Solution of inhomogeneous ordinary differential equations using Green's functions $G(x,x')$

Historical Introduction: **Green's functions** are auxiliary functions in the solution of linear partial differential equations. Green's function is named for the self-taught English mathematician George Green (1793 – 1841), who investigated electricity and magnetism in a thoroughly mathematical fashion. In 1828 Green published a privately printed booklet, introducing what is now called the Green function. This was ignored until William Thomson (Lord Kelvin) discovered it, recognized its great value and had it published nine years after Green's death. Bernhard Riemann gave it the name "Green function"
For example, in electrodynamics, we are concerned with finding solutions to the Poisson equation:

$$\nabla^2\Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o} \qquad \text{(I)}$$

and the Laplace equation:

$$\nabla^2\Phi(\vec{r}) = 0 \qquad \text{(II)}$$

In fact, the Laplace equation is the "homogeneous" version of the Poisson equation.
The Green's function allows us to determine the electrostatic potential from volume and surface integrals:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') - \qquad \text{(III)}$$

This general form can be used in 1, 2, or 3 dimensions. In general, the Green's function must be constructed to satisfy the appropriate boundary conditions. In some cases, it may be difficult or inconvenient to find a Green's function that generates a solution with the correct boundary conditions. In these situations, we can still use Eq. (III) to obtain a solution to the Poisson equation (I) and then add the appropriate linear combinations of solutions to the Laplace's equation (II) to adjust the boundary values. Here, we are going to discussed one method of constructing Green's functions that works for one-dimensional systems. Next, we discuss another method that is generalizable for higher dimensional systems.

1- Homogeneous Equation

Start with the second order linear homogeneous differential equations (Eigen value equations), which can be written as an eigenvalue problem of the form:

$$\hat{L}(x)y_n(x) = E_n y_n(x) \qquad (1)$$

where $\hat{L}$ is an operator involving derivatives w.r.t. x, E_n is an eigenvalue and y_n is an eigenfunction (which satisfies some specified boundary conditions). The general case that we are interested in is called a "Sturm-Liouville[1]" problem, for which one can show that the eigenvalues are real, and the eigenfunctions are orthonormal, i.e.

$$\langle m|n\rangle = \int_a^b y_m^*(x)y_n(x)dx = \delta_{mn} \qquad (2)$$

where a and b are the upper and lower limits of the region where we are solving the problem, and we have also "normalized" the solutions.

[1]The general Sturm-Liouville problem, see Arfeken, has a "weight function" w(x) multiplying the eigenvalue on the RHS of Eq. (1) and the same weight function multiplies the integrand shown in the LHS of the orthogonality and

normalization condition, Eq. (2). Furthermore the eigenfunctions may be complex, in which case one must take the complex conjugate of either yn or ym in in Eq. (2). Here, to keep the notation simple, we will just consider examples with w(x) = 1 and real eigenfunctions.

Example, Find the solution of the differential equation:

$$\frac{d^2y(x)}{dx^2}+\frac{1}{4}y(x)=\lambda y(x)$$

$$\Rightarrow \frac{d^2y(x)}{dx^2}=\underbrace{\left(\lambda-\frac{1}{4}\right)}_{-n^2}y(x) \tag{3}$$

in the interval $0 \le x \le \pi$, with the boundary conditions $y(0)=y(\pi)=0$.

Answer: This corresponds to

$$\hat{L}=\frac{d^2}{dx^2} \tag{4}$$

This is just the simple harmonic oscillator equation, and so the solutions are $\cos(nx)$ and $\sin(nx)$. The boundary condition $y(0)=0$ eliminates $\cos(nx)$ and the condition $y(\pi)=0$ gives n a positive integer. (Note: For n = 0 the solution vanishes and taking n < 0 just gives the same solution as that for the corresponding positive value of n because $\sin(-nx)=-\sin(nx)$). Hence we only need consider positive integer n.) The normalized eigenfunctions are therefore:

$$y_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx), \qquad n=1,2,3,\cdots \tag{5}$$

and the eigenvalues in Eq. (3) are:

$$\lambda_n=\frac{1}{4}-n^2 \tag{6}$$

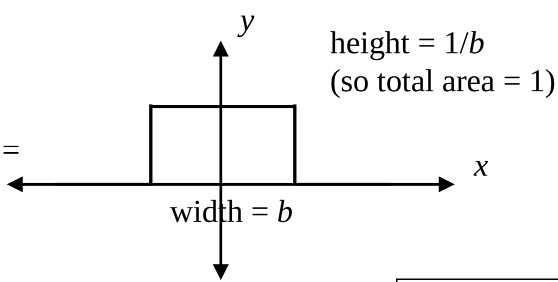
since the equation satisfied by $y_n(x)$ is $y''(x)+n^2y(x)=0$.

Dirac delta function

Definitions

1. Definition as limit. The Dirac delta function can be thought of as a rectangular pulse that grows narrower and narrower while simultaneously growing larger and larger.

rect(x, b) =



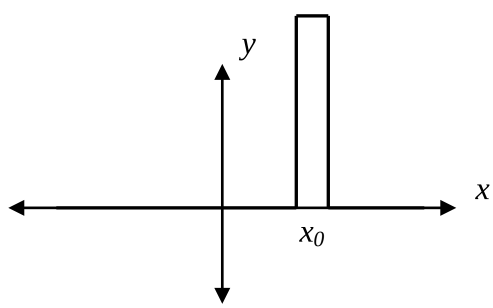
$$\boxed{\delta(x) = \lim(b \to 0)\ \mathrm{rect}(x, b)}$$

Note that the integral of the delta function is the area under the curve, and has been held constant at 1 throughout the limit process.

$$\boxed{\int_{-\infty}^{\infty} \delta(x) = 1}$$

Shifting the origin. Just as a parabola can be shifted away from the origin by writing $y = (x - x_0)^2$ instead of just $y = x^2$, any function can be shifted by plugging in $x - x_0$ in place of its usual argument x.

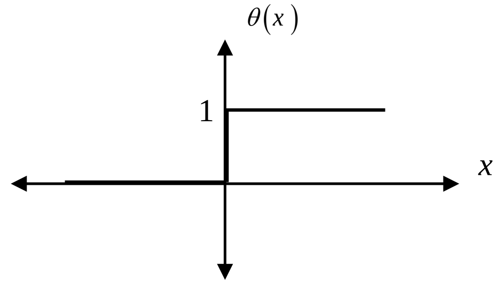
$\delta(x - x_0)$ =



Shifting the position of the peak doesn't affect the total area if the integral is taken from $-\infty$ to ∞.

$$\boxed{\int_{-\infty}^{\infty} \delta(x - x_0) = 1}$$

Disclaimer: Mathematicians will object that the Dirac delta function defined this way (or any other way, for that matter) is not a real function. That is true, but physicists recognize that for all practical purposes you really can just think of the delta function as a very large peak.



2. <u>Definition as derivative of step function</u>. The step function, also called the "Heaviside step function" is usually defined like this:

$$\theta(x)=\begin{cases}1, & if \;\; x>0 \\ 0, & if \;\; x\le 0\end{cases}$$

It's a function whose only feature is a step up from 0 to 1, at the origin:
What's the derivative of this function? Well, the slope is zero for $x < 0$ and the slope is zero for $x > 0$. What about right at the origin? The slope is infinite! So the derivative of this function is a function which is zero everywhere except at the origin, where it's infinite—and the integral of the derivative function from $-\infty$ to ∞ must be 1 because $\theta(x)$ is the anti-derivative and has a value of 1 at $x = \infty$ and a value of 0 at $x = -\infty$ (think Fundamental Theorem of Calculus).

