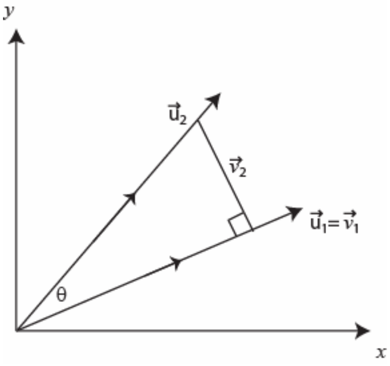
<u>Gram-Schmidt procedure</u> (10.3 Page 641)

To construct an orthonormalize vectors from un-orthonormalize vectors.
The first two steps of the Gram–Schmidt process

The sequence $\mathbf{v}_1, ..., \mathbf{v}_n$ is the required system of orthogonal vectors, and the normalized vectors $\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n$ form an orthonormal set. The calculation of the sequence $\mathbf{v}_1, ..., \mathbf{v}_k$ is known as *Gram–Schmidt orthogonalization*, while the calculation of the sequence $\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_n$ is known as *Gram–Schmidt orthonormalization* as the vectors are normalized.

To check that these formulas yield an orthogonal sequence, first compute $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ by substituting the above formula for $\mathbf{v}_2$: we get zero. Then use this to compute $\langle \mathbf{v}_1, \mathbf{v}_3 \rangle$ again by substituting the formula for $\mathbf{v}_3$: we get zero.

Geometrically, this method proceeds as follows: to compute $\mathbf{v}_i$, it projects $\mathbf{u}_i$ orthogonally onto the subspace V generated by $\mathbf{v}_1, ..., \mathbf{v}_{i-1}$, which is the same as the subspace generated by $\mathbf{u}_1, ..., \mathbf{u}_{i-1}$. The vector $\mathbf{v}_i$ is then defined to be the difference between $\mathbf{u}_i$ and this projection, guaranteed to be orthogonal to all of the vectors in the subspace V.



Step 1: Let $\vec{v}_1 = \vec{u}_1$,

Step 2: Graphically,

$$\vec{v}_2 = \vec{u}_2 - (|\vec{u}_2|\cos\theta)\underbrace{\hat{u}_1}_{\hat{v}_1} = \vec{u}_2 - (|\vec{u}_2|\cos\theta)\hat{v}_1$$

Where $(|\vec{u}_2|\cos\theta)\underbrace{\hat{u}_1}_{\hat{v}_1}$ is the projection of $\vec{u}_2$ on $\hat{u}_1$.

With the definition:

$$\hat{v}_1 = \frac{\vec{v}_1}{|\vec{v}_1|}, \quad \text{and} \quad \vec{u}_1 \cdot \vec{u}_2 = |\vec{u}_1||\vec{u}_2|\cos\theta = |\vec{v}_1||\vec{u}_2|\cos\theta \quad \Rightarrow |\vec{u}_2|\cos\theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{|\vec{v}_1|} = \frac{\vec{v}_1 \cdot \vec{u}_2}{|\vec{v}_1|}$$

Then:

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{v}_1 \cdot \vec{u}_2}{\|\vec{v}_1\|^2}\vec{v}_1,$$

In general:

$$\vec{v}_n = \vec{u}_n - \sum_{m=1}^{n-1} \frac{\vec{u}_n \cdot \vec{v}_m}{\|\vec{v}_m\|^2}\vec{v}_m,$$

And in Dirac's notation:

$$|v_n\rangle = |u_n\rangle - \sum_{m=1}^{n-1} \frac{\langle u_n | v_m \rangle}{|\langle v_m | v_m \rangle|^2}|v_m\rangle,$$

Example: Let $V = R^3$ with the Euclidian inner product. Apply Gram-Schmidt algorithm to orthogonalize the basis $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Ans:

$$\mathbf{v}_1 = \mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \qquad \|\mathbf{v}_1\|^2 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2 | \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \|\mathbf{v}_2\|^2 = \frac{2}{3}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3 | \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3 | \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right]}{\frac{2}{3}}\left[\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{6}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

You can verify that $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ forms an orthogonal basis for $V = R^3$.

