.

(Note: Hermit and Airy's equations will be given as an assignment)

Hermit's Equation

Hermite's Equation of order k has the form

$$y''-2ty'+2ky=0$$

where k is usually a non-negative integer.

H.W. Work out the Hermit's equation using the power series. Hermit's equation is an example of a differential equation, which has a polynomial solution. As usual, the generic form of a power series is

$$y(t)=\sum_{n=0}^{\infty}a_nt^n$$

We have to determine the right choice for the coefficients (a_n).

As in other techniques for solving differential equations, once we have a "guess" for the solutions, we plug it into the differential equation. Recall that

$$y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1},$$

and

$$y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}.$$

Plugging this information into the Differential equation we obtain:

$$\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}-2t\sum_{n=1}^{\infty}na_nt^{n-1}+2k\sum_{n=0}^{\infty}a_nt^n=0,$$

or after rewriting slightly:

$$\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}-\sum_{n=1}^{\infty}2na_nt^n+\sum_{n=0}^{\infty}2ka_nt^n=0.$$

Next we shift the first summation up by two units:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty} 2na_n t^n + \sum_{n=0}^{\infty} 2ka_n t^n = 0.$$

Before we can combine the terms into one sum, we have to overcome another slight obstacle: the second summation starts at *n*=1, while the other two start at *n*=0.

Evaluate the 0th term for the second sum: $2 \cdot 0 \cdot a_0 \cdot t^0 = 0$. Consequently, we do not change the value of the second summation, if we start at *n*=0 instead of *n*=1:

$$\sum_{n=1}^{\infty} 2na_n t^n = \sum_{n=0}^{\infty} 2na_n t^n.$$

Thus we can combine all three sums as follows:

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - 2na_n + 2ka_n \right) t^n = 0.$$

Therefore our recurrence relations become:

$$(n+2)(n+1)a_{n+2} - 2na_n + 2ka_n = 0 \text{ for all } n = 0,1,2,3,\ldots$$

After simplification, this becomes

$$a_{n+2} = \frac{2(n-k)}{(n+2)(n+1)}a_n \qquad \text{for all } n = 0,1,2,3,\ldots$$

Let us look at the special case, where *k* = 5, and the initial conditions are given as:

$$y(0) = a_0 = 0,\ y'(0) = a_1 = 1$$

.

In this case, all even coefficients will be equal to zero, since a_0=0 and each coefficient is a multiple of its second predecessor.

$$a_0 = a_2 = a_4 = a_6 = \ldots = 0.$$

What about the odd coefficients? a_1=1, consequently

$$a_3 = \frac{2(1-5)}{2\cdot 3} = -\frac{4}{3};$$

and

$$a_5 = \frac{2(3-5)}{4\cdot 5}a_3 = (-\frac{1}{5})(-\frac{4}{3}) = \frac{4}{15}.$$

What about a_7:

$$a_7 = \frac{2(5-5)}{6\cdot 7}a_5 = 0.$$

Since a_7=0, all odd coefficients from now on will be equal to zero, since each coefficient is a multiple of its second predecessor.

$$a_7 = a_9 = a_{11} = a_{13} = \ldots = 0.$$

Consequently, the solution has only 3 non-zero coefficients, and hence is a polynomial. This polynomial

$$H_5(t) = t - \frac{4}{3}t^3 + \frac{4}{15}t^5$$

(or a multiple of this polynomial) is called the **Hermit Polynomial of order 5**.

It turns out that the Hermit Equation of positive integer order *k* always has a polynomial solution of order *k*. We can even be more precise: If *k* is odd, the initial value problem

$a_0 = 0,\ a_1 = 1$ will have a polynomial solution, while for *k* even, the initial value problem $a_0 = 1,\ a_1 = 0$ will have a polynomial solution.