Example: Prove that $\dfrac{d\theta}{dx}=\delta(x)$, where $\theta(x)$ is the step function, see the figure.

$$\theta(x)=\begin{cases}1, & if \;\; x>0 \\ 0, & if \;\; x\le 0\end{cases}$$

Answer:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx &= f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\theta(x)\frac{df}{dx}dx \\ &= f(\infty) - \int_{0}^{\infty}\frac{df}{dx}dx = f(\infty) - \big(f(\infty) - f(0)\big) \\ &= f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx \quad \Rightarrow \quad \frac{d\theta}{dx}=\delta(x)\end{aligned}$$

An infinite peak at the origin whose integral is 1? Sound familiar? Therefore $\delta(x)$ can also be defined as the derivative of the step function.

Note that one of the uses of step function is to write $r_>$ (the larger of r and r') as:

$$r_> = r\theta(r-r') + r'\theta(r'-r).$$

Similarly, $r_<$ (the smaller of r and r') as:

$$r_< = r\theta(r'-r) + r'\theta(r-r')$$

Then

$$\begin{aligned}\frac{\partial}{\partial r}(r_>) &= \frac{\partial}{\partial r}[r\theta(r-r') + r'\theta(r'-r)] \\ &= \theta(r-r') + r\delta(r-r') - r'\delta(r'-r) = \theta(r-r')\end{aligned};$$

Where r' is treated as a constant. Also

$$\begin{aligned}\frac{\partial}{\partial r}(r_<) &= \frac{\partial}{\partial r}[r\theta(r'-r) + r'\theta(r-r')] \\ &= \theta(r'-r) - r\delta(r-r') - r'\delta(r'-r) = \theta(r'-r)\end{aligned}$$

Where r' is treated as a constant

It can be proved that $g\left(r,r'\right)=\sin kr_{<}\cos kr_{>}$ is a solution of the differential equation:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2\right]g\left(r,r'\right)=-\frac{1}{r^2}\delta\left(r-r'\right)$$

by direct substitution.

3. Definition as Fourier transform. We've seen that taking the Fourier transform of a function gives you the frequency components of the function. What about taking the Fourier transform of a pure sine or cosine wave oscillating at ω_0? There is only one frequency component, so the Fourier transform must be a single, very large peak at ω_0 (or possibly two peaks, one at ω_0 and one at $-\omega_0$). A delta function!

4. Definition as density. What's the density of a 1 kg point mass located at the origin? Well, it's a function that must be zero everywhere except at the origin—and it must be infinitely large at the origin because for a mass that truly occupies only a single point, the mass must have been infinitely compressed. How about the integral? The integral of the density must give you the mass, which is 1 kg. A function that is zero everywhere except at the origin, and has an integral equal to 1? Sounds like the delta function again!

More precisely, this would be a three-dimensional analog to the regular delta function $\delta(x)$, because the density must be integrated over three dimensions in order to give the mass. This is sometimes written $\delta(\mathbf{r})$ or as $\delta^3(\mathbf{r})$:

$$\delta^3(\mathbf{r}) = \delta(x)\ \delta(y)\ \delta(z)$$

Properties

1. Integral. One of the most important properties of the delta function has already been mentioned: it integrates to 1.

2. Sifting property. When a delta function $\delta(x - x_0)$ multiplies another function $f(x)$, the product must be zero everywhere except at the location of the infinite peak, x_0. At that location, the product is infinite like the delta function, but it might be a larger or smaller infinity (now you see why mathematicians don't like physicists), depending on whether the value of $f(x)$ at that point is larger or smaller than 1. In other words, the area of the product function is not necessarily 1 any more, it is modified by the value of $f(x)$ at the infinite peak. This is called the "sifting property" of the delta function:

$$\boxed{\int_{-\infty}^{\infty}\delta(x-x_0)\,f(x)dx = f(x_0)}$$

Mathematicians may call the delta function a "functional", because it is really only well-defined inside integrals like this, in terms of what it does to other functions. (A "functional" is something that operates on functions, rather than a "function" which is something that operates on variables.)

3. Symmetry. A few other properties can be readily seen from the definition of the delta function:

a. $\delta(-x) = \delta(x)$ (Note that $\delta(x)$ behaves as if it were an even function)

b. $\delta(x - x_0) = \delta(-x + x_0)$

4. Linear systems. If a physical system has linear responses and if its response to delta functions ("impulses") is known, then in theory the output of this system can be determined for almost *any* input, no matter how complex. This rather amazing property of linear systems is a result of the following: almost any arbitrary function can be decomposed into (or "sampled by") a linear combination of delta functions, each weighted appropriately, and each of which produces its own impulse response. Thus, by application of the superposition principle, the overall response to the arbitrary input can be found by adding up all of the individual impulse responses.

More Properties

1- $\delta^*(x) = \delta(x)$ it is a real function

2- $\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1$ It is normalized

3- $\delta(x-a) = 0, \quad x \neq a \qquad \int_{a-}^{a+} dx\delta(x-a) = 1$

4- $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$

5- $f(x)\delta(x-a) = f(a)\delta(x-a)$

6- $\int_{0-}^{0+} dx\ f(x)\left[x\frac{d}{dx}\delta(x)\right] = \int_{0-}^{0+} dx\ f(x)\delta(x) = f(0$

7- $\delta(ax) = \frac{1}{|a|}\delta(x)$

8- $\frac{d^2|x|}{dx^2} = 2\delta(x)$

9- $\frac{d}{dx}\left(\delta(x)\right) = -\frac{1}{x}\delta(x)$

Closure property: Consider a complete orthonormal set of function $\varphi_1(x), \varphi_2(x), \cdots$ then for any complete function $f(x) = \sum_n a_n \varphi_n(x)$

Where

$$a_n = \int f(x')\varphi_n^*(x')dx'$$

$$f(x) = \sum_n \left[\int f(x')\varphi_n^*(x')dx' \right] \varphi_n(x)$$

$$= \int f(x') \underbrace{\left\{ \sum_n \varphi_n^*(x')\varphi_n(x) \right\}}_{\delta(x-x')} dx', \quad \delta(x-x') = \sum_n \varphi_n^*(x')\varphi_n(x)$$

For

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

Then

$$\delta(x-x') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}$$

2- Inhomogeneous Equations

=======================8/11/2013===========================

Theorem: the solution of the inhomogeneous equation $\hat{L}(x)\, y(x) = f(x)$ is given by:

$$f(x) = \varphi_0 + \int G(x,x')f(x')dx'$$

where φ_0 is the solution of the equation:

$$\hat{L}(x)\, \varphi_0(x) = 0$$

and is the solution of the equation:

$$\hat{L}(x)\, G(x,x') = \delta(x-x')$$

The definition of the above equation is not unique. Some books are used $\delta(x-x')$ and others are used $-4\pi\delta(x-x')$.

