Integral Equations (Chapter 16)

Words from Arfken's book: Why do we bother about integral equations? After all, the differential equations have done a rather good job of describing our physical world so far. There are several reasons for introducing integral equations here."

We have placed considerable emphasis on the solution of differential equations **subject to particular boundary conditions**. For instance, the boundary condition at r = 0 determines whether the Neumann function $N_n(r)$ is present when Bessel's equation is solved. The boundary condition for $r\rightarrow\infty$ determines whether the $I_n(r)$ is present in our solution of the modified Bessel equation. The integral equation relates the unknown function not only to its values at neighboring points (derivatives) but also to its values throughout a region, including the boundary. In a very real sense the boundary conditions are built into the integral equation rather than imposed at the final stage of the solution. It can be seen that the Green's function are constructed, that the form of the function depends on the values on the boundary. The integral equation, then, is compact and may turn out to be a more convenient or powerful form than the differential equation. Mathematical problems such as existence, uniqueness, and completeness may often be handled more easily and elegantly in integral form. Finally, whether or not we like it, there are some problems, such as some scattering, diffusion and transport phenomena that cannot be represented by differential equations. If we wish to solve such problems, we are forced to handle integral equations. Finally, an integral equation may also appear as a matter of deliberate choice based on convenience or the need for the mathematical power of an integral equation formulation.

Classification

• Let *g(x)* be a given function, $K(x, t)$ be a given function of two variables, and *F(x)* be an unknown function. The **Volterra equation of the first kind** reads

$$g(x)=\int_a^x dt\, K(x,t)f(t)\,.$$

• **Volterra equation of the second kind** is

$$g(x)=f(x)+\int_a^x dt\, K(x,t)f(t)\,.$$

• The **Fredholm equation of the first kind** reads

$$g(x)=\int_a^b dt\, K(x,t)f(t)\,.$$

• **Fredholm equation of the second kind** is

$$g(x)=f(x)+\int_a^b dt\, K(x,t)f(t)\,.$$

1- Introduction (Classifications)

The most general type of **integral equation** is written in the form,

$$h(x)u(x)=f(x)+\lambda\int_a^{b(x)}K(x,y)u(y)dy\ . \tag{1}$$

We are given functions $h(x)$, $u(x)$, $K(x, y)$, and wish to determine $f(x)$. The quantity λ is a parameter (eigenvalue), which may be complex in general. The function $K(x,y)$ is called the **kernel** of the integral equation.

We shall assume that $h(x)$ and $u(x)$ are defined and continuous on the interval $a \le x \le b(x)$, and that the kernel is defined and continuous on $a \le x \le b$ and $a \le y \le b$. Here we will concentrate on the problem for real variables x and y. The functions may be complex-valued, although we will sometimes simplify the discussion by considering real functions. However, many of the results can be generalized in fairly obvious ways, such as relaxation to piecewise continuous functions, and generalization to multiple dimensions.

➔ If the **limits of integration are fixed**, we call the equation a **Fredholm** equation; if **one limit is variable**, it is a **Volterra** equation.

➔ If the **unknown function** appears **only under the integral** sign, we label it **first kind**. If it appears both **inside and outside** the integral, it is labeled **second kind**.

The Volterra integral equation, is derived when the general equation (1) has the property $b(x)=x$. When $h(x)=0$ we incur a Volterra equation of the **first kind**, shown as the following,

$$-f(x)=\int_a^x K(x,y)u(y)dy\ . \tag{2}$$

And a Volterra equation of the **second kind** is shown when $h(x)=1$.

$$u(x)=f(x)+\int_a^x K(x,y)u(y)dy \tag{3}$$

When $f(x)=0$, the equation is said to be homogenous.

The Fredholm integral equation is derived from the general equation (1) when $b(x)=b$, (where b is a constant). When $h(x)=0$, and $h(x)=1$, Fredholm equations of the **first** and **second kind** are obtained as follows,

$$-f(x)=\int_a^b K(x,y)u(y)dy \tag{4}$$

and

$$u(x)=f(x)+\int_a^b K(x,y)u(y)dy \tag{5}$$

Both the Volterra integral equation and the Fredholm integral equation are very similar in the way that they are presented. The only difference that distinguishes these integral equations is the fact that the limits of integration are different, as mentioned above.

2- Integral Transforms

If $h(x)=0$, we can take $\lambda = -1$ without loss of generality and obtain the integral equation:

$$g(x)=\int_a^b K(x,y)f(y)dy. \tag{2}$$

This is called a **Fredholm equation of the first kind** or an **integral transform**. Particularly important examples of integral transforms include the Fourier transform and the Laplace transform.