Normalize the vectors in the orthogonal basis we have:

$$\|\mathbf{v}_1\|^2 = A^2\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 \quad \Rightarrow \quad A = \sqrt{\frac{1}{3}}.$$

$$\|\mathbf{v}_2\|^2 = B^2\frac{1}{3}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right] = B^2\frac{1}{9}(6) = 1 \quad \Rightarrow \quad B = \sqrt{\frac{3}{2}}.$$

$$\|\mathbf{v}_3\|^2 = C^2\frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}\left[\frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right] = C^2\frac{1}{4}(2) = 1 \quad \Rightarrow \quad C = \sqrt{2}.$$

$$\text{V} = \left\{ \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \right\}$$

Example 10.3.1 (page 644): Legendre Polynomials by GS Orthogonalization.
Consider the set:

$$f_0(x)=1,\quad f_1(x)=x,\quad f_2(x)=x^2,\quad f_3(x)=x^3$$

Which form a basis for the space of all real polynomials of degree ≤ 3 on the interval $-1\le x\le 1$. This means that any polynomial $p(x)$ in this space can be written as:

$$p(x)=a_0+a_1x+a_2x^2+a_3x^3$$

a- Use the inner product:

$$\langle f_i|f_j\rangle=\int_{-1}^{1}f_i^*(x)f_j(x)dx$$

And the Gram-Schmidt procedure to form the corresponding set of orthogonal polynomials on the interval $-1\le x\le 1$.
b- Verify that the new set forms a mutually orthogonal set.
c- Calculate the normalized set.
Answer:
a-

$$\boxed{p_o(x)=f_0(x)=1,}$$

$$p_1(x)=f_1(x)-\frac{\langle f_1|p_0\rangle}{|p_0|^2}p_0=x-\frac{\langle f_1|p_0\rangle}{|p_0|^2}p_0;$$

$$\left.\begin{array}{l}\langle f_1|p_0\rangle=\int_{-1}^{1}x\,dx=0\\ |p_0|^2=\langle p_0|p_0\rangle=\int_{-1}^{1}1\,dx=2\end{array}\right\}\Rightarrow \boxed{p_1(x)=f_1(x)=x}$$

$$p_2(x)=f_2(x)-\frac{\langle f_2|p_0\rangle}{|p_0|^2}p_0-\frac{\langle f_2|p_1\rangle}{|p_1|^2}p_1=x^2-\frac{\langle f_2|p_0\rangle}{|p_0|^2}p_0-\frac{\langle f_2|p_1\rangle}{|p_1|^2}p_1;$$

$$\left.\begin{array}{l}\langle f_2|p_0\rangle=\int_{-1}^{1}x^2dx=\frac{2}{3},\quad |p_0|^2=\langle p_0|p_0\rangle=\int_{-1}^{1}1\,dx=2\\ \langle f_2|p_1\rangle=\int_{-1}^{1}x^3dx=0\\ |p_1|^2=\langle p_1|p_1\rangle=\int_{-1}^{1}x^2dx=\frac{2}{3}\end{array}\right\}\Rightarrow \boxed{p_2(x)=x^2-\frac{1}{3}}$$

$$p_3(x)=f_3(x)-\frac{\langle f_3|p_0\rangle}{|p_0|^2}p_0-\frac{\langle f_3|p_1\rangle}{|p_1|^2}p_1-\frac{\langle f_3|p_2\rangle}{|p_2|^2}p_2=x^3-\frac{\langle f_3|p_0\rangle}{|p_0|^2}p_0-\frac{\langle f_3|p_1\rangle}{|p_1|^2}p_1-\frac{\langle f_3|p_2\rangle}{|p_2|^2}p_2;$$

$$\left.\begin{array}{l}\langle f_3|p_0\rangle=\int_{-1}^{1}x^3dx=0\\ \langle f_3|p_1\rangle=\int_{-1}^{1}x^4dx=\frac{2}{5},|p_1|^2=\langle p_1|p_1\rangle=\int_{-1}^{1}x^2dx=\frac{2}{3}\\ \langle f_3|p_2\rangle=\int_{-1}^{1}x^3(x^2-\frac{1}{3})dx=0\\ |p_2|^2=\langle p_2|p_2\rangle=\int_{-1}^{1}(x^2-\frac{1}{3})^2dx=\frac{8}{45}\end{array}\right\}\Rightarrow \boxed{p_3(x)=x^3-\frac{(2/5)}{(2/3)}x=x^3-\frac{3}{5}x}$$