Proof:

Start with the equation:

$$f(x) = \varphi_0 + \int G(x,x')f(x')dx'$$

One finds:

$$\begin{aligned} L(x)f(x) &= L(x)\varphi_0 + \int \left[L(x)G(x,x')\right] f(x')dx' \\ &= 0 + \int \left[\delta(x,x')\right] f(x')dx' \\ &= f(x) \end{aligned}$$

===

Green's functions, the topic of this handout, appear when we consider the inhomogeneous equation analogous to Eq. (1)

$$\hat{L}(x)\, y(x) = f(x) \tag{7}$$

where $\hat{L}(x)$ is a linear, self-adjoint differential operator, $y(x)$ is the unknown function, and *f(x)* is a known non-homogeneous term. For a discussion of the concept of self-adjoint and non self-adjoint differential operators please refer, for example, to the text by "**Arfeken".** Operationally, we can write a solution to equation (1) as

$$y(x) = \hat{L}^{-1} f(x)$$

where $L^{-1}(x)$ is the inverse of the differential operator $\hat{L}(x)$. Since $\hat{L}(x)$ is a differential operator, it is reasonable to expect its inverse to be an integral operator. We expect the usual properties of inverses to hold,

$$\hat{L}\hat{L}^{-1} = \hat{L}^{-1}\hat{L} = \mathrm{I}$$

where I is the *identity* operator. More specifically, we define the inverse operator as

$$L^{-1} f(x) = \int G(x,x')f(x')dx'$$

where the kernel $G(x,x')$ is the *Green's Function* associated with the differential operator *L*. Note that $G(x,x')$ is a two-point function which depends on *x* and *x'*. To complete the idea of the inverse operator *L*, we introduce the Dirac delta function as the identity operator I.
Recall the properties of the Dirac delta function $\delta(x)$ are:

$$f(x)=\int_{-\infty}^{\infty}\delta(x-x')f(x')dx', \qquad \int_{-\infty}^{\infty}\delta(x')dx'=1$$

The Green's function $G(x,x')$ then satisfies

$$L(x)G(x,x')=\delta(x-x') \tag{8}$$

Important note: The definition in equation (8) is not unique. Some books are used $\delta(x-x')$ and others are used $-4\pi\delta(x-x')$.

The solution to equation (7) can then be written directly in terms of the Green's function as

$$y(x)=\int_{-\infty}^{\infty}G(x,x')f(x')dx' \tag{9}$$

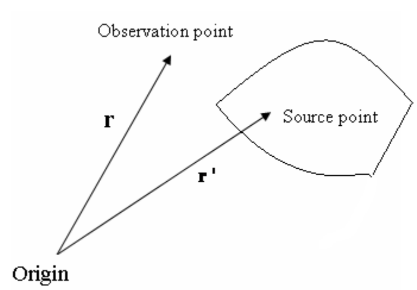
To prove that equation (9) is indeed a solution to equation (7), simply substitute as follows:

$$\begin{aligned} L(x)\,y(x)&=L(x)\int_{-\infty}^{\infty}G(x,x')f(x')dx'=\int_{-\infty}^{\infty}L(x)G(x,x')f(x')dx' \\ &=\int_{-\infty}^{\infty}\delta(x-x')f(x')dx'=f(x) \end{aligned} \tag{10}$$

Note that we have used the linearity of the differential and inverse operators in addition to equations (4), (5), and (6) to arrive at the final answer.

We emphasize that the same Green's function applies for any $f(x)$, and so it only has to be calculated once for a given differential operator L and boundary conditions.

Comment: The green's function $G(x,x')$ represents the response of the system to a unit impulse at $x=x'$.
$G(x,x')$ is the field at the observer's point $\mathbf{r}$ caused by a unit source at the source point $\mathbf{r}'$, then the field at r caused by a source distribution is the integral of over the whole range of occupied by the source.



3- A simple example

Example, Find the solution of the differential equation:

$$\frac{d^2y(x)}{dx^2}+\frac{1}{4}y(x)=\sin(2x) \tag{A}$$

in the interval $0\le x\le\pi$, with the boundary conditions $y(0)=y(\pi)=0$.

$$a=0\underline{\qquad\qquad\qquad\qquad}b=\pi$$

Solution:

1st method: The general solution of this equation is:

$$y(x)=\underbrace{A\cos(\tfrac{x}{2})+B\sin(\tfrac{x}{2})}_{\substack{\text{complementary solution of the equation}\\ \frac{d^2y(x)}{dx^2}+\frac{1}{4}y(x)=0}}\qquad\underbrace{-\frac{4}{15}\sin(2x)}_{\substack{\text{particular solution of the equation}\\ \frac{\mathrm{Im}(e^{2ix})}{D^2+\frac{1}{4}}}} \tag{B}$$

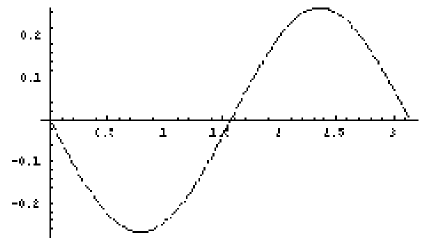
Note: The particular solution is given as: $y_p = \frac{\text{Im}(e^{2ix})}{D^2+\frac{1}{4}} = \frac{\sin(2x)}{(2i)^2+\frac{1}{4}} = -\frac{4}{15}\sin(2x)$

Using the above boundary conditions, one finds $A = B = 0$, and

$$y(x) = -\frac{4}{15}\sin(2x) \tag{C}$$

See the plotting of this solution

```
DE = y''[x] + 1/4 y[x] - Sin[2 x] == 0;
sol1=DSolve[{DE,y[0]== 0,y[π]==0},y,x];
Plot[y[x]/.sol1[[1]],{x,0, π}]
```

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Plot[-4/15 Sin[2 x], {x, 0, π}]
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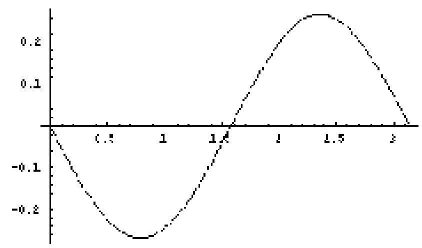

H.W. check that (C) sactisfies (A).

2nd **method:** Closed form expression for the Green's function

In many useful cases, one can obtain a closed form expression for the Green's function by starting with the defining equation, Eq. (8). We will illustrate this for the example in the previous section for which Eq. (8) is

$$G''(x,x') + \frac{1}{4}G(x,x') = \delta(x-x') \tag{11}$$

Remember that x' is fixed (and lies between 0 and π) while x is a variable, and the derivatives are with respect to x. We solve this equation separately in the two regions

i- $0 \le x < x'$, and

ii- $x' < x \le \pi$.

In each region separately the equation is G″ + (1/4)G = 0, for which the solutions are

$$G(x,x') = A(x')\cos(\tfrac{x}{2}) + B(x')\sin(\tfrac{x}{2}) \tag{12}$$

Where the constants *A* and *B* will depend on x'. Since $y(0) = 0$, we require $G(0,x') = 0$ and so, for the solution in the region $0 \le x < x'$, the cosine is eliminated. Similarly G(π, x′) = 0 and so, for the region $x' < x \le \pi$, the sine is eliminated. Hence the solution is:

$$G(x,x') = \begin{cases} G_I \equiv B(x')\sin(x/2) & (0 \le x < x') \\ G_{II} \equiv A(x')\cos(x/2) & (x' < x \le \pi) \end{cases} \tag{13}$$

$$\boxed{\begin{array}{c} \qquad G_I \qquad\qquad G_{II} \\ a = 0 \underline{\qquad\qquad} | \underline{\qquad\qquad} b = \pi \\ x' \end{array}}$$

Now: How do we determine the two coefficients *A* and *B*? The answer will be as follows:

