Partial Differential Equations

Partial differential equations (PDEs) are equations involving functions of more than one variable and their partial derivatives with respect to those variables.

Most (but not all) physical models in science and engineering that result in partial differential equations are of at most second order and are often linear. In this lecture we shall have time to look at only a very small subset of second order linear partial differential equations (wave eqution).

Major Classifications of Common PDEs

A general second order linear partial differential equation in two Cartesian variables can be written as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

Three main types arise, based on the value of $D = B^2 - 4AC$ (a discriminator):

➔ Hyperbolic, wherever (x, y) is such that $D > 0$;

➔ Parabolic, wherever (x, y) is such that $D = 0$;

➔ Elliptic, wherever (x, y) is such that $D < 0$.

Among the most important partial differential equations in science and engineering are:

☞ The **wave equation**: $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$

or its one-dimensional special case $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$,

where u is the displacement and c is the speed of the wave.

$A = 1,\ C = -1,\ B = 0 \quad \Rightarrow\ D = 0 + 4 > 0$. This PDE is hyperbolic everywhere.

☞ The **heat** (or **diffusion**) **equation**: $\mu\rho\dfrac{\partial u}{\partial t} = K\,\nabla^2 u + \vec{\nabla} K \bullet \vec{\nabla} u$

a one-dimensional special case of which is

$$\frac{\partial u}{\partial t} = \frac{K}{\mu\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le R, \quad 0 \le t \le T$$

which is parabolic everywhere. u is the temperature, μ is the specific heat of the medium, ρ is the density, K is the thermal conductivity and c is the);

☞ The potential (or **Laplace's**) **equation**: $\nabla^2 u = 0$

a special case of which is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$A = C = 1,\ B = 0 \quad \Rightarrow\ D = 0 - 4 < 0$. This PDE is **elliptic** everywhere.

The complete solution of a PDE requires additional information, such as:

1- Initial conditions (values of the dependent variable and its first partial derivatives at $t = 0$), 2- Boundary conditions (values of the dependent variable on the boundary of the domain) or some combination of these conditions.

Harmonic function

A function $f(x, y)$ is **harmonic** if and only if $\nabla^2 f = 0$ everywhere inside a domain Ω.

Example

Is $u = e^x \sin y$ harmonic on L^2?

$$\frac{\partial u}{\partial x} = e^x \sin y \quad \text{and} \quad \frac{\partial u}{\partial y} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$\Rightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \qquad \forall (x, y)$$

Therefore **yes**, $u = e^x \sin y$ is harmonic on L^2.

The Wave Equation – Vibrating Finite String

The wave equation is

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$$

If $u(x, t)$ is the vertical displacement of a point at location x on a vibrating string at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If $u(x, y, t)$ is the vertical displacement of a point at location (x, y) on a vibrating membrane at time t, then the governing PDE is

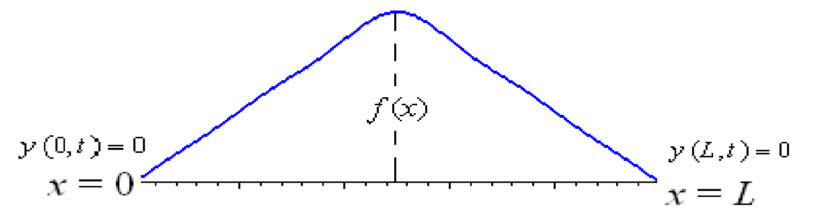
$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

or, in plane polar coordinates (r, θ), (appropriate for a circular drum),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

Example: of the Plucked String:

An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is displaced into the form $y = f(x)$ and is released from rest. Find the displacement $y(x,t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).



