Fourier transforms (Chapter 15)

Fourier integrals are generalizations of Fourier series. The series representation

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)]$$

of a function $f(x)$ is a periodic form on $-\infty<x<\infty$ obtained by generating the coefficients from the function's definition on the least period $[-L,L]$. If a function $f(x)$ defined on the set of all real numbers has no period, then an analogy to Fourier integrals can be envisioned as letting $L\to\infty$ and replacing the integer valued index, n, by a real valued function k. The coefficients a_n and b_n then take the form $a(k)$ and $b(k)$. This mode of thought leads to the following definition. We will assume the following conditions on $f(x)$

1- $f(x)$ satisfies the Dirichlet conditions in every finite interval $[-L,L]$.

2- $\int_{-\infty}^{+\infty}\left|f(x)dx\right|$ finite, converges, i.e. $f(x)$ is absolutely integrable in $[-\infty,\infty]$.

Fourier's formula for 2L-periodic functions using sines and cosines

For a $2L$-periodic function $f(x)$ that is integrable on $[-L,L]$, the numbers

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)] \tag{1}$$

is called the ***Fourier series*** of $f(x)$. Using the integrals:

$$\int_{-L}^{+L}\cos(\frac{m\pi x}{L})\cos(\frac{n\pi x}{L})dx=L\delta_{mn},\qquad \int_{-L}^{+L}\sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx=L\delta_{mn},$$

$$\int_{-L}^{+L}\sin(\frac{n\pi x}{L})\cos(\frac{n\pi x}{L})dx=0$$

one finds:

$$a_0=\frac{1}{L}\int_{-L}^{+L}f(x)dx,\qquad a_n=\frac{1}{L}\int_{-L}^{+L}f(x)\cos(\frac{n\pi x}{L})dx\qquad\text{and}\qquad b_n=\frac{1}{L}\int_{-L}^{+L}f(x)\sin(\frac{n\pi x}{L})dx;\quad n>0$$

are called the Fourier coefficients of f.

For continuous range, i.e. $L\to\pm\infty$, Equation (1) reduces to (note that $f(x)\to 0$ as $L\to\pm\infty$):

$$\begin{aligned}f(x)&=\frac{1}{2L}\int_{-L}^{+L}f(x')dx'+\sum_{n=1}^{\infty}\frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)\int_{-L}^{+L}f(x')\cos(\frac{n\pi x'}{L})dx'\\&\quad+\sum_{n=1}^{\infty}\frac{1}{L}\sin\left(\frac{n\pi x}{L}\right)\int_{-L}^{+L}f(x')\sin(\frac{n\pi x'}{L})dx'\\&=\cancel{\frac{1}{2L}\int_{-L}^{+L}f(x')dx'}+\frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{+L}f(x')\cos\left\{\frac{n\pi x'}{L}(x-x')\right\}dx'\end{aligned} \tag{2}$$

Let, $k=\frac{n\pi}{L}$; $\Rightarrow$ $\Delta k=\frac{\pi}{L}\Delta n=\frac{\pi}{L}$. If $f(x)$ is finite, then ($\left\{\text{using}\sum_{n=1}^{\infty}=\int_{0}^{\infty}\frac{L}{\pi}dk\right\}$)

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^{\infty} dk \int_{-\infty}^{+\infty} f(x')\cos\{k(x-x')\}dx' \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{+\infty} f(x')\cos\{k(x-x')\}dx' \end{aligned} \tag{3}$$

Since:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{+\infty} f(x')\sin\{k(x-x')\}dx' = 0$$

then, we can have:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(x')e^{ik(x-x')}dx' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{ikx}\underbrace{\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-ikx'}dx'\right\}}_{g(k)}$$

Where $g(k)$ is known as the ***Fourier transform*** of $f(x)$.

Applications:

Example: Calculate the Fourier transformation of the Gaussian function $f(x) = e^{-x^2}$.