I- We can get one relation between them by requiring that the solution is continuous at $x = x'$, i.e. the limit as $x \to x'$ from below is equal to the limit as $x \to x'$ from above. This gives:

$$B(x')\sin(x'/2) = A(x')\cos(x'/2) \tag{14}$$

II- The second relation between *A* and *B* is obtained by integrating Eq. (11) from $x \to x'-\varepsilon$ to $x'+\varepsilon$, and taking the limit $\varepsilon \to 0$, which gives:

$$\lim_{\epsilon\to 0}\left[\frac{dG}{dx}\right]_{x'-\epsilon}^{x'+\epsilon} + \frac{1}{4}\lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon} G(x,x')\,dx = \lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon}\delta(x-x')\,dx \tag{15}$$

so

$$\lim_{\varepsilon\to 0}\left(\left.\frac{dG}{dx}\right|_{x'+\varepsilon} - \left.\frac{dG}{dx}\right|_{x'-\varepsilon}\right) + 0 = 1 \tag{16}$$

Where

$$\frac{dG(x,x')}{dx} = \begin{cases} G_I' \equiv \dfrac{B(x')}{2}\cos(\dfrac{x}{2}) & (0 \le x < x') \\ G_{II}' \equiv -\dfrac{A(x')}{2}\sin(\dfrac{x}{2}) & (x' < x \le \pi) \end{cases} \tag{17}$$

Hence $\dfrac{dG(x,x')}{dx}$ has a discontinuity of 1 at $x = x'$, i.e.

$$G_{II}' - G_I' = 1 \quad \Rightarrow \quad -\frac{A}{2}\sin(\frac{x'}{2}) - \frac{B}{2}\cos(\frac{x'}{2}) = 1 \tag{18}$$

Solving Eqs. (14) and (18) gives

$$\begin{aligned} B(x') &= -2\cos(x'/2) \\ A(x') &= -2\sin(x'/2) \end{aligned} \tag{19}$$

Substituting into Eq. (3) gives

$$G(x,x') = \begin{cases} -2\cos(\dfrac{x'}{2})\sin(\dfrac{x}{2}) & (0 \le x < x') \\ -2\sin(\dfrac{x'}{2})\cos(\dfrac{x}{2}) & (x' < x \le \pi) \end{cases} \tag{20}$$

A sketch of the solution is shown in the figure below. The discontinuity in slope at $x = x'$ (I took $x' = \varphi = 3\pi/4$) is clearly seen.

```
In[9]:= φ := 3*π/4

In[10]:= g[x_] := Which[0 ≤ x < φ, -2 Cos[φ/2]*Sin[x/2], φ < x ≤ π, -2 Sin[φ/2]*Cos[x/2]]

In[13]:= Plot[g[x], {x, 0, π}, Frame → True]
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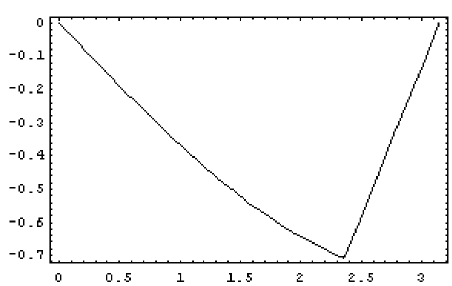

```
In[9]:= φ := (3*π)/4

In[16]:= gg[x_] := If[x < φ, -2 Cos[φ/2]*Sin[x/2], -2 Sin[φ/2]*Cos[x/2]]

In[17]:= Plot[gg[x], {x, 0, π}, Frame → True]
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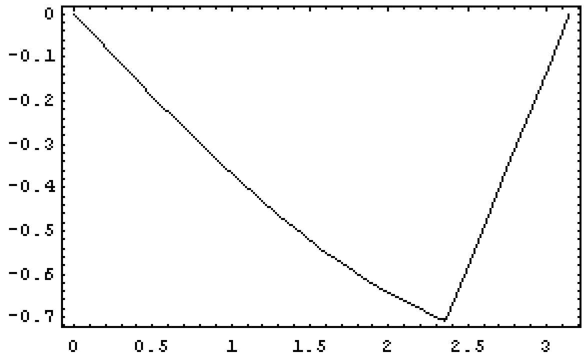


It is instructive to rewrite Eq. (20) in terms of $x_<$, the smaller of x and x', and $x_>$, the larger of x and x'. One has

$$G(x,x') = -2\cos(x_>/2)\sin(x_</2) \tag{21}$$

Irrespective of which is larger, which shows that **G is symmetric under interchange of x and x'**.
We now apply the closed form expression for G in Eq. (20) to solve our simple example, Eq. (A), with the function:

$$y(x) = -2\cos(x/2)\int_0^x \sin(\ell/2) f(\ell)\,d\ell - 2\sin(x/2)\int_x^\pi \cos(\ell/2) f(\ell)\,d\ell$$

with $f(\ell) = \sin(2\ell)$, and using formulae for sines and cosines of sums of angles and integrating gives:

$$\begin{aligned} y(x) &= -2\cos(x/2)\left(\frac{\sin(3x/2)}{3} - \frac{\sin(5x/2)}{5}\right) - 2\sin(x/2)\left(\frac{\cos(3x/2)}{3} + \frac{\cos(5x/2)}{5}\right) \\ &= -2\left(\frac{1}{3}\sin 2x\right) + 2\left(\frac{1}{5}\sin 2x\right) \;=\; -\frac{4}{15}\sin 2x\,, \end{aligned}$$

where we again used formulae for sums and differences of angles. This result is in agreement with Eq. (C).

```
f[t_] := Sin[2 t]
y = - 2 Cos[x / 2] ∫_0^x Sin[t / 2] f[t] ⅆt - 2 Sin[x / 2] ∫_x^π Cos[t / 2] f[t] ⅆt // Simplify
-8/15 Cos[x] Sin[x]
```

4- Summary

We have shown how to solve linear, inhomogeneous, ordinary differential equations by using Green's functions. These can be represented in terms of eigenfunctions, see next section, and in many cases can alternatively be evaluated in closed form, see Sec. 6. The advantage of the Green's function approach is that the Green's function only needs to be computed once for a given differential operator L and boundary conditions, and this result can then be used to solve for any function $f(x)$ on the RHS of Eq. (9) by using Eq. (20). The advantages of Green's functions may not be readily apparent from the simple examples presented here. However, they are used in many advanced applications in physics.

We can summarize the properties of the one-dimensional Green's function as follows:

1- $G(x,\xi)$ is symmetric under interchange of x and ξ.

2- Both $G_I(x,\xi)$ and $G_2(x,\xi)$ satisfy the homogeneous equation

$$LG_I(x,\xi)=\delta(x-\xi), \qquad a\le x<\xi$$
$$LG_{II}(x,\xi)=\delta(x-\xi), \qquad \xi<x\le b$$

3- $G_I(x,\xi)$ satisfies the boundary condition at $x=a$. Similarly $G_{II}(x,\xi)$ satisfies the boundary condition at $x=b$.

4- $G(x,\xi)$ is a continuous function of x, i.e. $\lim_{x\to\xi}G_I(x,\xi)=\lim_{x\to\xi}G_{II}(x,\xi)$.

5- $\frac{dG(x,\xi)}{dx}$ is a discontinuous and the discontinuity is given by.

$$\left.\frac{dG_{II}}{dx}\right|_{x=\xi}-\left.\frac{dG_I}{dx}\right|_{x=\xi}=1$$

6- Generates a superposition principle for the solution under general forcing functions:

$$y(x)=\int_{-\infty}^{\infty}G(x,x')f(x')dx'$$

Green's Functions for self-adjoint operator, Sturm-Liouville equation
Arfken, 10.5, page 663

H.W. Problems 10.5. 1,2,3,10,11

For the self-adjoint operator, Sturm-Liouville equation is defined as:

$$L=\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x), \tag{1}$$

Define the Green's equation by the equation:

$$\left\{\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right\}G(x,\xi)=-\delta(x-\xi) \tag{2}$$

where $p(x)$, $q(x)$ and $G(x,\xi)$ are continuous functions in the interval [a,b].