The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Boundary conditions:

$$\left.\begin{aligned} y(0,t) &= 0 \qquad \text{(I)} \\ y(L,t) &= 0 \qquad \text{(II)} \end{aligned}\right\} \quad \text{for} \quad t \geq 0$$

Initial conditions of string:

$$\left.\begin{aligned} y(x,0) &= f(x) \qquad \text{(III)} \\ \left.\frac{\partial y}{\partial t}\right|_{(x,0)} &= 0 \qquad \text{(IV)} \end{aligned}\right\} \quad \text{for} \quad 0 \leq x \leq L$$

Where (III) and (IV) represent the initial displacement and the initial transverse speed, respectively:

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for } 0 \leq x \leq L$$

Separation of Variables

Attempt a solution of the form:

$$y(x, t) = X(x)\, T(t)$$

Substitute $y(x, t) = X(x)\, T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \quad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} \qquad \text{(A)}$$

The left hand side of equation (A) is a function of t only. At any instant t (snap shot) it must have the same value at all values of x. Therefore the right hand side, which is a function of x only, must at any one instant have that same value at all values of x.

By a similar argument, the right hand side of this equation is a function of x only. At any location x it must have the same value at all times t. Therefore the left hand side, which is a function of t only, must at any one location have that same value at all times t. Thus both sides of this differential equation must be the same absolute constant, which we shall represent for now by k, then we have the pair of ODE"s

$$\frac{1}{X}\frac{d^2X}{dx^2} - k = 0 \qquad \text{and} \qquad \frac{1}{c^2T}\frac{d^2T}{dt^2} - k = 0$$

or $\left(\text{use } k = \lambda^2\right)$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \qquad \text{and} \qquad \frac{d^2T}{dt^2} - \lambda^2 c^2 T = 0$$

The above two equations have the following solutions:

➔ $\lambda^2 = 0$

$X = c_1 x + c_2$, and $T = c_3 T + c_4$

➔ $\lambda^2 > 0$

$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$, and $T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}$;

$X(x) = c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x)$ and $T(t) = c_3 \cosh(\lambda c t) + c_4 \sinh(\lambda c t)$

Note that: It is not possible for both ends of the string to be fixed for all time in the above two cases (unless we admit the trivial solution $y(x, t) \equiv 0$, a string that never moves from its equilibrium position). So, the following last condition will be acceptable:

➔ $\lambda^2 < 0$

$X = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$, and $T = c_3 e^{ic\lambda t} + c_4 e^{-ic\lambda t}$;

$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$ and $T(t) = C\cos(\lambda c t) + D\sin(\lambda c t)$

respectively, where A, B, C and D are arbitrary constants.

Now, back to our example boundary conditions. Consider the boundary condition (I):

$$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$$

For a non-trivial solution, this requires $X(0) = 0 \;\Rightarrow\; A = 0$.

Consider the boundary condition (II):

$$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L) = 0$$

$$\Rightarrow \quad B\sin(\lambda L) = 0 \quad \Rightarrow \quad \lambda_n = \frac{n\pi}{L}, \quad (n \in \mathbb{Z})$$

We now have a solution only for a discrete set of eigenvalues λ_n, with the corresponding eigenfunctions:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

Consider the initial condition (IV):

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = X(x)T'(0) = 0 \quad \forall x \quad \Rightarrow \quad T'(0) = 0$$

$$T'(t) = -C\lambda c\sin(\lambda c t) + D\lambda c\cos(\lambda c t) \quad \Rightarrow \quad T'(0) = D\lambda c = 0 \quad \Rightarrow \quad D = 0$$

Therefore our complete solution for $y(x,t)$ is now some linear combination of

$$y_n(x,t) = X_n(x)T_n(t) = C_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

And the general solution will be:

$$y(x,t)=\sum_n y_n(x,t)=\sum_n X_n(x)T_n(t)=\sum_n C_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right),\quad (n=1,2,3,\ldots)$$

There is one condition remaining to be satisfied.
The initial configuration (III) of the string:

$$y(x,0)=f(x)\ \text{ for }\ 0\le x\le L.$$

$$\Rightarrow\quad y(x,0)=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi x}{L}\right)=f(x)$$

This is precisely the Fourier sine series expansion of $f(x)$ on $[0, L]$!
From Fourier series theory, the coefficients C_n are

$$C_n=\frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du$$

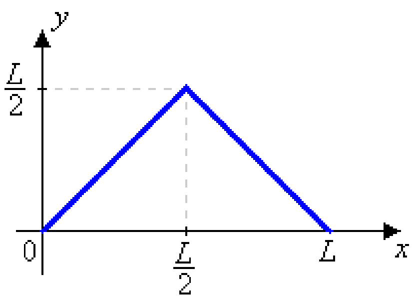
Therefore our complete solution is

$$\boxed{y(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)}$$

This solution is valid for any initial displacement function $f(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $f(0)=f(L)=0$.