c- The normalized set

$$p_0(x)=\frac{p_o(x)}{|p_0|}=\frac{1}{\sqrt{2}},\qquad p_1(x)=\frac{p_1(x)}{|p_1|}=\frac{x}{\sqrt{2/3}}=\sqrt{\frac{3}{2}}x,$$

$$p_2(x)=\frac{p_2(x)}{|p_2|}=\frac{\left(x^2-\frac{1}{3}\right)}{\sqrt{8/45}}=\sqrt{\frac{5}{8}}\left(3x^2-1\right),$$

$$p_3(x)=\frac{p_3(x)}{|p_3|}=\frac{\left(x^2-\frac{3}{5}x\right)}{\sqrt{8/175}}=\sqrt{\frac{7}{8}}\left(5x^3-3x\right),$$

Example: Let $V = R^3$ with the Euclidian inner product. Apply Gram-Schmidt algorithm to orthogonalize the basis $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$.

Ans:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2 | \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3 | \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3 | \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 3 & 7 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 & 7 & -1 \end{pmatrix}\left[\frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}\right]}{\frac{1}{5}\begin{pmatrix} 1 & 2 & -5 \end{pmatrix}\left[\frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}\right]}\left[\frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} - \frac{(-1)}{5}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{(22/5)}{(30/25)}\begin{pmatrix} 1/5 \\ 2/5 \\ -5/5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8 \\ 16 \\ 8 \end{pmatrix}$$

The normalization could be calculated as follows:

$$\|\mathbf{v}_1\|^2 = A^2\begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 1 \quad \Rightarrow \quad A = \sqrt{\frac{1}{5}}.$$

$$\|\mathbf{v}_2\|^2 = B^2\frac{1}{5}\begin{pmatrix} 1 & 2 & -5 \end{pmatrix}\left[\frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}\right] = B^2\frac{1}{25}(30) = 1 \quad \Rightarrow \quad B = \sqrt{\frac{5}{6}}.$$

$$\|\mathbf{v}_3\|^2 = C^2\frac{1}{3}\begin{pmatrix} 8 & 16 & 8 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} 8 \\ 16 \\ 8 \end{pmatrix}\right] = C^2\frac{1}{9}(??) = 1 \quad \Rightarrow \quad C = \sqrt{?}.$$

2- **a)**

Given that $|f_o\rangle = 1$, $|f_1\rangle = x$, $|f_2\rangle = x^2$, $|f_3\rangle = x^3$

To produce a set of orthoginal polynomials using G.S process:

let $|w_1\rangle = |f_o\rangle = 1$

$$|w_2\rangle = |f_1\rangle - \frac{\langle w_1|f_1\rangle}{\langle w_1|w_1\rangle}|w_1\rangle \quad \text{But} \quad \langle w_1|f_1\rangle = \int_{-1}^{1} x \; dx = 0 \;\Rightarrow\; |w_2\rangle = |f_1\rangle = x$$

$$|w_3\rangle = |f_2\rangle - \frac{\langle w_1|f_2\rangle}{\langle w_1|w_1\rangle}|w_1\rangle - \frac{\langle w_2|f_2\rangle}{\langle w_2|w_2\rangle}|w_2\rangle \quad \text{But} \quad \langle w_2|f_2\rangle = \int_{-1}^{1} x^3 \; dx = 0$$

$$\text{And} \quad \langle w_1|f_2\rangle = \int_{-1}^{1} x^2 \; dx = \frac{2}{3} \quad \& \quad \langle w_1|w_1\rangle = \int_{-1}^{1} dx = 2 \quad \Rightarrow \quad |w_3\rangle = x^2 - \frac{1}{3}$$