Integrating both sides of equation (2) from $\xi-\varepsilon$ to $\xi+\varepsilon$, one obtains:

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon}\frac{d}{dx}\left(p(x)\frac{dG(x,\xi)}{dx}\right)dx+\int_{\xi-\varepsilon}^{\xi+\varepsilon}q(x)G(x,\xi)dx=-1 \tag{3}$$

If we now let $\varepsilon\to 0$, the integral $\int_{\xi-\varepsilon}^{\xi+\varepsilon}q(x)G(x,\xi)dx$ vanishes because $q(x)$ and $G(x,\xi)$ are continuous functions of *x*. Then equation (3) reduces to:

$$\left[p(x)\frac{dG(x,\xi)}{dx}\right]_{\xi-\varepsilon}^{\xi+\varepsilon}=-1 \tag{4}$$

or

$$\left.\frac{dG(x,\xi)}{dx}\right|_{\xi+\varepsilon}-\left.\frac{dG(x,\xi)}{dx}\right|_{\xi-\varepsilon}=-\frac{1}{p(\xi)} \tag{5}$$

Where it is implied that $\varepsilon\to 0$.

Since there is a discontinuity in the derivative of the Green's function at $x=\xi$, it is convenient to consider the two intervals $a\le x<\xi$ and $\xi<x\le b$ separately and write the Green's function in the following form:

$$G(x,\xi)=\begin{cases}c_1u(x), & a\le x<\xi\\ c_2v(x), & \xi<x\le b\end{cases} \tag{6}$$

We can summarize the following properties of the one-dimensional Green's function.

1- Both $G_1(x,\xi)$ and $G_2(x,\xi)$ satisfy the homogeneous equation

$$LG_1(x,\xi), \qquad a\le x<\xi$$
$$LG_2(x,\xi), \qquad \xi<x\le b$$

2- $G_1(x,\xi)$ satisfies the boundary condition at $x=a$. Similarly $G_2(x,\xi)$ satisfies the boundary condition at $x=b$.

3- $G(x,\xi)$ is a continuous function of x, i.e. $\lim_{x\to\xi}G_1(x,\xi)=\lim_{x\to\xi}G_2(x,\xi)$.

4- $\frac{dG(x,\xi)}{dx}$ is a discontinuous and the discontinuity is given by.

$$\left.\frac{dG_2}{dx}\right|_{x=\xi}-\left.\frac{dG_1}{dx}\right|_{x=\xi}=-\frac{1}{p(\xi)} \tag{7}$$

These properties can be used for the explicit construction of the Green's function. Let $u(x)$ be a solution of $Lu(x)=0$ satisfying the boundary condition at $x=a$. Similarly, Let $v(x)$ be a solution of $Lv(x)=0$ satisfying the boundary condition at $x=b$. Following the definition of the Green's function it follows that:

$$G(x,\xi)=\begin{cases} c_1 u(x), & a \le x < \xi \\ c_2 v(x), & \xi < x \le b \end{cases} \tag{8}$$

Since $G(x,\xi)$ is a continuous function of $x=\xi$, we have

$$c_2 v(\xi)-c_1 u(\xi)=0 \tag{9}$$

and equation (7) implies

$$c_2 v'(\xi)-c_1 u'(\xi)=-\frac{1}{p(\xi)} \tag{10}$$

Equations (9) and (10) can be solved to obtain c_1 and c_2. A non-trivial solution exists only if the determinant

$$\begin{vmatrix} v(\xi) & -u(\xi) \\ v'(\xi) & -u'(\xi) \end{vmatrix} \neq 0 \tag{11}$$

The left hand side of (11) is the Wronskain of $v(\xi)$ and $u(\xi)$, $W(\xi)=u\ v'-u'\ v$. $W(\xi)$ will be non-zero if $v(\xi)$ and $u(\xi)$ are linearly independent. If this condition is satisfied, from (9) and (10), we get:

$$c_1=-\frac{v(\xi)}{p(\xi)W(\xi)} \tag{12}$$

Similarly,

$$c_2=-\frac{u(\xi)}{p(\xi)W(\xi)} \tag{13}$$

Using: $A=W(\xi)p(\xi)$, we have

$$c_1=-\frac{v(\xi)}{A}\quad, \tag{14}$$

$$c_2=-\frac{u(\xi)}{A} \tag{15}$$

And (8) will be

$$G(x,\xi)=\begin{cases} -\dfrac{v(\xi)u(x)}{A}, & a \le x < \xi \\ -\dfrac{u(\xi)v(x)}{A}, & \xi < x \le b \end{cases} \tag{16}$$

H.W. : From equation (1), $Ly=\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y=0$, check the following:

$$Lu=\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]u=0$$

$$\Rightarrow p(x)u''+u'p'(x)+q(x)u=0 \quad \Rightarrow \boxed{u''+u'\frac{p'(x)}{p(x)}+\frac{q(x)}{p(x)}u=0},$$

$$Lv=\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]v=0$$

$$\Rightarrow p(x)v''+v'p'(x)+q(x)v=0 \quad \Rightarrow \boxed{v''+v'\frac{p'(x)}{p(x)}+\frac{q(x)}{p(x)}v=0}$$

1- $W(x)=uv'-u'v \quad \Rightarrow \quad \frac{dW(x)}{dx}=uv''-u''v=-\frac{p'(x)}{p(x)}W$

2- $W(x)=uv'-u'v \quad \Rightarrow \quad W(x)=u^2\left(\frac{uv'-u'v}{u^2}\right)=u^2\frac{d}{dx}\left(\frac{v}{u}\right)$

H.W.

1- For the D.E.

$$y''(x)-y(x)=f(x)$$

such that $y(0)=y(\ell)=0$, calculate $G(x,x')$.

2- Construct the Green function for the differential equation :

$$y''(x)+3y'(x)=F(x),$$

Subject to the boundary conditions: $y(0)=y(1)=0$.

3- For the boundary value problem

$$y''(x)+\omega^2 y(x)=f(x),$$

where $f(x)$ is a known function and y satisfies the boundary conditions $y(0)=y(\ell)=0$, calculate the Green's function.

4- For the D.E.

$$y''(x)-y(x)=x$$

such that $y(0)=y(\ell)=0$. Calculate $G(x,x')$.

Green's function step by step with example

Example: For the non-homogeneous differential equation.

$$\underbrace{\frac{d^2}{dx^2}}_{L(x)} y(x) = -e^{-\alpha x} \qquad \text{(A)}$$

such that $y(0) = y'(1) = 0$.

a- Calculate $G(x, x')$. b- Solve for $y(x)$ using part b. c- Check your answer by solving (A) directly.