3- Two useful Identities

To change from differential equation to integral equation, or vise versa, we need the following identities:

☞ $$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)dt = \int_{G(x)}^{H(x)} \frac{\partial F(x,t)}{\partial x}dt + F\left[x,H(x)\right]\frac{dH(x)}{dx} - F\left[x,G(x)\right]\frac{dG(x)}{dx} \tag{I}$$

Comment: If H and G are constants, we can have $\frac{d}{dx}\int_G^H F(x,t)dt = \int_G^H \frac{\partial F(x,t)}{\partial x}dt$

☞ $$\int_a^x y(t)dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}y(t)dt \tag{II}$$

Example: use identity (I) to calculate:

a- $\frac{d}{dx}\int_0^x \sin(x-t)y(t)dt$, b- $\frac{d}{dx}\int_0^x \cos(x-t)y(t)dt$

Answer:

$$a-\frac{d}{dx}\int_0^x \underbrace{\sin(x-t)y(t)}_{\text{F(x,t)}}dt = \int_0^x \cos(x-t)y(t)dt + \underbrace{\sin(x-x)}_{=0}y(x)\frac{dx}{dx} - \sin(x-0)y(x)\underbrace{\frac{d0}{dx}}_{=0}$$

$$=\int_0^x \cos(x-t)y(t)dt$$

$$b-\frac{d}{dx}\int_0^x \underbrace{\cos(x-t)y(t)}_{\text{F(x,t)}}dt = -\int_0^x \sin(x-t)y(t)dt + \underbrace{\cos(x-x)}_{=1}y(x)\frac{dx}{dx} - \cos(x-0)y(x)\underbrace{\frac{d0}{dx}}_{=0}$$

$$=-\int_0^x \sin(x-t)y(t)dt + y(x)$$

Example: use the identity (II) to express $\int_0^x y(t)dt^2$.

Answer:

$$\int_0^x y(t)dt^2 = \int_0^x (x-t)y(t)dt$$

Example: Show that the function $y(x) = \dfrac{1}{(1+x^2)^{3/2}}$ is a solution of the Volterra integral equation

$$y(x) = \frac{1}{(1+x^2)} - \int_0^x \frac{t}{(1+x^2)} y(t)dt$$

Answer: In the second term, if we put $y(t) = \dfrac{1}{(1+t^2)^{3/2}}$, then we can find

$$y(x) = \frac{1}{(1+x^2)} - \int_0^x \frac{t}{(1+x^2)} y(t)dt = \frac{1}{(1+x^2)} - \frac{1}{(1+x^2)}\int_0^x \frac{t}{(1+t^2)^{3/2}}dt = \frac{1}{(1+x^2)^{3/2}}$$

$$y(x) = \frac{1}{(1+x^2)} - \frac{1}{(1+x^2)}\int_0^x \frac{t}{(1+t^2)^{3/2}} \,dt \ \text{// Simplify}$$

$$= \frac{1}{(1+x^2)^{3/2}}$$

Example: Show that the function $y(x) = xe^x$ is a solution of the Volterra integral equation

$$y(x) = \sin(x) + 2\int_0^x \cos(x-t)y(t)dt$$

Answer: In the second term, if we put $y(t) = te^t$, then we can find

$$y(x) = \sin(x) + 2\int_0^x \cos(x-t)t\,e^t dt = xe^x$$

With MATHEMATICA

$$y(x) = \text{Sin}[x] + 2\int_0^x \text{Cos}[x-t]\, t\, e^t \,dt \ \text{// Simplify} \ = e^x x$$

Example: Show that the function $y(x) = \cos(2x)$ is a solution of the Volterra integral equation

$$y(x) = \cos(x) + 3\int_0^\pi K(x,t)y(t)dt$$

where $K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x < t \\ \sin t \cos x, & 0 \le x < \pi \end{cases}$

Answer: $y(x) = \cos(2x) \quad \Rightarrow \quad y(t) = \cos(2t)$

$$y(x) = \cos(x) + 3\left[\sin(x)\int_0^x \sin(t)\cos(2t)dt + \cos(x)\int_x^\pi \sin(t)\cos(2t)dt\right] = \cos(2x)$$

With MATHEMATICA

$$\text{y} = \text{Cos}[x] + 3\left(\text{Cos}[x]\int_0^x \text{Sin}[t]\,\text{Cos}[2\,t]\,dt + \text{Sin}[x]\int_x^\pi \text{Cos}[t]\,\text{Cos}[2\,t]\,dt\right) \ \text{// Simplify}$$

$\cos(2x)$

H.W. Show that the function $y(x)=(x+1)^2$ is a solution of the Volterra integral equation

$$y(x)=e^{-x}+2x+\int_0^x e^{t-x}\,y(t)\,dt$$

4- Conversion of ordinary differential equations into integral equations

Example: Convert

$$y''(x)+y(x)=0$$

with initial conditions $y(0)=1$, $y'(0)=0$ into integral equation.