Exercise 1:
Find the Hermit Polynomials of order 1 and 3.
Answer. Recall that the recurrence relations are given by

$$a_{n+2} = \frac{2(n-k)}{(n+2)(n+1)} a_n \qquad \text{for all } n = 0, 1, 2, 3, \ldots$$

We have to evaluate these coefficients for k=1 and k=3, with initial conditions a_0=0, a_1=1.
When k=1,

$$a_3 = \frac{2(1-1)}{2 \cdot 3} a_1 = 0.$$

Consequently all odd coefficients other than a_1 will be zero. Since a_0=0, all even coefficients will be zero, too. Thus

$$H_1(t) = t$$

When k=3,

$$a_3 = \frac{2(1-3)}{2 \cdot 3} a_1 = -\frac{2}{3},$$

and

$$a_5 = \frac{2(3-3)}{4 \cdot 5} a_3 = 0.$$

Consequently all odd coefficients other than a_1 and a_3 will be zero. Since a_0=0, all even coefficients will be zero, too. Thus

$$H_3(t) = t - \tfrac{2}{3} t^3$$

Exercise 2:
Find the Hermit Polynomials of order 2, 4 and 6.
Answer.
Recall that the recurrence relations are given by

$$a_{n+2} = \frac{2(n-k)}{(n+2)(n+1)} a_n \qquad \text{for all } n = 0, 1, 2, 3, \ldots$$

We have to evaluate these coefficients for k=2, k=4 and k=6, with initial conditions a_0=1, a_1=0.
When k=2,

$$a_2 = \frac{2(0-2)}{1 \cdot 2} a_0 = -2,$$

while

$$a_4 = \frac{2(2-2)}{3 \cdot 4} a_2 = 0.$$

Consequently all even coefficients other than a_2 will be zero. Since a_1=0, all odd coefficients will be zero, too. Thus

$$H_2(t) = 1 - 2t^2.$$

When k=4,

$$
\begin{aligned}
a_2 &= \frac{2(0-4)}{1\cdot 2}a_0 = -4, \\
a_4 &= \frac{2(2-4)}{3\cdot 4}a_2 = \left(-\frac{4}{12}\right)(-4) = \frac{4}{3} \\
a_6 &= \frac{2(4-4)}{5\cdot 6}a_4 = 0.
\end{aligned}
$$

Consequently all even coefficients other than a_2 and a_4 will be zero. Since a_1=0, all odd coefficients will be zero, too. Thus

$$H_4(t) = 1 - 4t^2 + \frac{4}{3}t^4.$$

You can check that

$$H_6(t) = 1 - 6\,t^2 + 4\,t^4 - \frac{8}{15}t^6.$$

H.W. Discuss the SHM as an application of Hermit's in quantum mechanics. What is the condition to have $E_n = \left(n + \frac{1}{2}\right)h\omega$?.

Airy's Equation

Airy's differential equation:

$$y'' - ty = 0$$

is used in physics to model the diffraction of light.
We want to find power series solutions for this second-order linear differential equation. The generic form of a power series is

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

We have to determine the right choice for the coefficients (a_n).
As in other techniques for solving differential equations, once we have a "guess" for the solutions, we plug it into the differential equation. Recall that

$$y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}.$$

Plugging this information into the Differential equation we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=0}^{\infty} a_n t^n = 0,$$

or equivalently

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0.$$

Our next goal is to simplify this expression such that (basically) only one summation sign "$\sum$ " remains. The obstacle we encounter is that the powers of both sums are different, t^{n-2} for the first sum and t^{n+1} for the second sum. We make them the same by shifting the index of the first sum up by 2 units and the index of the second sum down by one unit to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty} a_{n-1}t^n = 0.$$

Now we run into the next problem: the second sum starts at n=1, while the first sum has one more term and starts at n=0. We split off the 0th term of the first sum: (FIRST ONE SHOUD START FROM $n=1$)

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n = 2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}t^n.$$

Now we can combine the two sums as follows:

$$2a_2 + \sum_{n-1}^{\infty}\left((n+2)(n+1)a_{n+2}t^n - a_{n-1}t^n \right) = 0,$$

and factor out t^n:

$$2a_2 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} - a_{n-1} \right)t^n = 0.$$

The power series on the left is identically equal to zero, consequently all of its coefficients are equal to 0:

$$\begin{cases} 2a_2 & = 0 \\ (n+2)(n+1)a_{n+2} - a_{n-1} & = 0 \text{ for all } n = 1, 2, 3, \ldots \end{cases}$$