Answer:

To find the solution, using the Green's function, of the non-homogeneous differential equation:

$$L(x)y(x) = f(x), \qquad (1)$$

with the given boundary conditions, we have to do the following:

1- It is the most important step to have the general solution of the homogeneous equation: $L(x)y(x) = 0$. For example: the solution of homogeneous equation of (1)

$\frac{d^2}{dx^2} y(x) = 0 \quad \Rightarrow$

$$y(x) = cx + d \qquad (2)$$

2- Use the general solution in (2) as a general solution of $L(x)G(x, \xi) = 0$, with the constants defined in the two regions. For example:

$$G(x,\xi) = \begin{cases} G_I(x,\xi) = c_1 x + d_1, & a \le x < \xi \\ G_{II}(x,\xi) = c_2 x + d_2, & \xi < x \le b \end{cases} \qquad (3)$$

3- Define the non-homogeneous equation $L(x)G(x,\xi) = \delta(x - \xi)$

4- Use the

i- boundary conditions given in the equation, $y(0) = 0$ and $y'(1) = 0$, to have $d_1 = 0,$ and $c_2 = 0$, then

$$G(x,\xi) = \begin{cases} c_1 x & (0 \le x < \xi) \\ d_2 & (\xi < x \le 1) \end{cases} \qquad (4)$$

ii- Use the continuity of the Green's functions $G_I(x,\xi) = G_{II}(x,\xi)$ to have $c_1 \xi = d_2$, and

iii- Use the discontinuity $\left.\frac{dG_2}{dx}\right|_{x=\xi} - \left.\frac{dG_1}{dx}\right|_{x=\xi} = 1$ to have $c_1 - 0 = 1 \Rightarrow c_1 = 1$. From step (ii) one has $d_2 = \xi$

iv- Finally, we have the target Green's function in the for:

$$G(x,\xi)=\begin{cases} G_I(x,\xi)=x & (0\le x<\xi) \\ G_{II}(x,\xi)=\xi & (\xi<x\le 1) \end{cases} \tag{5}$$

Finally, calculate the integral

$$y(x)=\int_a^b G(t,x)f(t)dt=\int_a^x G_I(t,x)f(t)dt+\int_x^b G_{II}(t,x)f(t)dt$$

to have, using $f(x)=-e^{-ax}$, then

$$y(x)=\int_0^1 G(t,x)f(t)dt=\int_0^x te^{-at}dt+x\int_x^1 e^{-at}dt=-\frac{e^{-ax}}{a^2}-\frac{e^{-a}}{a}x+\frac{1}{a^2}$$

c- With simple integration, one finds:

$$y''(x)=-e^{-ax}\Rightarrow y'(x)=\frac{e^{-ax}}{a}+b\quad\Rightarrow\quad y(x)=-\frac{e^{-ax}}{a^2}+bx+c\,,$$

Use the boundary conditions $y(0)=y'(1)=0$, one finds $c=\frac{1}{a^2}, b=-\frac{e^{-a}}{a}$, Then

$$\boxed{y(x)=-\frac{e^{-ax}}{a^2}-\frac{e^{-a}}{a}x+\frac{1}{a^2}}$$

Arfeken method

Example: For the D.E.

$$y''(x) + e^{-ax} = 0$$

such that $y(0) = y'(1) = 0$.

a- Solve for $y(x)$. b- Calculate $G(x, x')$. c- Solve for $y(x)$ using part b.

a- $$y''(x) = -e^{-ax} \Rightarrow y'(x) = \frac{e^{-ax}}{a} + b \quad \Rightarrow \quad y(x) = -\frac{e^{-ax}}{a^2} + bx + c,$$

Use the boundary conditions $y(0) = y'(1) = 0$, one finds $c = \frac{1}{a^2}, b = -\frac{e^{-a}}{a}$, Then

$$\boxed{y(x) = -\frac{e^{-ax}}{a^2} - \frac{e^{-a}}{a}x + \frac{1}{a^2}}$$

b- For the equation $y''(x) = 0 \quad \Rightarrow \quad y(x) = cx + d$ then $G''(x) = 0 \quad \Rightarrow \quad G(x) = cx + d$

$$G(x, x') = \begin{cases} c_1 x + d_1 & (0 \le x < x') \\ c_2 x + d_2 & (x' < x \le 1) \end{cases}$$

With the B.C. $y(0) = 0 \quad \Rightarrow \quad d_1 = 0, \quad y'(1) = 0 \quad \Rightarrow \quad c_2 = 0$, then,

$$G(x, x') = \begin{cases} c_1 x = c_1 u(x) \Rightarrow \ u(x) = x & (0 \le x < x') \\ d_2 = d_2 v(x) \ \Rightarrow \ v(x) = 1 & (x' < x \le 1) \end{cases},$$

The Wronskain at any convenient point gives:

$$W(x) = uv' - vu' = x \times 0 - 1(1) = -1$$

$p(x) = 1 \quad \Rightarrow \quad A = W(x')\, p(x') = -1, \ c_1 = \frac{v(x')}{A} = -1, \quad d_2 = \frac{u(x')}{A} = -x'$ and the Green's function in the form:

$$G(x, x') = \begin{cases} -x & (0 \le x < x') \\ -x' & (x' < x \le 1) \end{cases}$$

Using $f(x) = -e^{-ax}$, then

$$y(x) = -\int_0^1 G(x, x') f(x') dx' = \int_0^x t e^{-at} dt + x \int_x^1 e^{-at} dt = -\frac{e^{-ax}}{a^2} - \frac{e^{-a}}{a}x + \frac{1}{a^2}$$

aa = DSolve[{y''[x] == $-e^{-a x}$, y'[1] == 0, y[0] == 0}, {y[x]}, x] // ExpandAll

$\{\{y[x] \to \frac{1}{a^2} - \frac{e^{-a x}}{a^2} - \frac{e^{-a} x}{a}\}\}$

bb = $\int_0^x t\, e^{-a t}\, dt + x \int_x^1 e^{-a t}\, dt$ // ExpandAll

$\frac{1}{a^2} - \frac{e^{-a x}}{a^2} - \frac{e^{-a} x}{a}$

Green's function for Bessel' function

(Afken 10.5.8, 10.5.3)

For reading

Find the Green' function for the equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = f(x)$$

with the boundary conditions $y(0) =$ finite, $y(a) = 0$.

Butting the above equation in Sturm-Liouville in the form:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (k^2x - \frac{n^2}{x})y = \frac{f(x)}{x}$$

where $P(x) = x$ .The Green' function $G_n(x,\xi)$ in the form:

$$\frac{d}{dx}\left(x\frac{dG_n}{dx}\right) + (k^2x - \frac{n^2}{x})G_n = -\delta(x-\xi)$$

With the general solution $c_1 J_n(kx) + c_2 N_n(kx)$. Hence:

$$G(x,x') = \begin{cases} d_1 J_n(kx) + d_2 N_n(kx) & (0 \le x < \xi) \\ c_1 J_n(kx) + c_2 N_n(kx) & (\xi < x \le a) \end{cases}$$

With the boundary conditions $y(0) = \text{finite} \Rightarrow d_2 = 0$ and $u(x) = J_n(kx)$. With $y(a) = 0$, we have $c_1 J_n(ka) + c_2 N_n(ka) = 0 \quad \Rightarrow \quad c_2 = -\frac{J_n(ka)}{N_n(ka)}c_1$ and

$$v(x) = \frac{J_n(kx)N_n(ka) - J_n(ka)N_n(kx)}{N_n(ka)}.$$

With knowing the $u(x)$ and $v(x)$, we have to evaluate the Wronskian $W(\xi) = u\,v' - u'\,v$. To do so, we can choose any convenient point such as $x \to 0$.

$$\lim_{x\to 0} J_n(kx) \to \frac{1}{n!}\left(\frac{kx}{2}\right)^n, \qquad \lim_{x\to 0} N_n(kx) \to -\frac{(n-1)!}{\pi}\left(\frac{2}{kx}\right)^n$$

H.W. Prove that

$$W(\xi) = u\,v' - u'\,v = -\frac{J_n(ka)}{N_n(ka)}\left(W\left[J_n(kx), N_n(kx)\right]\right) = -\frac{J_n(ka)}{N_n(ka)}\left(\frac{2}{\pi x}\right)$$

$$p(\xi) = \xi \quad \Rightarrow \quad A = W(\xi)p(\xi) = --\frac{J_n(ka)}{N_n(ka)}\left(\frac{2}{\pi\xi}\right)\xi$$

Hence

$$G(x,x') = \begin{cases} \dfrac{\pi}{2}\dfrac{J_n(k\xi)N_n(ka) - J_n(ka)N_n(k\xi)}{N_n(ka)} J_n(kx) & (0 \le x < \xi) \\[2ex] \dfrac{\pi}{2}\dfrac{J_n(kx)N_n(ka) - J_n(ka)N_n(kx)}{N_n(ka)} J_n(k\xi) & (\xi < x \le a) \end{cases}$$

Green's Function in Free Space

Example: Solve $\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$ using the Fourier's transformation.