Answer:

1- Use the original equation $y''(x)+y(x)=0$, rearrange and integrate it in the form $y''(z)=-y(z)$ to give

$$\int_0^x y''(z)dz=-\int_0^x y(z)dz \quad \Rightarrow \left[y'(z)\right]_0^x=-\int_0^x y(z)dz \quad \Rightarrow y'(x)-\underbrace{y'(0)}_{0}=-\int_0^x y(z)dz$$

$$\Rightarrow y'(x)=-\int_0^x y(z)dz$$

2- Integrate both sides of the last equation,

$$\int_0^x y'(z)dz=-\int_0^x y(z)dz^2 \quad \Rightarrow \left[y(z)\right]_0^x=-\int_0^x y(z)dz^2 \quad \Rightarrow y(x)-\underbrace{y(0)}_{1}=-\int_0^x (x-z)y(z)dz$$

Then the required integral equation is $y(x)=1-\int_0^x (x-z)y(z)dz$.

Example: Show that the function $y(x)=\cos(x)$ is a solution of the Volterra integral equation

$$y(x)=1-\int_0^x (x-t)y(t)dt$$

Answer: Given $y(x)=\cos(x) \Rightarrow y(t)=\cos(t)$, and

$$y(x)=1-\int_0^x (x-t)\cos(t)dt=1-x\sin(x)+[-1+\cos(x)+x\sin(x)]=\underline{\cos(x)}$$

With MATHEMATICA (as an integral equation)

```
In[12]:= 1 - ∫_0^x (x - z) Cos[z] dz

Out[12]= Cos[x]
```

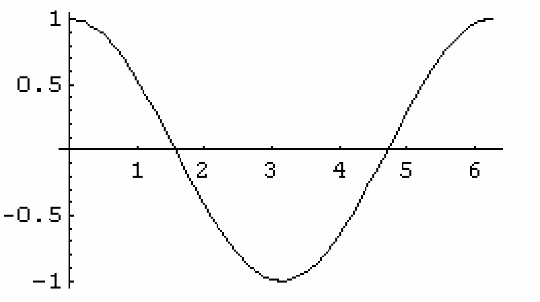
With MATHEMATICA (as a differential equation)

```
DE = ∂{x,2} y[x] + k y[x] == 0
 k y[x] + y''[x] == 0
 sol1=DSolve[{DE,y[0]== 1, y'[0]== 0},y,x]            /.k→1
{{y → Function [{x}, Cos [√1 x]]}}

Table[ {x, y[x], y'[x]} /. sol1[[1]], {x,0,2 π,0.5} ] // TableForm
```

0	1	0
0.5	0.877583	-0.479426
1.	0.540302	-0.841471
1.5	0.0707372	-0.997495
2.	-0.416147	-0.909297
2.5	-0.801144	-0.598472
3.	-0.989992	-0.14112
3.5	-0.936457	0.350783
4.	-0.653644	0.756802
4.5	-0.210796	0.97753
5.	0.283662	0.958924
5.5	0.70867	0.70554
6.	0.96017	0.279415

```
Plot[y[x]/.sol1[[1]],{x,0,2 π}]
```



5- Successive approximation (Neumann solution)

We will use the expression:

$$\varphi_n(x) = f(x) + \int_0^x K(t,x)\, \varphi_{n-1}(t)dt, \qquad n = 1,2,3,\cdots$$

and define $\varphi_o(x) = f(x)$.

Example: Use the successive approximation method to find the solution of the integral equation:

$$y(x) = 1 - \int_0^x (x-t)y(t)dt$$

Answer:

Define the 1st approximation: $\varphi_o(x) = 1$ as the first term in the above equation $\Rightarrow \varphi_o(t) = 1$

Then $\varphi_1(x)$ will be given by:

$$\varphi_1(x) = 1 - \int_0^x (x-t)\, \varphi(t)dt = 1 - \int_0^x (x-t)\, \varphi_0(t)dt$$

$$= 1 - \int_0^x (x-t)[1]dt = 1 - \frac{x^2}{2}$$

With $\varphi_1(t) = 1 - \frac{t^2}{2}$, then $\varphi_2(x)$ will be given by:

$$\varphi_2(x) = 1 - \int_0^x (x-t)\varphi_1(t)dt = 1 - \int_0^x (x-t)\left[1 - \frac{t^2}{2}\right]dt = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

With $\varphi_2(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24}$, then $\varphi_3(x)$ will be given by:

$$\varphi_3(x) = 1 - \int_0^x (x - t)\,\varphi_2(t)dt$$

$$= 1 - \int_0^x (x - t)\left[1 - \frac{t^2}{2} + \frac{t^4}{24}\right]dt = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

Which is the same as: $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots, \quad |x| < 1$