Answer:

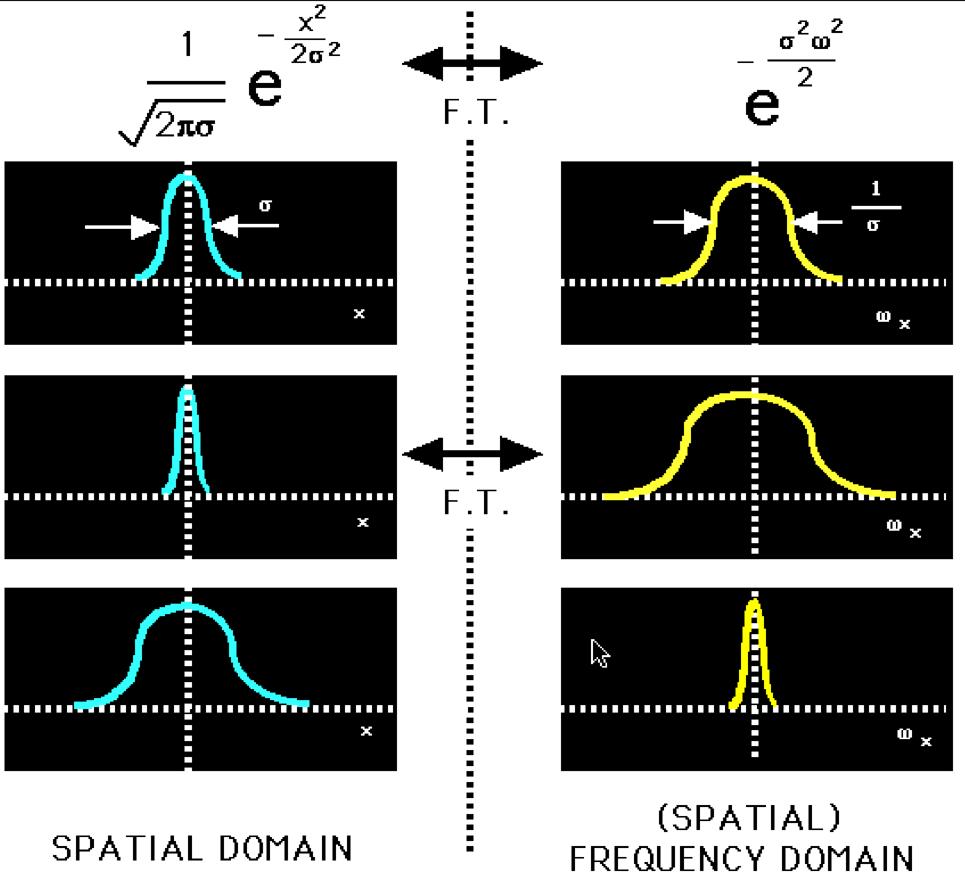
$$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2-ikx}dx = \frac{e^{-k^2/4}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-\left(x+\frac{ik}{2}\right)^2}dx}_{\sqrt{\pi}} = \frac{e^{-k^2/4}}{\sqrt{2}}$$

Note that: The Fourier transform of a Gaussian functions is a Gaussian function.

H.W. Plot both functions.

Prove that if $f(x) = Ne^{-\alpha x^2}$, then $g(k) = \frac{N}{\sqrt{2\alpha}}e^{-k^2/(4\alpha)}$, and vice versa.

The Fourier transform of the Gaussian function is another Gaussian:



Note that the width sigma is oppositely positioned in the arguments of the exponentials. This means **the narrower a Gaussian is in one domain, the broader it is in the other domain**.