Answer: Start with the Fourier transformation in one dimension:

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} G(k)\,d\,k_x \quad \Rightarrow \quad \nabla^2 G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (ik)^2 e^{ikx} G(k)\,d\,k_x$$

In three dimensions:

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\,\mathbf{k}\cdot\mathbf{r}} G(\mathbf{k})\,d\,\mathbf{k} \quad \Rightarrow \quad \nabla^2 G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} (ik)^2 e^{i\,\mathbf{k}\cdot\mathbf{r}} G(\mathbf{k})\,d\,\mathbf{k}$$

For a continuous function in three dimensions, one finds:

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\,\vec{\mathrm{k}}.(\mathbf{r}-\mathbf{r}')} d^3k$$

Thus $\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$ will be:

$$\nabla^2 G(\mathbf{r}) + \delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} (ik)^2 e^{i\,\mathbf{k}\cdot\mathbf{r}} G(\mathbf{k})\,d\,\mathbf{k} + \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\,\vec{\mathrm{k}}.(\mathbf{r}-\mathbf{r}')} d^3k = 0$$

This implies:

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \left\{(-k^2)G(\mathbf{k}) + e^{-i\,\mathbf{k}\cdot\mathbf{r}'}\right\} e^{i\,\mathbf{k}\cdot\mathbf{r}} d^3k = 0$$

Consequently,

$$\Rightarrow \quad -k^2 G(\mathbf{k}) + e^{-i\,\mathbf{k}\cdot\mathbf{r}'} = 0 \quad \Rightarrow \quad \boxed{G(\mathbf{k}) = \frac{e^{-i\,\mathbf{k}\cdot\mathbf{r}'}}{k^2}}$$

Thus:

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\int e^{i\,\mathbf{k}.\mathbf{r}} G(\mathbf{k})\,d\,\mathbf{k} = \frac{1}{(2\pi)^3}\int e^{i\,\mathbf{k}.\mathbf{r}} \frac{e^{-i\,\mathbf{k}\cdot\mathbf{r}'}}{k^2} d\,\mathbf{k}$$

$$= \frac{1}{(2\pi)^3}\int \frac{e^{i\,\mathbf{k}\cdot\mathbf{R}}}{k^2} d\,\mathbf{k}^*, \qquad\qquad \mathbf{R} = \mathbf{r} - \mathbf{r}'$$

H.W. Do the integration with the notations:

$d\,\mathbf{k} = k^2 dk\,d\Omega = k^2 dk \sin\theta\, d\theta\, d\phi$ and $\mathbf{k}\cdot\mathbf{R} = kr\cos\theta$, one finds:

Answer:

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{2\pi^2 R}\underbrace{\int_0^{\infty} \frac{\sin(kr)}{k} dk}_{\pi/2} = \frac{1}{4\pi R} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

[*Help: prove the standard integral $I_1 = \int \frac{e^{-br \pm i\,\mathbf{q}\cdot\mathbf{r}}}{r} d\tau = \frac{4\pi}{b^2+q^2}$]

H.W. Prove that the Green's function $G(\mathbf{r}) = \dfrac{e^{\pm ikr}}{r}$ is the solution of the scalar wave equation:

$$\left(\nabla^2 \pm k^2\right) G(\mathbf{r}) = -4\pi\delta(\mathbf{r}) \tag{1}$$

Proof: We have two cases:

Case I: $r \neq 0$, we have to prove that $\left(\nabla^2 + k^2\right)\left(\dfrac{e^{ikr}}{r}\right) = 0$

Start with $\nabla^2\left(\dfrac{e^{ikr}}{r}\right) = \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d}{dr}(\dfrac{e^{ikr}}{r})\right) = -\dfrac{k^2}{r}e^{ikr}, \Rightarrow \quad \left(\nabla^2 + k^2\right)\left(\dfrac{e^{ikr}}{r}\right) = 0,$

Case II: $r = 0$, we have to prove that $\int\left(\nabla^2 + k^2\right)G(\mathbf{r})d^3r = -4\pi\int\delta(\mathbf{r})d^3r = -4\pi$

In this case let us construct the solution of a scalar wave equation in any volume V of free space having an arbitrary small radius ξ and having an arbitrary source $\rho(\mathbf{r})$, then

a)

$$\begin{aligned}\int\limits_{\text{sphere of radius } \xi} \nabla^2 G(\mathbf{r})d^3r &= \int\nabla\cdot\left(\nabla G(\mathbf{r})\right)d^3r \underbrace{=}_{\text{using the Green's identity}} \int\nabla G(\mathbf{r})\cdot d\vec{\mathrm{s}} \\ &= \int\hat{\mathrm{r}}\frac{d}{dr}\left(\frac{e^{ikr}}{r}\right)\cdot\hat{\mathrm{r}}\, r^2 d\Omega = 4\pi\left[ik\,\xi e^{ik\xi} - e^{ik\xi}\right]\end{aligned} \tag{A}$$

b)

$$\begin{aligned}k^2\int G(\mathbf{r})d^3r &= k^2\int\frac{e^{ikr}}{r}r^2 dr d\Omega = 4\pi\left[\left(\frac{e^{ikr}}{ik}\right)\Bigg|_0^{\xi} - \frac{1}{ik}\int_0^{\xi} e^{ikr}dr\right] \\ &= 4\pi\left[-ik\,\xi e^{ik\xi} + e^{ik\xi} - 1\right]\end{aligned} \tag{B}$$

From the final results of equations A and B, one finds

$$\int\left(\nabla^2 + k^2\right)G(\mathbf{r})d^3r = -4\pi$$

Applications of Green's function
The Born Approximation
(9.7.1, 2)

Exercise: 9.7.18: Integral scattering equation for stationary states

The time independent Schrodinger equation in the form:

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\Psi(\mathbf{r}) = E\,\Psi(\mathbf{r}), \qquad E = \frac{\hbar^2 k^2}{2\mu} > 0$$

could be changed to the non-homogeneous equation:

$$\left[\nabla^2 + k^2\right]\Psi(\mathbf{r}) = U(r)\Psi(\mathbf{r}), \qquad U(r) = \frac{2\mu}{\hbar^2}V(r)$$

We claim that the solutions may be written in the form:

$$\Psi(\mathbf{r}) = \varphi_i(r) + \int G(\mathbf{r},\mathbf{r}')U(r')\Psi(\mathbf{r}')d^3r',$$

where $\varphi_i(r)$ is a solution of the homogeneous equation:

$$\left[\nabla^2 + k^2\right]\varphi_i(r) = 0,,$$

and $G(\boldsymbol{r})$ is a solution of

$$\left[\nabla^2 + k^2\right]G(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}'), \qquad .$$

H. W. Proof the claim:

Exercise: 9.7.16: The solutions of $\left(\nabla^2 + k^2\right)G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}')$ are $G^{\pm}(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$.