Example: Use the successive approximation method to find the solution of the integral equation:

$$y(x) = 1 - \int_0^x (t - x)y(t)dt$$

```
In[1]:= 1 - ∫_0^x (z - x) Cosh[z] ⅆz
Out[1]= Cosh[x]
```

Answer:

Define the 1st approximation: $\varphi_o(x) = 1$ as the first term in the above equation $\Rightarrow \varphi_o(t) = 1$

Then $\varphi_1(x)$ will be given by:

$$\varphi_1(x) = 1 - \int_0^x (t - x)\varphi(t)dt = 1 - \int_0^x (t - x)\varphi_0(t)dt$$

$$= 1 - \int_0^x (t - x)[1]dt = 1 + \frac{x^2}{2}$$

With $\varphi_1(t) = 1 + \frac{t^2}{2}$, then $\varphi_2(x)$ will be given by:

$$\varphi_2(x) = 1 - \int_0^x (t - x)\varphi_1(t)dt = 1 - \int_0^x (t - x)\left[1 + \frac{t^2}{2}\right]dt = 1 + \frac{x^2}{2} + \frac{x^4}{24}$$

With $\varphi_2(t) = 1 + \frac{t^2}{2} + \frac{t^4}{24}$, then $\varphi_3(x)$ will be given by:

$$\varphi_3(x) = 1 - \int_0^x (t - x)\varphi_2(t)dt$$

$$= 1 - \int_0^x (t - x)\left[1 + \frac{t^2}{2} + \frac{t^4}{24}\right]dt = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$$

Which is the same as: $\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$

Example: Use Neumann series method to solve the integral equation:

$$y(x) = x + \lambda \int_0^1 x\, z\, y(z)\, dz$$

Answer: Start by $y_0(x) = x \Rightarrow y_0(z) = z$, we will have:

$$y(x) = x + \lambda \int_0^1 x\, z\, y_o(z)\, dz = x + \lambda \int_0^1 x\, z^2\, dz = x + x\left(\frac{\lambda}{3}\right)$$

Repeating the procedure once more, with $y_1(z) = z + z\left(\frac{\lambda}{3}\right)$, we obtain:

$$y_2(x) = x + \lambda \int_0^1 xzy_1(z)\, dz$$

$$= x + \lambda \int_0^1 xz\left(z + \frac{\lambda z}{3}\right) dz = x + \left(\frac{\lambda}{3} + \frac{\lambda^2}{9}\right) x.$$

For this simple example, it is easy to see that by continuing this process the solution is obtained as

$$y(x) = x + \left[\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + \cdots\right] x.$$

Clearly the expression in brackets is an infinite geometric series with first term $\lambda/3$ and common ratio $\lambda/3$. Thus, *provided* $|\lambda| < 3$, this infinite series converges to the value $\lambda/(3-\lambda)$, and the solution is

$$y(x) = x + \frac{\lambda x}{3-\lambda} = \frac{3x}{3-\lambda}.$$

[The geometric series: $a + ar + ar^2 + \cdots = \frac{a}{1-r}$, where a is the first term and r is the common ration.]

Example: Derive a Fredholm integral equation corresponding to:

$$y''(x) - \lambda y(x) = 0, \qquad a < x < b$$

with initial conditions $y(a) = 0$, $y(b) = 0$.

Answer: Given that

$$y''(x) - \lambda y(x) = 0 \quad \Rightarrow \quad y''(x) = \lambda y(x)$$

with initial conditions: $y(a) = 0$, $y(b) = 0$

1- Integrate the equation $y''(x) = \lambda y(x)$, gives

$$\int_a^x y''(z)dz = \lambda \int_a^x y(z)dz \quad \Rightarrow \left[y'(z) \right]\Big|_a^x = \lambda \int_a^x y(z)dz \quad \Rightarrow y'(x) - c_1 = \lambda \int_0^x y(z)dz$$

$$\Rightarrow y'(x) = \lambda \int_0^x y(z)dz + c_1$$

2- Integrate both sides of the last equation,

$$\int_a^x y'(z)dz = \lambda \int_a^x y(z)dz^2 + c_1 \quad \Rightarrow \left[y(z) \right]\Big|_a^x = \lambda \int_a^x y(z)dz^2 + c_1 \int_a^x dz$$

$$\Rightarrow y(x) - y(a) = \lambda \int_a^x (x - z) y(z)dz + c_1 x + c_2$$

Use the initial conditions $y(a) = 0$

$$y(a) - y(a) = \lambda \int_a^a (a - z) y(z)dz + c_1 a + c_2 = 0 + c_1 a + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 a$$

$$\Rightarrow y(x) = \lambda \int_a^x (x - z) y(z)dz + c_1 (x - a)$$

Use the initial condition $y(b) = 0$.