If the initial displacement is itself sinusoidal $\left(f(x)=a\sin\left(\frac{n\pi x}{L}\right)\ \text{ for some }\ n\in\mathbb{N}\right)$, then the complete solution is a single term from the infinite series,

$$y_n(x,t)=a\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right).$$



Example: Suppose that the initial configuration is triangular:

$$y(x,0) = f(x) = \begin{cases} x & \left(0 \le x \le \tfrac{1}{2}L\right) \\ L-x & \left(\tfrac{1}{2}L < x \le L\right) \end{cases}$$

Then the Fourier sine coefficients are:

$$C_n = \frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du = \frac{2}{L}\int_0^{L/2} u\sin\left(\frac{n\pi u}{L}\right)du + \frac{2}{L}\int_{L/2}^{L}(L-u)\sin\left(\frac{n\pi u}{L}\right)du$$

$$= \frac{2}{L}\cdot\left(\frac{L}{n\pi}\right)^2\left\{\left[-\left(\frac{n\pi u}{L}\right)\cos\left(\frac{n\pi u}{L}\right) + \sin\left(\frac{n\pi u}{L}\right)\right]_0^{L/2} + \left[-\left(\frac{n\pi(L-u)}{L}\right)\cos\left(\frac{n\pi u}{L}\right) - \sin\left(\frac{n\pi u}{L}\right)\right]_{L/2}^{L}\right\}$$

$$= \frac{2L}{(n\pi)^2}\left\{\left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right) - (0-0) + (-0+0) - \left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)\right\}$$

$$= \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

But $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & (n \text{ even}) \\ \pm 1 & (n \text{ odd}) \end{cases}$

and

$$\sin\left(\frac{(2k-1)\pi}{2}\right) = (-1)^{k+1}, \quad (k \in \mathbb{N})$$

Therefore sum over the odd integer values of n only ($n = 2k - 1$).

$$C_k = \frac{4L}{\left((2k-1)\pi\right)^2}(-1)^{k+1}$$

and

$$\boxed{y(x,t) = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\left(\frac{(2k-1)\pi x}{L}\right)\cos\left(\frac{(2k-1)\pi c t}{L}\right)}$$

See the web page "www.engr.mun.ca/~ggeorge/5432/demos/ex431.html" for an animation of this solution

Example: An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is initially in its equilibrium state [$y(x, 0) = 0$ for all x] and is released with the initial velocity $\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \text{ and } t > 0$$

Boundary conditions:

$$\left.\begin{array}{ll} y(0,t) = 0 & \text{(I)} \\ y(L,t) = 0 & \text{(II)} \end{array}\right\} \quad \text{for} \quad t \geq 0$$

Initial conditions of string:

$$\left.\begin{array}{ll} y(x,0) = 0 & \text{(III)} \\ \left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x) & \text{(IV)} \end{array}\right\} \quad \text{for} \quad 0 \leq x \leq L$$

As before, the method of the **separation of variables gives** the general solutions as:

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

where A, B, C and D are arbitrary constants.

Consider the boundary condition (I):

$$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$$

For a non-trivial solution, this requires $X(0) = 0 \Rightarrow A = 0$.

Consider the boundary condition (II):

$$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \quad \Rightarrow X(L) = 0$$

$$\Rightarrow B\sin(\lambda L) = 0 \Rightarrow \lambda_n = \frac{n\pi}{L}, \quad (n \in \mathbb{Z})$$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

and

$$y_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)T_n(t), \quad (n = 1, 2, 3, \ldots)$$

So far, the solution has been identical to the previous example.
Consider the initial condition (III) :

$$y(x,0) = 0 \Rightarrow X(x)T(0) = 0 \quad \forall x \Rightarrow T(0) = 0$$

The initial value problem for $T(t)$ is now

$$T'' + \lambda^2 c^2 T = 0, \quad T(0) = 0, \qquad \text{where} \quad \lambda = \frac{n\pi}{L}$$

the solution to which is

$$T_n(t) = C_n \sin\left(\frac{n\pi c\, t}{L}\right), \qquad (n \in \mathbb{N})$$