These are two linearly independent solutions of a second order differential equation. One is representing the outgoing wave, e^{+ikr}, and the other the incoming wave, e^{-ikr}. We will be interest in outgoing wave.

The Born approximation

$$\Psi(\mathbf{r}) = \varphi_i(r) + \int G(\mathbf{r},\mathbf{r}')U(r')d^3r'\left[\varphi_i(r') + \int G(\mathbf{r}',\mathbf{r}'')U(r'')\varphi_i(r'')\cdots\right].$$

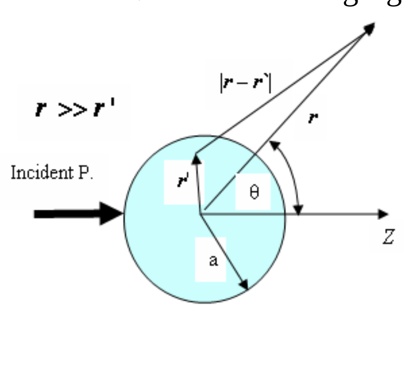
with $d^3\mathrm{r} = r^2 dr\,d\Omega = r^2 dr \sin\theta\,d\theta\,d\phi$. This procedure can be repeated and yields the **Born expansion**. The Born approximation is the first term in the Born expansion.

$$\Psi(\mathbf{r}) \approx \varphi_i(r) + \int G(\mathbf{r},\mathbf{r}')U(r')\varphi_i(r')d^3r' = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{1}{4\pi}\int\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}U(r')\Psi(\mathbf{r}')d^3r',$$

This yields an integral expression for the scattering amplitude $f\left(\theta,\phi\right)$!.

Exercise: Simplify the first Born approximation in the case that $r >> r'$.

Answer: In the following figure, one finds:



$$|\vec{r}-\vec{r}\,'| = \sqrt{\vec{r}^{\,2} + \vec{r}\,'^2 - 2\vec{r}\cdot\vec{r}\,'} = r\left(1 - 2\frac{\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{\vec{r}\,'^2}{r^2}\right)^{1/2}$$

$$\xrightarrow[r\to\infty]{} r\left(1 - \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) \approx r - \frac{\vec{r}}{r}\cdot\vec{r}\,' \approx r - \hat{r}\cdot\vec{r}\,', \qquad \text{(a)}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\frac{1}{r}\left(1-2\frac{\vec{r}\cdot\vec{r}\,'}{r^2}+\frac{\vec{r}\,'^2}{r^2}\right)^{-1/2}\xrightarrow[r\to\infty]{}\frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right)\approx\frac{1}{r}\text{ (b)}$$

From the above two equations we have:

$$\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\approx\frac{e^{ik(r-\hat{r}\cdot\vec{r}\,')}}{r}=\frac{e^{ikr}e^{-ik(\hat{r}\cdot\vec{r}\,')}}{r}=\frac{e^{ikr}}{r}e^{-i\mathbf{k}_f\cdot\mathbf{r}'}$$

Then

$$\Psi(r)\approx\varphi_i(r)+\left\{-\frac{1}{4\pi}\int e^{-i\mathbf{k}_f\cdot\mathbf{r}'}U(r')\varphi_i(r')d^3r'\right\}\frac{e^{ikr}}{r}$$

The term in curly bracket is called the “Scattering amplitude $f_B(\theta,\phi)$”,

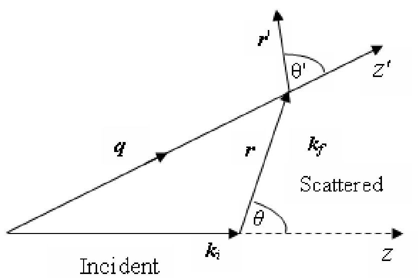
$$f_B(\theta,\phi)=-\frac{1}{4\pi}\int e^{-i\vec{k}_f\cdot\vec{r}\,'}U(r')\varphi_i(r')d^3r'=-\frac{1}{4\pi}<\varphi_f\,|U\,|\varphi_i>$$

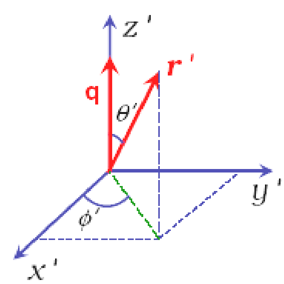
Using $\varphi_i(r')=e^{ik_i r'}$, then

$$f_B(\theta,\phi)=-\frac{1}{4\pi}\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}U(r')d^3r',\qquad \boldsymbol{q}=\boldsymbol{k}_i-\boldsymbol{k}_f$$

$\hbar\boldsymbol{q}\equiv$ momentum transfer. In case of elastic collision: $k_i=k_f$

$$|\boldsymbol{q}|^2=\left(\boldsymbol{k}_i-\boldsymbol{k}_f\right)^2=k_i^2+k_f^2-2k_ik_f\cos\theta=2k^2\left(1-\cos\theta\right)=4k^2\sin^2(\theta/2)$$





H.W. For central potential, use the relation $\boldsymbol{q}\cdot\boldsymbol{r}'=q\,r'\cos\theta'$ to derive the relation:

$$f_B(\theta)=-\frac{2\mu}{q\hbar^2}\int_0^\infty V(r')\sin(qr')r'dr'$$

Answer:

$$f_B(\theta)=-\frac{\mu}{2\pi\hbar^2}\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}V(r')d^3r'$$

$$=-\frac{\mu}{2\pi\hbar^2}\int_0^\infty V(r')r'^2dr'\underbrace{\int_0^{2\pi}d\varphi'}_{2\pi}\underbrace{\int_{-1}^{1}e^{iqr'\cos\theta'}d\cos\theta'}_{\frac{2\sin(qr')}{qr'}}=-\frac{2\mu}{q\hbar^2}\int_0^\infty V(r')\sin(qr')r'dr'$$

H.W. Do exercise 9.7.19 for Yukawa potential. You may need $I_1=\int\frac{e^{-br\pm i\boldsymbol{q}\cdot\boldsymbol{r}}}{r}d\boldsymbol{r}=\frac{4\pi}{b^2+q^2}$.

$$I_1 = \int \frac{e^{-br \pm i\,\boldsymbol{q}\cdot\boldsymbol{r}}}{r} d\boldsymbol{r} = \frac{4\pi}{b^2 + q^2};$$

$$I_2 = \int e^{-br \pm i\,\boldsymbol{q}\cdot\boldsymbol{r}} d\boldsymbol{r} = -\frac{\partial I_1}{\partial b} = \frac{8\pi b}{(b^2 + q^2)^2};$$

$$I_4 = \int \frac{e^{\pm i\,\boldsymbol{q}\cdot\boldsymbol{r}}}{|\,\boldsymbol{r} - \boldsymbol{r}'\,|} d\boldsymbol{r} = \frac{4\pi}{q^2} e^{\pm i\boldsymbol{q}\cdot\boldsymbol{r}'}$$

$$I_5 = \iint \frac{e^{-2b(r_1 + r_2)}}{r_1} d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \iint \frac{e^{-2b(r_1 + r_2)}}{r_2} d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \frac{\pi^2}{b^5}$$

$$I_6 = \iint \frac{e^{-2b(r_1 + r_2)}}{r_{12}} d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \frac{5\pi^2}{8b^5}, \qquad r_{12} = |\,\boldsymbol{r}_2 - \boldsymbol{r}_1\,|$$

$$I_7 = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t^2 + \tau^2} dt = \frac{\pi}{\tau} e^{-\omega\tau}$$

$$I_8 = \left| \int e^{-br + i\,\omega r} dr \right|^2 = \frac{1}{b^2 + \omega^2}$$