We can slightly rewrite as

$$\begin{cases} a_2 & = 0 \\ a_{n+2} & = \dfrac{a_{n-1}}{(n+1)(n+2)} \text{ for all } n = 1, 2, 3, \ldots \end{cases}$$

These equations are known as the **"recurrence relations"** of the differential equations. The recurrence relations permit us to compute all coefficients in terms of a_0 and a_1.
We already know from the 0th recurrence relation that a_2=0. Let's compute a_3 by reading off the recurrence relation for n=1:

$$a_3 = \frac{a_0}{2\cdot 3}.$$

Let us continue:

$$\begin{aligned} a_4 &= \frac{a_1}{3\cdot 4} \\ a_5 &= \frac{a_2}{4\cdot 5} = 0 \\ a_6 &= \frac{a_3}{5\cdot 6} = \frac{a_0}{(2\cdot 3)(5\cdot 6)} \\ a_7 &= \frac{a_4}{6\cdot 7} = \frac{a_1}{(3\cdot 4)(6\cdot 7)} \\ a_8 &= \frac{a_5}{7\cdot 8} = 0 \\ a_9 &= \frac{a_6}{8\cdot 9} = \frac{a_0}{(2\cdot 3)(5\cdot 6)(8\cdot 9)} \end{aligned}$$

The hardest part, as usual, is to recognize the patterns evolving; in this case we have to consider three cases:

1. All the terms $a_2, a_5, a_8, \cdots$ are equal to zero. We can write this in compact form as

$$a_{3k+2} = 0 \text{ for all } k = 0, 1, 2, 3, \ldots$$

2. All the terms $a_3, a_6, a_9, \cdots$ are multiples of a_0. We can be more precise:

$$a_{3k} = \frac{1}{(2\cdot 3)(5\cdot 6)\cdots((3k-1)\cdot(3k))}\cdot a_0 \text{ for all } k = 1, 2, 3, \ldots$$

(Plug in k = 1, 2, 3, 4 to check that this works!)

3. All the terms $a_4, a_7, a_{10}, \cdots$ are multiples of a_1. We can be more precise:

$$a_{3k+1} = \frac{1}{(3\cdot 4)(6\cdot 7)\cdots((3k)\cdot(3k+1))}\cdot a_1 \text{ for all } k = 1, 2, 3, \ldots$$

(Plug in k = 1, 2, 3, 4 to check that this works!)

Thus the general form of the solutions to Airy's Equation is given by

$$\begin{aligned} y(t) &= a_0\left(1+\sum_{k=1}^{\infty}\frac{t^{3k}}{(2\cdot 3)(5\cdot 6)\cdots((3k-1)\cdot(3k))}\right) \\ &\quad + a_1\left(t+\sum_{k=1}^{\infty}\frac{t^{3k+1}}{(3\cdot 4)(6\cdot 7)\cdots((3k)\cdot(3k+1))}\right). \end{aligned}$$

Note that, as always, $y(0) = 1$ and $y'(0) = 1$. Thus it is trivial to determine a_0 and a_1 when you want to solve an initial value problem.

In particular

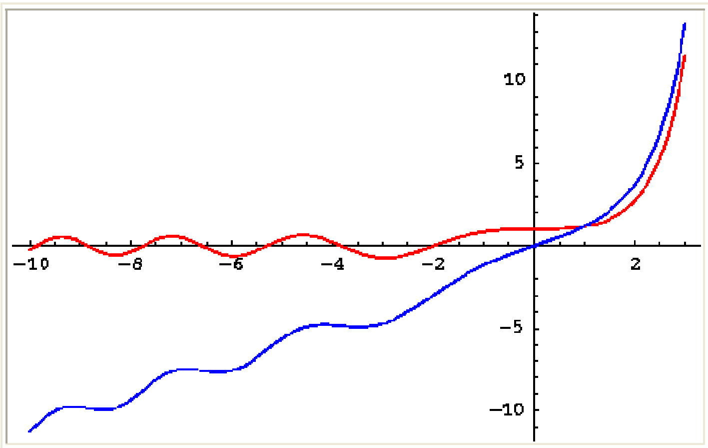
$$y_1(t) = 1+\sum_{k=1}^{\infty}\frac{t^{3k}}{(2\cdot 3)(5\cdot 6)\cdots((3k-1)\cdot(3k))}$$

and

$$y_2(t) = t+\sum_{k=1}^{\infty}\frac{t^{3k+1}}{(3\cdot 4)(6\cdot 7)\cdots((3k)\cdot(3k+1))}$$

form a fundamental system of solutions for Airy's Differential Equation.