IMAGE QUALITY

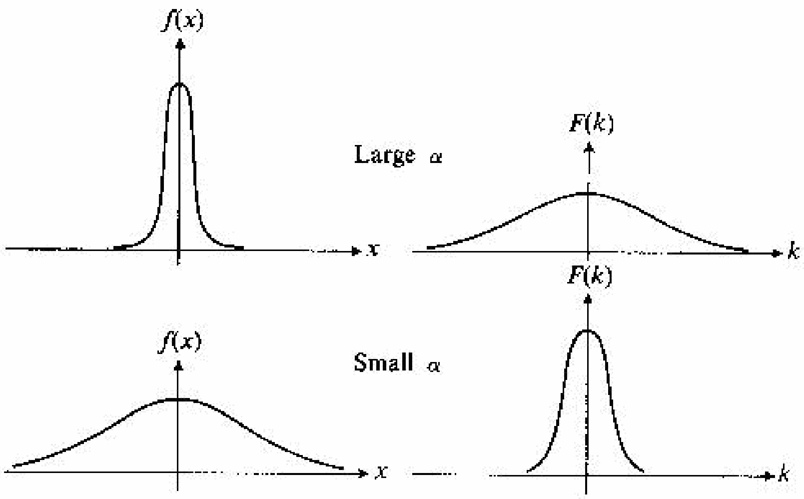
The Gaussian function can approximate the behavior of an imaging system. In particular, if we think of a very narrow slit of x-rays as being a line of delta functions, an x-ray screen will blur this delta line into a broader "ridge". It should be obvious that we want this ridge to be as narrow as possible. The imaging system's response to a delta function line input is called the **LINE SPREAD FUNCTION, or LSF** in the spatial domain. The *magnitude* of the complex function which is the Fourier transform of the LSF is the frequency-dependent function known as the **MODULATION TRANSFER FUNCTION, or MTF**.

Using what we have just learned about Gaussian functions, we conclude that the narrower the LSF, the broader the MTF in frequency space. Since we want narrow LSF's to produce sharper images, we want MTF's to stay high until a high spatial frequency is reached before it falls to zero. High frequencies are associated with sharp features in the image, and the MTF is the system's ability to record information as a function of frequency.

HEISENBERG UNCERTAINTY PRINCIPLE

In Quantum Mechanics, the Heisenberg uncertainty principle states that we cannot simultaneously know a particle's position and momemtum (or direction of motion). This is because the position wave function and the momentum wave function are Fourier transform pairs. The narrower one function becomes, the wider the pair becomes. The better we know position, the worse we know momentum.

It is of interest to observe that $F(k)$ is also a Gaussian probability function with a peak at the origin, monotone decreasing as $k \to \pm\infty$. Note, however, that if $f(x)$ is sharply peaked (large α), then $F(k)$ is flattened, and vice versa (Fig.). This is a general feature in the theory of Fourier transforms. In quantum-mechanical applications it is related to the Heisenberg uncertainty principle.



Note that both $f(x)$ and $F(k)$ are Gaussian distribution functions with peaks at the origin. The **standard deviation**, width, is defined as the range of the variable x (or k) for which the function $f(x)$ [or $F(k)$] drops by a factor of $e^{-1/2} = 0.606$ of its maximum value. For $f(x) = Ne^{-\alpha x^2}$, the standard deviation is given by

$$\sigma_x = \frac{\Delta x}{2} = \frac{1}{\sqrt{2\alpha}}.$$

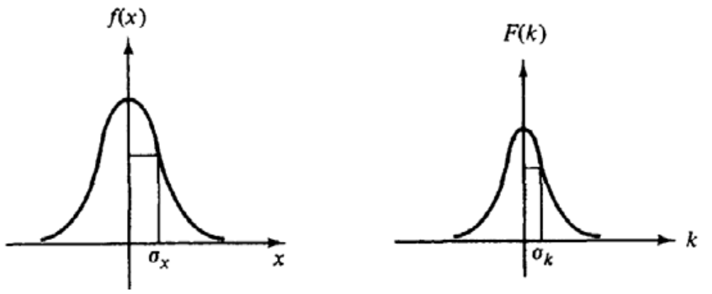
For

$$F(k) = \frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$$

the standard deviation is given by

$$\sigma_k = \frac{\Delta k}{2} = \sqrt{2\alpha}.$$

Note that $\Delta x\, \Delta k = (2/\sqrt{2\alpha})(2\sqrt{2\alpha}) = 4$. If $\alpha \to 0$ (small), then $\Delta x \to \infty$ and $\Delta k \to 0$. For $\alpha \to \infty$ (large), $\Delta x \to 0$ and $\Delta k \to \infty$ (see Fig.).