$$|w_4\rangle = |f_3\rangle - \frac{\langle w_1|f_3\rangle}{\langle w_1|w_1\rangle}|w_1\rangle - \frac{\langle w_2|f_3\rangle}{\langle w_2|w_2\rangle}|w_2\rangle - \frac{\langle w_3|f_3\rangle}{\langle w_3|w_3\rangle}|w_3\rangle$$

$$\left.\begin{array}{l} \langle w_1|f_3\rangle = \int_{-1}^{1} x^3 \; dx = 0 \quad \& \quad \langle w_3|f_3\rangle = \int_{-1}^{1} \left(x^5 - \frac{x^3}{3} \right) dx = 0 \\ \langle w_2|f_3\rangle = \int_{-1}^{1} x^4 \; dx = \frac{2}{5} \quad \& \quad \langle w_2|w_2\rangle = \int_{-1}^{1} x^2 \; dx = \frac{2}{3} \end{array}\right\} \Rightarrow \quad |w_4\rangle = x^3 - \frac{3}{5}x$$

$\therefore$ The set of orthogonal polynomials:

$$\left\{ |w_1\rangle = 1 \;,\; |w_2\rangle = x \;,\; |w_3\rangle = x^2 - \frac{1}{3} \;,\; |w_4\rangle = x^3 - \frac{3}{5}x \right\}$$

b)

$$\left.\begin{aligned}
\langle w_1|w_2\rangle &= \int_{-1}^{1} x\ dx = \frac{1}{2}-\frac{1}{2}=0\\
\langle w_1|w_3\rangle &= \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)dx = \frac{1}{3}-\frac{1}{3}-\frac{1}{3}+\frac{1}{3}=0\\
\langle w_1|w_4\rangle &= \int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)dx = \frac{1}{4}-\frac{3}{10}-\frac{1}{4}+\frac{3}{10}=0\\
\langle w_2|w_3\rangle &= \int_{-1}^{1}\left(x^3-\frac{1}{3}x\right)dx = \frac{1}{4}-\frac{1}{6}-\frac{1}{4}+\frac{1}{6}=0\\
\langle w_2|w_4\rangle &= \int_{-1}^{1}\left(x^4-\frac{3}{5}x^2\right)dx = \frac{1}{5}-\frac{1}{5}+\frac{1}{5}-\frac{1}{5}=0\\
\langle w_3|w_4\rangle &= \int_{-1}^{1}\left(x^5-\frac{14}{15}x^3+\frac{1}{5}x\right)dx = 0
\end{aligned}\right\} \Rightarrow |w_1\rangle,\ |w_2\rangle,\ |w_3\rangle,\ |w_4\rangle \text{ are orthogonal}$$

c)

$$\left.\begin{aligned}
|v_1\rangle &= \frac{|w_1\rangle}{\sqrt{\langle w_1|w_1\rangle}}, \quad \langle w_1|w_1\rangle = 2 &&\Rightarrow |v_1\rangle = \frac{1}{\sqrt{2}}\\
|v_2\rangle &= \frac{|w_2\rangle}{\sqrt{\langle w_2|w_2\rangle}}, \quad \langle w_2|w_2\rangle = \int_{-1}^{1}\left(x^2\right)dx = \frac{2}{3} &&\Rightarrow |v_2\rangle = \frac{x}{\sqrt{2/3}}\\
|v_3\rangle &= \frac{|w_3\rangle}{\sqrt{\langle w_3|w_3\rangle}}, \quad \langle w_3|w_3\rangle = \int_{-1}^{1}\left(x^4-\frac{2}{3}x^2+\frac{1}{9}\right)dx = \frac{8}{45} &&\Rightarrow |v_3\rangle = \frac{x^2-(1/3)}{\sqrt{8/45}}\\
|v_4\rangle &= \frac{|w_4\rangle}{\sqrt{\langle w_4|w_4\rangle}}, \quad \langle w_4|w_4\rangle = \int_{-1}^{1}\left(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2\right)dx = \frac{4}{175} &&\Rightarrow |v_4\rangle = \frac{x^3-(3/5)x}{\sqrt{4/175}}
\end{aligned}\right\} \text{Normalized set}$$