$$y(b) = \lambda \int_a^b (b - z) y(z)dz + c_1 (b - a) = 0 \quad \Rightarrow \quad c_1 = \frac{\lambda}{a - b} \int_a^b (b - z) y(z)dz$$

then:

$$y(x) = \lambda \int_a^x (x - z) y(z)dz + \lambda \frac{x - a}{a - b} \int_a^b (b - z) y(z)dz$$

To change the integral equation to Fredholm type, we have to split the limits of the second integral from $[a,b]$ to $[a,x]$ and $[x,b]$, to have the final expression:

$$y(x) = \lambda \int_a^x K_1(x,t) y(t)dt + \lambda \int_x^b K_2(x,t) y(t)dt = \lambda \int_a^b K(x,t) y(t)dt,$$

With*

$$K(x,t)=\begin{cases} K_1=\dfrac{(x-b)(t-a)}{(b-a)}, & a\le t\le x \\ K_2=\dfrac{(t-b)(x-a)}{(b-a)}, & x\le t\le b \end{cases},$$

[*To calculate the Kernel we use: $(x-z)+\frac{x-a}{a-b}(b-z)=\frac{(x-b)(z-a)}{b-a}$]

H.W. Do example 16.1.2 in Arfken's book page 1009

Example: Convert

$$y''(x)-3y'(x)+2y(x)=4\sin(x)$$

with initial conditions $y(0)=1$, $y'(0)=-2$ into a Volterra integral equation of the second kind. Conversely, derive the original differential equation with initial conditions from the integral equation obtained.

Answer: Given

$$y''(x)-3y'(x)+2y(x)=4\sin(x) \quad\Rightarrow\quad y''(x)=4\sin(x)+3y'(x)-2y(x)$$

with initial conditions: $y(0)=1$, $y'(0)=-2$. Integrate the equation $y''(z)=4\sin(z)+3y'(z)-2y(z)$, gives

$$\int_0^x y''(z)dz=4\int_0^x\sin(z)dz+3\int_0^x y'(z)dz-2\int_0^x y(z)dz$$

$$\Rightarrow \left[y'(z)\right]\Big|_0^x=4\left[-\cos(z)\right]_0^x+3\left[y(z)\right]_0^x-2\int_0^x y(z)dz$$

$$\Rightarrow y'(x)-\underbrace{y'(0)}_{-2}=4\left[-\cos(x)-1\right]+3[y(x)-\underbrace{y(0)}_{1}]-2\int_0^x y(z)dz$$

$$\Rightarrow y'(x)=-1-4\cos(x)-3y(x)-2\int_0^x y(z)dz$$

2- Integrate both sides of the last equation,

$$\int_0^x y'(x)dz=-\int_0^x dz-4\int_0^x\cos(z)dz+3\int_0^x y(z)dz-2\int_0^x y(z)dz^2$$

$$\Rightarrow \left[y(z)\right]\Big|_0^x=-x-4\left[\sin(z)\right]_0^x+3\int_0^x y(z)dz-2\int_0^x y(z)dz^2$$

$$\Rightarrow y(x)-\underbrace{y(0)}_{1}=-x-4\sin(x)+3\int_0^x y(z)dz-2\int_0^x (x-z)y(z)dz$$

Then $y(x)=1-x-4\sin(x)+\int_0^x[3-2(x-z)]y(z)dz$ is the required Volterra equation of the second kind.

Example: Solve the integral equation: $y(x)=e^{-x}+2x+\int_0^x e^{t-x}y(t)dt$ by taking the derivative with the boundary conditions $y(0)=1$.

Answer: Given $y(x)=e^{-x}+2x+\int_0^x e^{t-x}y(t)dt$, we have

$$y'(x)=-e^{-x}+2-\int_0^x e^{t-x}y(t)dt+e^{x-x}y(x)\frac{dx}{dx}$$
$$=-e^{-x}+2-\int_0^x e^{t-x}y(t)dt+y(x)=2x+2$$

$y'(x)=(2x+2) \quad \Rightarrow \quad y(x)=x^2+2x+\text{constant}$
The initial condition is $y(0)=1 \Rightarrow y(0)=0+0+\text{constant} = 1 \quad \Rightarrow \quad \text{constant}=1$,
thus the solution is $y(x)=\underline{(x+1)^2}$.

H.W. Solve the integral equation: $y(x)=1-x-4\sin(x)+\int_0^x[3-2(x-z)]y(z)dz$ by taking the derivative. Check the boundary conditions $y(0)=1$, $y'(0)=-2$.

Answer: check the first derivative: $y'(x)=-1-4\cos(x)+3y(x)-2\int_0^x y(z)dz$

And the second: $y''(x)-4\sin(x)-3y'(x)+2y(x)=0$.