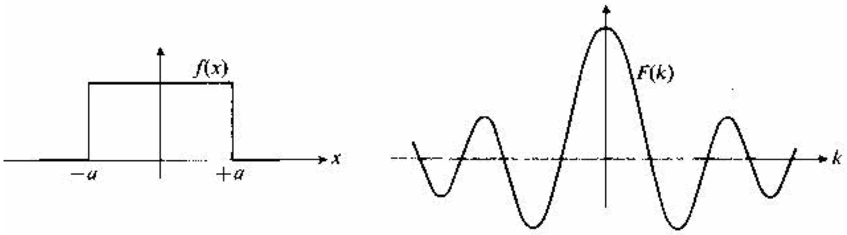
Example (15.5a): Calculate the Fourier transformation of the function

$$f(x)=\begin{cases}1, & |x|\le a\\ 0, & |x|>a\end{cases}, \qquad (a>0).$$

Answer: Its Fourier transform reads

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx'}dx'=\frac{1}{\sqrt{2\pi}}\frac{e^{-ika}-e^{ika}}{(-ik)}=\sqrt{\frac{2}{\pi}}\frac{\sin(ak)}{k}$$

The functions $f(k)$ and $F(k)$ are shown in the following figure.



Comment: This is the single-slit diffraction problem of physical optics. The slit is described by $f(x)$. The diffraction pattern amplitude is given by the Fourier transform $F(k)$, where $F(0)=a\sqrt{\frac{2}{\pi}}$.

Example: Dirac Delta function

Start with $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{ikx}dk$, and using $g(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-ikx'}dx'$ then

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-ikx'}dx'\right)e^{ikx}dk=\int_{-\infty}^{\infty} f(x')\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk\right)}_{\delta(x-x')}dx'=f(x)$$

For a continuous function in three dimensions, one finds:

$$\boxed{\delta(\mathbf{r}-\mathbf{r'})=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\,\vec{k}.(\mathbf{r}-\mathbf{r'})}d^3k} \qquad (15.21d)$$

1- Find the Fourier transformation of the triangular pulse.

$$f(x)=\begin{cases}1-|x| & |x|<1;\\ 0 & \text{Otherwise}\end{cases}$$

Answer: For the given function (it is even function), the Fourier transform is:

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{ikx}dx=\sqrt{\frac{2}{\pi}}\int_0^1(1-x)\cos kx\,dx$$

This can be integrated by parts to obtain

$$F(k)=\sqrt{\frac{2}{\pi}}\left[\frac{(1-x)\sin kx}{k}-\frac{\cos kx}{k^2}\right]_0^1=\sqrt{\frac{2}{\pi}}\frac{1-\cos k}{k^2}$$

2- Find the Fourier transformation of the function $f\,(\mathbf{r})=\dfrac{e^{-r}}{r}$.{Hint: use the spherical coordinates where $d\tau = r^2 \sin\theta d\theta d\varphi dr$]

Answer:

$$f\,(\mathbf{r})=\frac{e^{-r}}{r}\Rightarrow \quad f\,(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\int\frac{e^{-r+i\ \mathbf{q}\cdot\mathbf{r}}}{r}d\tau, \qquad d\tau = r^2 \sin\theta d\theta d\varphi dr$$

$$I_1=\int\frac{e^{-br+i\,\boldsymbol{q}\bullet\boldsymbol{r}}}{r}d\boldsymbol{r} = \int_0^{2\pi}d\phi\int_0^{\infty}r^2dr\left[\int_{-1}^{1}d\cos\theta e^{iqr\cos\theta}\right]\frac{e^{-br}}{r}$$