Our eigenfunctions for y are now

$$y_n(x,t) = X_n(x) T_n(t) = C_n \sin\left(\frac{n\pi\, x}{L}\right) \sin\left(\frac{n\pi c\, t}{L}\right), \qquad (n \in \mathbb{N})$$

Differentiate term by term and impose the initial velocity condition (IV):

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi\, x}{L}\right) = g(x)$$

which is just the Fourier sine series expansion for the function $g(x)$.
The coefficients of the expansion are

$$C_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(u) \sin\left(\frac{n\pi\, u}{L}\right) du$$

which leads to the complete solution

$$\boxed{y(x,t) = \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^L g(u) \sin\left(\frac{n\pi\, u}{L}\right) du \right) \sin\left(\frac{n\pi\, x}{L}\right) \sin\left(\frac{n\pi c\, t}{L}\right)}$$

This solution is valid for any initial velocity function g(x) that is continuous with a piece-wise continuous derivative on $[0, L]$ with g(0) = $g(L)$ = 0.

The solutions for Examples 1 and 2 may be superposed.
Let $y_1(x, t)$ be the solution for initial displacement $f(x)$ and zero initial velocity.
Let $y_2(x, t)$ be the solution for zero initial displacement and initial velocity $g(x)$.

Then $y(x, t) = y_1(x, t) + y_2(x, t)$ satisfies the wave equation
(the sum of any two solutions of a linear homogeneous PDE is also a solution),
and satisfies the boundary conditions $y(0, t) = y(L, t) = 0$.

$y(x, 0) = y_1(x, 0) + y_2(x, 0) = f(x) + 0$,
which satisfies the condition for initial displacement $f(x)$.

$y_t(x, 0) = y_{1t}(x, 0) + y_{2t}(x, 0) = 0 + g(x)$,
which satisfies the condition for initial velocity $g(x)$.

Therefore the sum of the two solutions is the complete solution for initial displacement $f(x)$ **and** initial velocity g(x):

$$\boxed{\begin{aligned} y(x,t) &= \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(u) \sin\left(\frac{n\pi\, u}{L}\right) du \right) \sin\left(\frac{n\pi\, x}{L}\right) \cos\left(\frac{n\pi c\, t}{L}\right) \\ &+ \frac{2}{\pi c} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^L g(u) \sin\left(\frac{n\pi\, u}{L}\right) du \right) \sin\left(\frac{n\pi\, x}{L}\right) \sin\left(\frac{n\pi c\, t}{L}\right) \end{aligned}}$$

See the following web for more examples
http://tutorial.math.lamar.edu/Classes/DE/SolvingHeatEquation.aspx

Example: Find a solution to the following partial differential equation that will also satisfy the boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

Okay the first thing we technically need to do here is apply separation of variables. Even though we did that in the previous section let's recap here what we did.

First, we assume that the solution will take the form,

$$u(x,t) = \varphi(x)G(t)$$

and we plug this into the partial differential equation and boundary conditions. We separate the equation to get a function of only t on one side and a function of only x on the other side and then introduce a separation constant. This leaves us with two ordinary differential equations.

We did all of this in the wave equation and the two ordinary differential equations are,

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

The time dependent equation can really be solved at any time, but since we don't know what λ is yet let's hold off on that one. Also note that in many problems only the boundary value problem can be solved at this point so don't always expect to be able to solve either one at this point.

The spatial equation is a boundary value problem and we know from our work in the previous chapter that it will only have non-trivial solutions (which we want) for certain values of λ , which we'll recall are called eigenvalues. Once we have those we can determine the non-trivial solutions for each λ , *i.e.* eigenfunctions.

Now, we actually solved the spatial problem,

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

in the wave equation. So, because we've solved this once for a specific L and the work is not all that much different for a general L we're not going to be putting in a lot of explanation here and if you need a reminder on how something works or why we did something go back to the solution of the wave equation.

We've got three cases to deal with so let's get going.