Example (Arfken 16.3.1): Find the solution of the integral equation

$$\varphi(x)=x+\frac{1}{2}\int_{-1}^{1}(t-x)\varphi(t)dt$$

Answer:

Define the 1st approximation: $\varphi_o(x)=x$ as the first term in the above equation $\Rightarrow \varphi_o(t)=t$
Then $\varphi_1(x)$ will be given by:

$$\varphi_1(x)=x+\frac{1}{2}\int_{-1}^{1}(t-x)\varphi(t)dt=x+\frac{1}{2}\int_{-1}^{1}(t-x)\varphi_0(t)dt$$
$$=x+\frac{1}{2}\int_{-1}^{1}(t-x)[t]dt=x+\frac{1}{3}$$

With $\varphi_1(t)=t+\frac{1}{3}$, then $\varphi_2(x)$ will be given by:

$$\varphi_2(x)=x+\frac{1}{2}\int_{-1}^{1}(t-x)\varphi_1(t)dt=x+\frac{1}{2}\int_{-1}^{1}(t-x)\left[t+\frac{1}{3}\right]dt=x+\frac{1}{3}-\frac{x}{3}$$

With $\varphi_2(t)=t+\frac{1}{3}-\frac{t}{3}$, then $\varphi_3(x)$ will be given by:

$$\varphi_3(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)\varphi_2(t)dt$$
$$= x + \frac{1}{2}\int_{-1}^{1}(t-x)\left[t + \frac{1}{3} - \frac{t}{3}\right]dt = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2}$$

And $\varphi_{2n}(x)$ will be given by:

$$\varphi_{2n}(x) = x + \underbrace{\sum_{s=1}^{n}(-1)^{s-1}3^{-s}}_{1/4} - x\underbrace{\sum_{s=1}^{n}(-1)^{s-1}3^{-s}}_{1/4} \xrightarrow[n\to\infty]{} \frac{3}{4}x + \frac{1}{4}$$

6- Degenerate Kernels (Separable kernels)

The most straightforward integral equations to solve are Fredholm equations with separable (or degenerate) kernels. A kernel is separable if it has the form

$$K(x,z) = \sum_{i=1}^{n} \phi_i(x)\psi_i(z), \quad (23.8)$$

where $\phi_i(x)$ and $\psi_i(z)$ are respectively functions of x only and of z only and the, number of terms in the sum, n, is finite.

Let us consider the solution of the (inhomogeneous) Fredholm equation of the second kind,

$$y(x) = f(x) + \lambda \int_a^b K(x,z)y(z)\,dz, \quad (23.9)$$

which has a separable kernel of the form (23.8). Writing the kernel in its separated form, the functions $\phi_i(x)$ may be taken outside the integral over z to obtain

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} \phi_i(x) \int_a^b \psi_i(z)y(z)\,dz.$$

Since the integration limits a and b are constant for a Fredholm equation, the integral over z in each term of the sum is just a constant. Denoting these constants by

$$c_i = \int_a^b \psi_i(z)y(z)\,dz, \quad (23.10)$$

the solution to (23.9) is found to be

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} c_i\phi_i(x), \quad (23.11)$$

where the constants c_i can be evaluated by substituting (23.11) into (23.10).

Example: For the integral equation: $y(x) = \lambda \int_0^1 e^{x+t} y(t)dt$, Find λ and then $y(x)$, using the degenerate kernel approximation. Is there any restriction on λ?

Answer:

$$y(x) = \lambda \int_0^1 e^{x+t} y(t)dt = \lambda e^x \int_0^1 e^t y(t)dt = \lambda e^x C,$$

Where,

$$C = \int_0^1 e^t y(t)dt, \qquad (A)$$

$$\Rightarrow \quad y(t) = \lambda e^t C, \qquad (B)$$

Using (B) in (A), one finds:

$$C = \int_0^1 e^t y(t)dt = \int_0^1 e^t \lambda e^t C dt = \lambda C \int_0^1 e^{2t} dt = \lambda C \left.\frac{e^{2t}}{2}\right|_0^1 = \frac{\lambda C}{2}(e^2 - 1)$$

Then

$$C\left[1 - \frac{\lambda}{2}(e^2 - 1)\right] = 0.$$

Since $C \neq 0$, then $\left[1-\frac{\lambda}{2}(e^2-1)\right]=0 \Rightarrow \lambda=\frac{2}{(e^2-1)}$

So, no restriction on λ.

$$y(x)=\lambda e^x C=\left\{\frac{2C}{\left(e^2-1\right)}\right\}e^x$$

With the boundary condition $y(0)=2 \Rightarrow C=e^2-1$ and the integral equation $y(x)=\lambda\int_0^1 e^{x+t}y(t)dt$ has the form $y(x)=2e^x$.