Below you see a picture of these two solutions. Note that for negative t, the solutions behave somewhat like the oscillating solutions of $y''+y=0$, while for positive t, they behave somewhat like the exponential solutions of the Differential equation $y''-y=0$.



In the next section we will investigate what one can say about the radius of convergence of power series solutions.

LEGENDRE'S DIFFERENTIAL EQUATION

The differential equation of the form

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

where n is a real constant is called Legendre's differential equation. The singularities of Legendre's equation are x = ± 1. Legendre's equation can also be written as

$$[(1-x^2)y']'+n(n+1)y=0$$

1 SOLUTION OF LEGENDRE'S EQUATION

The Legendre's differential equation is

$$(1-x^2)y''-2xy'+n(n+1)y=0 \qquad (1)$$

we set the solution of equation (1), about x = 0.
Let us assume that the power series solution of equation (1) is of the form

$$y=\sum_{m=0}^{\infty} c_m x^m;\ c_0\neq 0 \qquad (2)$$

Then we get

$$y'=\sum_{m=1}^{\infty} m c_m x^{m-1} \quad , \qquad y''=\sum_{m=2} m(m-1)\,c_m x^{m-2}$$

Substituting the values of y, y' and y" in equation (1), we get

$$\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2}-\sum_{m=2}^{\infty} m(m-1)c_m x^m-2\sum_{m=1}^{\infty} mc_m x^m+n(n+1)\sum_{m=0}^{\infty} c_m x^m=0$$

or

$$(2c_2+n(n+1)c_0)+\{3.2c_3+(-2+n(n+1)]c_1\}x+\{4.3c_4+[-2(1+2)+n(n+1)c_2\}x^2$$
$$+\{(m+2)(m+1)c_{m+2}+[-m(m-1)-2m+n(n+1)c_m]x^m+\ldots=0$$

Equating the coefficients of various powers of x to zero, we get

$$2c_2+n(n+1)c_0=0$$
$$6c_3-2c_1+n(n+1)c_1=0$$
$$\ldots$$
$$\ldots$$
$$(m+2)(m+1)c_{m+2}+[-m(m-1)-2m+n(n+1)]c_m=0$$

Now

$$2c_2+n(n+1)c_0=0 \text{ gives } c_2=-\frac{n(n+1)}{2!}c_0$$

$$6c_3-2c_1+n(n+1)c_1=0 \text{ gives } c_3=\frac{1}{3!}(n-1)(n+2)c_1$$

And in general

$$(m+1)(m+2)c_{m+2}+[-m(n-1)-2m+n(n+1)]c_m=0$$

yields

$$c_{m+2}=-\frac{(n-m)(n+m+1)}{(m+1)(m+2)}c_m \qquad m\geq 2$$

We have

$$c_4=-\frac{(n-2)(n+3)}{4.3}c_2 = \frac{(n-1)n(n+1)(n+3)}{4.3.2!}c_0$$
$$=\frac{1}{4!}(n-2)\,n(n+1)(n+3)$$

and

$$c_5 = -\frac{(n-3)(n+4)}{5.3}c_3$$

$$= \frac{1}{5!}(n-3)(n-1)(n+2)(n+4)c_1$$

$$\ldots$$

$$\ldots$$

The solution of Legendre's equation is

$$y = c_o y_1 + c_1 y_2$$

where

$$y_0 = 1 - \frac{1}{2!}n(n+1)x^2 + \frac{1}{4!}(n-2)\,n(n+1)(n+3)x^4 - \ldots$$

And

$$y_1 = x - \frac{1}{3!}(n-1)(n+2)x^3 + \frac{1}{5!}(n-3)(n-1)(n+2)(n+4)x^5 - \ldots$$

2 LEGENDRE'S POLYNOMIALS

The singularities of Legendre's equation are x = ± 1. The distance between the point x = 0 and the nearest singularity is 1. Therefore, the power series solution is convergent in $|x| < 1$. The solution y_0 contains even powers of x and the solution y_1 contains odd powers of x. **The solutions y_0 and y_1 are the linearly independent solution of the Legendre's differential equation**.