$\underline{\lambda > 0}$

In this case we know the solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$\varphi(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

Now applying the second boundary condition, and using the above result of course, gives,

$$0 = \varphi(L) = c_2 \sin\left(L\sqrt{\lambda}\right)$$

Now, we are after non-trivial solutions and so this means we must have,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \quad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Note that we don't need the c_2 in the eigenfunction as it will just get absorbed into another constant that we'll be picking up later on.

$\underline{\lambda = 0}$

The solution to the differential equation in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the boundary conditions gives,

$$0 = \varphi(0) = c_1 \qquad 0 = \varphi(L) = c_2 L \qquad \Rightarrow \qquad c_2 = 0$$

So, in this case the only solution is the trivial solution and so $\lambda = 0$ is not an eigenvalue for this boundary value problem.

$\underline{\lambda < 0}$

Here the solution to the differential equation is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

and applying the second gives,

$$0 = \varphi(L) = c_2 \sinh\left(L\sqrt{-\lambda}\right)$$

So, we are assuming $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$ and this means $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$.
We therefore we must have $c_2 = 0$ and so we can only get the trivial solution in this case.

Therefore, there will be no negative eigenvalues for this boundary value problem. The complete list of eigenvalues and eigenfunctions for this problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Now let's solve the time differential equation,

$$\frac{dG}{dt} = -k\lambda_n G$$

and note that even though we now know λ we're not going to plug it in quite yet to keep the mess to a minimum. We will however now use λ_n to remind us that we actually have an infinite number of possible values here.

This is a simple linear (and separable for that matter) 1st order differential equation and so we'll let you verify that the solution is,

$$G(t) = c\mathrm{e}^{-k\lambda_n t} = c\mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Okay, now that we've gotten both of the ordinary differential equations solved we can finally write down a solution. Note however that we have in fact found infinitely many solutions since there are infinitely many solutions (*i.e.* eigenfunctions) to the spatial problem.
Our product solutions are then,

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1,2,3,\ldots$$

We've denoted the product solution u_n to acknowledge that each value of n will yield a different solution. Also note that we've changed the c in the solution to the time problem to B_n to denote the fact that it will probably be different for each value of n as well and because had we kept the c_2 with the eigenfunction we'd have absorbed it into the c to get a single constant in our solution.

So, there we have it. The function above will satisfy the heat equation and the boundary condition of zero temperature on the ends of the bar.

Now, let's extend the idea out that we used in the second part of the previous example a little to see how we can get a solution that will satisfy any sufficiently nice initial condition. The Principle of Superposition is, of course, not restricted to only two solutions. For instance the following is also a solution to the partial differential equation.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and notice that this solution will not only satisfy the boundary conditions but it will also satisfy the initial condition,

$$u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This solution will satisfy any initial condition that can be written in the form,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

To calculate the constant B_n one has to use the Fourier trick on the interval $0 \le x \le L$, and the coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

So, we finally can completely solve a partial differential equation.

Example 2 Solve the following heat problem for the given initial conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

(a) $$f(x) = 6\sin\left(\frac{\pi x}{L}\right)$$

(b) $$f(x) = 12\sin\left(\frac{9\pi x}{L}\right) - 7\sin\left(\frac{4\pi x}{L}\right)$$

Solution

(a) This is actually easier than it looks like. All we need to do is choose $n = 1$ and $B_1 = 6$ in the product solution above to get,

$$u(x,t) = 6\sin\left(\frac{\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{\pi}{L}\right)^2 t}$$

and we've got the solution we need. This is a product solution for the first example and so satisfies the partial differential equation and boundary conditions and will satisfy the initial condition since plugging in $t = 0$ will drop out the exponential.

(b) This is almost as simple as the first part. Recall from the Principle of Superposition that if we have two solutions to a linear homogeneous differential equation (which we've got here) then their sum is also a solution. So, all we need to do is choose *n* and B_n as we did in the first part to get a solution that satisfies each part of the initial condition and then add them up. Doing this gives,

$$u(x,t) = 12\sin\left(\frac{9\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{9\pi}{L}\right)^2 t} - 7\sin\left(\frac{4\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{4\pi}{L}\right)^2 t}$$

We'll leave it to you to verify that this does in fact satisfy the initial condition and the boundary conditions.