Example: Solve the integral equation

$$y(x)=x+\lambda\int_0^1 (xz+z^2)y(z)dz$$

Answer: The Kernel could be separable as follows:

$$K(x,z)=(xz+z^2)$$

Then:

$$y(x)=x+\lambda x\int_0^1 z\ y(z)dz+\lambda\int_0^1 z^2 y(z)dz$$

The kernel for this equation is $K(x,z)=xz+z^2$, which is clearly separable, and using the notation in (23.8) we have $\phi_1(x)=x$, $\phi_2(x)=1$, $\psi_1(z)=z$ and $\psi_2(z)=z^2$. From (23.11) the solution to (23.12) has the form

$$y(x)=x+\lambda\left(c_1x+c_2\right) \Rightarrow y(z)=z+\lambda\left(c_1z+c_2\right)$$

where the constants c_1 and c_2 are given by (23.10) as

$$c_1=\int_0^1 z[z+\lambda(c_1z+c_2)]\,dz = \tfrac{1}{3}+\tfrac{1}{3}\lambda c_1+\tfrac{1}{2}\lambda c_2,$$

$$c_2=\int_0^1 z^2[z+\lambda(c_1z+c_2)]\,dz = \tfrac{1}{4}+\tfrac{1}{4}\lambda c_1+\tfrac{1}{3}\lambda c_2.$$

These two simultaneous linear equations may be straightforwardly solved for c_1 and c_2 to give

$$c_1=\frac{24+\lambda}{72-48\lambda-\lambda^2} \quad \text{and} \quad c_2=\frac{18}{72-48\lambda-\lambda^2},$$

so that the solution to (23.12) is

$$y(x)=\frac{(72-24\lambda)x+18\lambda}{72-48\lambda-\lambda^2}. \blacktriangleleft$$

(23.12)

In the above example, we see that (23.12) has a (finite) unique solution provided that λ is not equal to either root of the quadratic in the denominator of y(x). The roots of this quadratic are in fact the eigenvalues of the corresponding homogeneous equation, as mentioned in the previous section. In general, if the separable kernel contains n terms, as in (23.8), there will be *n* such eigenvalues, although they may not all be different.

H.W. Solve the two simultaneous linear equations using the Matrix and using the determinant.

Kernels consisting of trigonometric (or hyperbolic) functions of sums or differences of x and z are also often separable.

Example: Find the eigenvalues and the corresponding eigenvalues of the homogeneous Fredholm equation:

$$y(x) = \lambda \int_0^\pi \sin(x+z) y(z) dz \qquad (23.13)$$

Answer:
The kernel of this integral equation can be written in separated form as

$$K(x,z) = \sin(x+z) = \sin x \cos z + \cos x \sin z,$$

so, comparing with (23.8), we have $\phi_1(x) = \sin x$, $\phi_2(x) = \cos x$, $\psi_1(z) = \cos z$ and $\psi_2(z) = \sin z$.

Thus, from (23.11), the solution to (23.13) has the form

$$y(x) = \lambda(c_1 \sin x + c_2 \cos x),$$

where the constants c_1 and c_2 are given by

$$c_1 = \lambda \int_0^\pi \cos z\,(c_1 \sin z + c_2 \cos z)\,dz = \frac{\lambda\pi}{2} c_2, \qquad (23.14)$$

$$c_2 = \lambda \int_0^\pi \sin z\,(c_1 \sin z + c_2 \cos z)\,dz = \frac{\lambda\pi}{2} c_1. \qquad (23.15)$$

Combining these two equations we find $c_1 = (\lambda\pi/2)^2 c_1$, and, assuming that $c_1 \neq 0$, this gives $\lambda = \pm 2/\pi$, the two eigenvalues of the integral equation (23.13).

By substituting each of the eigenvalues back into (23.14) and (23.15), we find that the eigenfunctions corresponding to the eigenvalues $\lambda_1 = 2/\pi$ and $\lambda_2 = -2/\pi$ are given respectively by

$$y_1(x) = A(\sin x + \cos x) \quad \text{and} \quad y_2(x) = B(\sin x - \cos x), \qquad (23.16)$$

where A and B are arbitrary constants. ◄

H.W. Calculate the above problem using matrix method.