If n takes even positive integral values, y_0 reduces to polynomial of even powers. In this case y_1 remains as an infinite series. If n takes odd positive, y_1 reduces to a polynomial of odd powers, whereas y_0 remains an infinite series. These polynomials multiplied by suitable constants are called Legendre's polynomials. The Legendre polynomials are of degree n and are denoted by $P_n(x)$ Therefore, when n takes integral values one of the linearly independent solutions of the Legendre's differential equation is a Legendre polynomial and the second independent solution is an infinite series. These infinite series are called Legendre's functions of second kind and are denoted by $Q_n(x)$.

In order to evaluate the multiplicative constants of
Legendre polynomials we get $P_n(1) = 1$.

Legendre's polynomials are also called Legendre's functions of the first kind and are given by:

$$P_n(x) = (-1)^{n/2}\frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \ldots\right]$$

when n is even, and

$$P_n(x) = (-1)^{(n-1)/2}\frac{1.3.5\ldots n}{2.4.6\ldots(n-1)}\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}x^5 + \ldots\right]$$

when n is odd.

$P_n(x)$ is a terminating series.

Selected values

n	$P_n(x \equiv \cos\theta)$	x^n
0	1	P_0
1	μ	P_1
2	$\frac{1}{2}\left(3x^2-1\right)$	$\frac{1}{3}\left(2P_2+1\right)$
3	$\frac{1}{2}\left(5x^3-3x\right)$	$\frac{1}{5}\left(2P_3+3P_1\right)$
4	$\frac{1}{8}\left(35x^4-30x^2+3\right)$	$\frac{1}{35}\left(8P_4+20P_2+7\right)$

```
:= Do[Print["n = ", n, "   ", "Lagendre = ",   LegendreP[n, x]], {n, 0, 4}]

n = 0   Lagendre = 1
n = 1   Lagendre = x
n = 2   Lagendre = -1/2 + 3 x^2/2
n = 3   Lagendre = -3 x/2 + 5 x^3/2
n = 4   Lagendre = 3/8 - 15 x^2/4 + 35 x^4/8
```

3 Properties of the Legendre Polynomials:

1- It is self adjoint (Hermitian) , i.e. if we have:

$$\underbrace{\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}\right]}_{L}P_\ell(x)=-\ell(\ell+1)P_\ell(x)$$

Then $L=L^\dagger$ (Prove that). Hence:

i- L has real eigenvalue $\ell(\ell+1)$.

ii- Eigenfunction corresponding to different eigenvalues must be orthogonal.

2- The function $P_\ell(x)$ constitutes a complete orthonormal set of functions on the interval $-1\le x\le 1$. So we can use them to expanding any function on that interval.

3- $P_\ell(1)=1$ for all ℓ

4- if ℓ is even: $P_\ell(\mu)=P_\ell(-\mu)$

5- if ℓ is odd: $P_\ell(\mu)=-P_\ell(-\mu)$

6- $$\int_0^\pi P_{\ell'}(\cos\theta)P_\ell(\cos\theta)\sin\theta\, d\theta=\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx=\begin{cases}0 & \text{if } \ell'\neq\ell\\ \dfrac{2}{2\ell+1} & \text{if } \ell'=\ell\end{cases}$$

Example 1: The potential at the surface of a sphere of radius R is given by $V(\theta)=V_o\cos(2\theta)$, where V_o is a constant. Show that $V=\frac{V_o}{3}(4P_2-P_o)$.