$$=2\pi\int_0^{\infty}r^2dr\left[\frac{e^{iqr}-e^{-iqr}}{iqr}\right]\frac{e^{-br}}{r}=\frac{2\pi}{iq}\int_0^{\infty}dr\left[e^{(iq-b)r}-e^{-(iq+b)r}\right]$$

$$=\frac{2\pi}{iq}\left[\frac{e^{(iq-b)r}\Big|_0^{\infty}}{iq-b}+\frac{e^{-(iq+b)r}\Big|_0^{\infty}}{iq+b}\right]=-\frac{2\pi}{iq}\left[\frac{1}{iq-b}+\frac{1}{iq+b}\right]=\frac{4\pi}{b^2+q^2}$$

$$f\,(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\int\frac{e^{-r+i\ \mathbf{q}\cdot\mathbf{r}}}{r}d\tau=\frac{1}{(2\pi)^{3/2}}\frac{4\pi}{1+q^2}=\sqrt{\frac{2}{\pi}}\frac{1}{1+q^2}$$

Example: Find the Fourier transformation of the function $f\,(\mathbf{r})=e^{-br}$. [Hint: $I_1=\int\dfrac{e^{-br\pm i\,\boldsymbol{q}\bullet\boldsymbol{r}}}{r}d\tau=\dfrac{4\pi}{b^2+q^2}$] , $d\tau = r^2 \sin\theta d\theta d\varphi dr$

$$f\,(\mathbf{r})=e^{-r}\Rightarrow f\,(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\int e^{-br+i\ \mathbf{q}\cdot\mathbf{r}}d\tau$$

$$=\frac{1}{(2\pi)^{3/2}}\frac{-\partial}{\partial b}\left[\int\frac{e^{-br\pm i\,\boldsymbol{q}\bullet\boldsymbol{r}}}{r}d\tau\right]=\frac{1}{(2\pi)^{3/2}}\frac{-\partial}{\partial b}\frac{4\pi}{b^2+q^2}=\frac{1}{(2\pi)^{3/2}}\frac{8\pi b}{(b^2+q^2)^2}$$

H.W. Calculate the following integrals:

$$I=\int\frac{e^{i\,\boldsymbol{k}\bullet\boldsymbol{R}}}{k^2}d\boldsymbol{k}=\frac{2\pi^2}{R}$$

$$I_2=\int e^{-br\pm i\,\boldsymbol{q}\bullet\boldsymbol{r}}d\tau=-\frac{\partial I_1}{\partial b}=\frac{8\pi b}{(b^2+q^2)^2};$$

$$I_3=\int\frac{e^{\pm i\,\boldsymbol{q}\bullet\boldsymbol{r}}}{|\text{r-r}'|}d\boldsymbol{r}=e^{\pm i\boldsymbol{q}\bullet\boldsymbol{r}'}\int\frac{e^{\pm i\,\boldsymbol{q}\bullet(\text{r-r}')}}{|\text{r-r}'|}d\boldsymbol{r}=e^{\pm i\boldsymbol{q}\bullet\boldsymbol{r}'}\lim_{b\to 0}I_1=\frac{4\pi}{q^2}e^{\pm i\boldsymbol{q}\bullet\boldsymbol{r}'}$$

Table 3.1. **Properties of Fourier transforms**

Property	If $f(x)$ is	Then its Fourier transform
Complex conjugation	Real	$g^*(k)=g(-k)$
	Real and even	$g^*(k)=g(-k)=g(k)$
	Real and odd	$g^*(k)=g(-k)=-g(k)$
Translation	$f(x-a)$	$e^{\ ika}g(k)$
Attenuation	$f(x)e^{ax}$	$g(k+ai)$
Derivatives	$\frac{d}{dx}f(x)$	$ikg(k)$
	$\frac{d^n}{dx^n}f(x)=f^{(n)}(x)$	$(ik)^n g(k)$