Example: Find the eigenvalues of the integral equation

$$f(x) = x + \lambda \int_0^1 (xy^2 + x^2 y) f(y) dy$$

Answer: Expand the function as:

$$f(x) = x + \lambda \int_0^1 (xy^2 + x^2 y) f(y) dy = x + \lambda x \int_0^1 y^2 f(y) dy + \lambda x^2 \int_0^1 y f(y) dy$$

And define:

$$A = \int_0^1 y^2 f(y) dy, \quad B = \int_0^1 y f(y) dy,$$

Then:

$$f(x)=x+\lambda x\int_0^1 y^2 f(y)dy+\lambda x^2\int_0^1 yf(y)dy=x+\lambda Ax+\lambda Bx^2$$

With $f(t)=t+\lambda At+\lambda Bt^2$, then

$$A=\int_0^1 y^2 f(y)dy=\int_0^1 y^2\left[y+\lambda Ay+\lambda By^2\right]dy=\frac{1}{4}+\frac{1}{4}\lambda A+\frac{1}{5}\lambda B$$

and

$$B=\int_0^1 yf(y)dy=\int_0^1 y\left[y+\lambda Ay+\lambda By^2\right]dy=\frac{1}{3}+\frac{1}{3}\lambda A+\frac{1}{4}\lambda B$$

Solve the above two equations for A and B, one gets:

$$A=\frac{60+\lambda}{240-120\lambda-\lambda^2},\qquad B=\frac{80}{240-120\lambda-\lambda^2}$$

Solve$\left[\left\{A==\frac{1}{4}+\frac{1}{4}\lambda A+\frac{1}{5}\lambda B,\ B==\frac{1}{3}+\frac{1}{3}\lambda A+\frac{1}{4}\lambda B\right\},\ \{A,\ B\}\right]$

$\left\{\left\{A\to-\frac{60+\lambda}{-240+120\lambda+\lambda^2},\ B\to-\frac{80}{-240+120\lambda+\lambda^2}\right\}\right\}$

And

$$f(x)=x+\lambda Ax+\lambda Bx^2=\frac{(240-60\lambda)x+80\lambda x^2}{240-120\lambda-\lambda^2}$$

Notes: Last equation f(x) = infinite at the two roots:

Solve$\left[240-120\lambda-\lambda^2==0,\ \lambda\right]$

$\left\{\left\{\lambda\to 4\left(-15-4\sqrt{15}\right)\right\},\ \left\{\lambda\to 4\left(-15+4\sqrt{15}\right)\right\}\right\}$

These are the two eigenvalues of the integral equation.

Example: Find the solution of the integral equation

$$\varphi(x)=x^2+\lambda\int_0^1 xt\varphi(t)dt$$

1st method (Neumann): Use the expression:

$$\varphi_n(x)=f(x)+\int_0^x K(t,x)\,\varphi_{n-1}(t)dt,\qquad n=1,2,3,\cdots$$

and define $\varphi_o(x)=f(x)$.

In Define the 1st approximation: $\varphi_o(x)=x^2$

With $\varphi_o(x)=x^2$, then $\varphi_o(t)=t^2$ and $\varphi_1(x)$ will be given by:

$$\varphi_1(x) = x^2 + \lambda\int_0^1 x\ t\ \varphi_0(t)dt = x^2 + \lambda\int_0^1 x\ t\ t^2 dt = x^2 + \frac{1}{4}x\,\lambda$$

With $\varphi_1(x) = x^2 + \frac{\lambda}{4}x$, then $\varphi_1(t) = t^2 + \frac{\lambda}{4}t$ and $\varphi_2(x)$ will be given by:

$$\varphi_2(x) = x^2 + \lambda\int_0^1 x\ t\ \varphi_1(t)dt = x^2 + x\,\lambda\int_0^1 t\left(t^2 + \frac{1}{4}t\,\lambda\right)dt = x^2 + \lambda\frac{x}{4}\left(1 + \frac{1}{3}\lambda\right)$$

2nd method (Degenerate Kernel):

$$\varphi(x) = x^2 + \lambda\int_0^1 xt\varphi(t)dt = x^2 + \lambda x\underbrace{\int_0^1 t\varphi(t)dt}_{A} \quad\Rightarrow\quad \varphi(x) = x^2 + \lambda x A$$

With

$$A = \int_0^1 t\varphi(t)dt = \int_0^1 t\left(t^2 + \lambda t A\right)dt = \frac{1}{4} + \frac{1}{3}\lambda A \quad\Rightarrow\quad A = \frac{1}{4\left(1 - \frac{\lambda}{3}\right)}$$

Then

$$\varphi(x) = x^2 + \frac{\lambda}{4}\frac{x}{\left(1 - \frac{\lambda}{3}\right)} \approx x^2 + \frac{\lambda}{4}x\left(1 + \frac{\lambda}{3} + \cdots\right)$$

H.W. Find the solution of the integral equation

$$\varphi(x) = x + \frac{1}{2}\int_{-1}^{1}(t - x)\varphi(t)dt$$

KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Mathematical Physics (Phys-571)
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Issued:	**Assignment # 8 Part A**	Due date

Using MATHEMATICA will give you more credit
Part A
From Arfken's book, solve the following problems:

1- 16.1.1, 2 and 4
2- 16.3.1, 2, 3 4, 7(a and b) and 9(a and b)
3- 16.3.2