Answer: $\cos2\theta=2\cos^2\theta-1=2\left[\frac{2P_2-1}{3}\right]-1=\frac{1}{3}\left[4P_2-1\right]$

Example 2: Express the function $f(x) = x^3 + x^2 + x + 1$, in terms of Legendre polynomials.
Answer:

$$f(x) = x^3 + x^2 + x + 1 = \frac{1}{5}(2P_3 + 3P_1) + \frac{1}{3}(2P_2 + P_o) + P_1 + P_o$$
$$= 2\left(\frac{1}{5}P_3 + \frac{1}{3}P_2 + \frac{4}{5}P_1 + \frac{1}{3}P_o\right)$$

Legendre series representation

Arbitrary function $f(x)$ can be expanded in Legendre polynomials as:

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x), \quad \Rightarrow \quad A_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Ex. 2. *Expand $f(x) = x^2$ in a series of the form $\Sigma c_r P_r(x)$*
Sol. Since x^2 is a polynomial of degree two, from Legendre series, we have

$$x^2 = \sum_{r=0}^{2} c_r P_r(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x), \qquad \text{...(1)}$$

where
$$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} x^2 P_r(x)\, dx. \qquad \text{...(2)}$$

But $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. ...(3)
Putting $r = 0, 1, 2$ successively in (2) and using (3), we have

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^2\, dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3},\ c_1 = \frac{3}{2}\int_{-1}^{1} x^3\, dx = 0,\ c_2 = \frac{1}{2} \times \frac{5}{2}\int_{-1}^{1} x^2(3x^2 - 1)\, dx = \frac{5}{4}\left[3 \times \frac{x^5}{5} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

With the above values of c_0, c_1 and c_2, (1) gives $x^2 = (1/3) \times P_0(x) + (2/3) \times P_2(x)$.

Expand $f(x)$, where $f(x) = \begin{cases} +1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$, as an infinite series of Legendre polynomial $P_n(x)$.

Solution:

We have $f(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) + \ldots$

where

$$c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx$$

Hence

$$c_0 = \frac{2.0+1}{2} \int_{-1}^{1} f(x) P_0(x) dx = \frac{1}{2}\left[\int_{-1}^{0} f(x) \cdot 1 \cdot dx + \int_{0}^{1} f(x) \cdot 1 \cdot P dx\right]$$
$$= \frac{1}{2}\left[\int_{-1}^{0} 0 \cdot dx + \int_{0}^{1} 1 dx\right] = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

and

$$c_1 = \frac{3}{2}\left[\int_{-1}^{0} 0\cdot P_1(x)\,dx + \int_0^1 P_1(x)\,dx\right] = \frac{3}{2}\left[0+\int_0^1 x\,dx\right] = \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{4}$$

$$c_2 = \frac{5}{2}\left[\int_{-1}^{0} 0\cdot dx + \int_0^1 P_2(x)\,dx\right] = \frac{5}{2}\left[0+\int_0^1 \frac{1}{2}(3x^2-1)\,dx\right]$$

$$= \frac{5}{2}\cdot\frac{1}{2}\int_0^1 (3x^2-1)\,dx = \frac{5}{4}\left[3\left[\frac{x^3}{3}\right]_0^1 - [x]_0^1\right] = \frac{5}{4}\left[(1^3-0)-(1-0)\right] = 0$$

$$c_3 = \frac{7}{2}\left[\int_{-1}^{0} 0\,dx + \int_0^1 P_3(x)\,dx\right] = \frac{7}{2}\left[0+\int_0^1 \frac{1}{2}(5x^3-3x)\,dx\right]$$

$$= \frac{7}{4}\left[5\left[\frac{x^4}{4}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1\right] = \frac{7}{4}\left[\frac{5}{4}-\frac{3}{2}\right] = -\frac{7}{16}$$

Substituting the values of c_0, c_1, c_2,... in

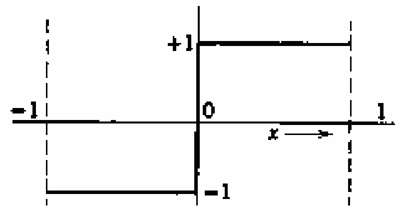
$$f(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + \ldots$$

We get

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \ldots$$

Example: Jackson's book page 99

Expand $f(x)$, where $f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$, as an infinite series of Legendre polynomial $P_n(x)$.



$$A_l = \frac{2l+1}{2}\left[\int_0^1 P_l(x)\,dx \quad \int_{-1}^{0} P_l(x)\,dx\right]$$

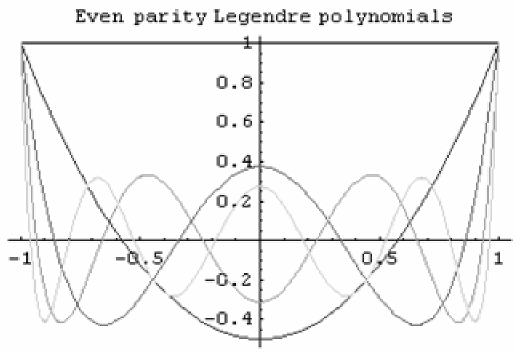
Since $P_l(x)$ is odd (even) about $x = 0$ if l is odd (even), only the odd l coefficients are different from zero. Thus, for l odd,

$$A_l = (2l+1)\int_0^1 P_l(x)\,dx$$

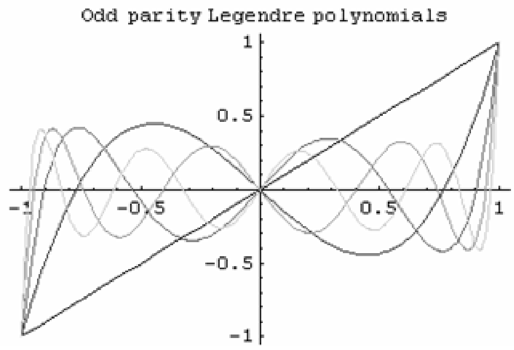
Answer:

$$f(x) = \tfrac{3}{2}P_1(x) - \tfrac{7}{8}P_3(x) + \tfrac{11}{16}P_5(x) - \cdots$$

```
In[2]:= eL = Table[LegendreP[n, x], {n, 0, 8, 2}] ; (* For even parity  *)

In[3]:= oL = Table[LegendreP[n, x], {n, 1, 9, 2}] ;(* For odd parity  *)

In[4]:= shades = Table[2 (i - 1)/10, {i, 5}];

In[5]:= GrayLevel /@ shades;

In[6]:= labe = "Even parity Legendre polynomials";

In[7]:= ge = Plot[Evaluate[eL], {x, -1, 1}, PlotLabel → labe, PlotStyle → GrayLevel /@ shades];
```




```
In[8]:= labe = "Odd parity Legendre polynomials";

In[9]:= ge = Plot[Evaluate[oL], {x, -1, 1}, PlotLabel → labe, PlotStyle → GrayLevel /@ shades];
```



Bessel's Equation

Bessel's equation in the form

$$x^2 y'' + xy' + \left\{x^2 - \nu^2\right\} y = y'' + \frac{1}{x} y' + \left\{1 - \frac{\nu^2}{x^2}\right\} y = 0 \quad (1)$$

has $x = 0$ as a regular singular point, so we can write:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \quad y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \quad (2)$$

And Bessel's equation (1) reduces to:

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) x^{n+r-2} + \frac{1}{x}(n+r) x^{n+r-1} + \left\{1 - \frac{\nu^2}{x^2}\right\} x^{n+r} \right] a_n = 0 \quad (3A)$$

or

$$\sum_{n=0}^{\infty} \left[\left\{ (n+r)(n+r-1) + (n+r) - \nu^2 \right\} x^{n+r-2} + x^{n+r} \right] a_n = 0 \quad (3B)$$

or

$$\sum_{n=0}^{\infty} \left[\left\{ (r+n)^2 - \nu^2 \right\} x^{r+n-2} + x^{r+n} \right] a_n = 0 \quad (3C)$$

1- From (3C) equating the coefficient of lowest power of x, i.e. x^{r-2} by putting $n = 0$, we have:

$$\left(r^2 - \nu^2\right) a_0 = 0 \qquad \because a_0 \neq 0 \;\Rightarrow\; r = \pm\nu . \quad (I)$$

Now equating to zero the coefficient of x^{r-1} by putting $n = 1$ in (3C), we get:

$$\left\{ (r+1)^2 - \nu^2 \right\} a_1 = 0 \quad (II)$$

Since, $r = \pm\nu$, $\left[(r+1)^2 - \nu^2\right] \neq 0$ therefore $a_1 = 0*$. $r = -\frac{1}{2}$ is a special case and has to be consider separately. Then, the indicial equation (II) will implies, $a_1 = a_3 = a_5 = \cdots = a_{2n+1} = 0$, i.e. no term with odd values will be given. So, for $n = 2, 4, 6, \cdots$ the recurrence relation of (3C) will be:

$$a_n = -\frac{1}{(n+r)^2 - \nu^2} a_{n-2}, \qquad n \geq 2 \quad (4)$$

With $r = +\nu$, (4) will be reduced to:

$$a_n = -\frac{1}{(n+\nu)^2 - \nu^2} a_{n-2} = -\frac{1}{n(n+2\nu)} a_{n-2}, \qquad n \geq 2$$

$$a_2 = -\frac{1}{2^2 \cdot 1 \cdot (+\nu)} a_0, \qquad a_4 = \frac{1}{2^4 \cdot 2! \cdot (\nu+1)(\nu+2)} a_0, \qquad a_6 = -\frac{1}{2^6 \cdot 3! \cdot (\nu+1)(\nu+2)(\nu+3)} a_0$$

And in general the coefficients in equation (4) reduce to:

$$a_{2n} = (-1)^n \frac{1}{2^{2n} n!(n+1)(n+2)\cdots(\nu+n)} a_0, \qquad n = 1, 2, 3, \cdots \quad (5)$$

Since a_0 is an unknown constant, which has different values for different problems as determined by the boundary conditions for the problem, we can redefine a_0 as follows:

$$a_0 = A \frac{1}{2^\nu \nu!} \quad (6)$$

where A is the constant that is selected to fit the boundary conditions. (This is a convention used to obtain an equation that is used for computation and tabulation of Bessel functions.) With this substitution we can write equation (5) as follows.

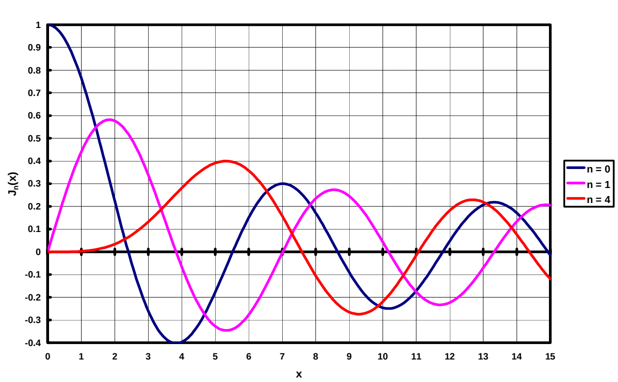
$$a_{2n} = (-1)^n \frac{1}{2^{2n+\nu} n!(n+1)(n+2)\cdots(\nu+n)\nu!} A, \qquad n = 1, 2, 3, \cdots$$

And the power series solution will be:

$$y(x) = \boxed{(x)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(\nu+j)!2^{2j+\nu}} (x)^{2j} = J_\nu(x)} \qquad (7)$$

The A coefficient is dropped with the understanding that any final solution can be multiplied by a constant to satisfy the boundary conditions. For integer ν, the relation $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$ is hold. This is implies that the two solutions are not independent. The Bessel function of the first kind of integer order ν, $J_\nu(x)$ is defined by equation (7), with the arbitrary constant, A, omitted.

Plots of Bessel functions Equation (7) for some low values of n are shown below. Note that $J_0(0) = 1$ while $J_n(0) = 0$ for all $n > 0$.



Example: the solution of the equation $x^2 y'' + xy' + \{x^2 - 0\} y = 0$ is

$$y = c_1 J_0(x)$$

Example: the solution of the equation $x^2 y'' + xy' + \{x^2 - 1\} y = 0$ is

$$y = c_1 J_1(x)$$

Example: $$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Example: $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^4} + \frac{x^5}{2^7 \cdot 3} + \cdots$$

Zeros of Bessel functions

It is clear from (7) that the Bessel function $J_\nu(x)$ has an infinite amount of zeros for the half axis 0 < x < 1. Let us denote these zeros as $J_\nu(x) = 0$.

<< NumericalMath`BesselZeros`

BesselJZeros[0,5]

{2.40483,5.52008,8.65373,11.7915,14.9309}

Table 1: ROOTS of the FUNCTION $J_n(x)$ are given in the following table.

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178