Two-dimensional Ising model

Reminder: The standard 2D Ising model consists of an $n \times n$ square lattice, at each site i of which is a spin $s_i = \pm 1$, with partition function (in the absence of an external field)

$$Z(K,N) = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} e^{K(S_1S_2+S_2S_3+\cdots+S_NS_1)} \tag{7.28}$$

Mean-field theory approximation gives a second-order phase transition in all dimensions D , at

$$\beta_c J = \frac{1}{2D} \qquad \Rightarrow k_B T_c = 2D\, J$$

Also for the two-dimensional Ising model there is an exact solution, due to Onsager (1944), which is, however, considerably more complicated than for the one-dimensional system. The partition function is given by

$$Z = \left[2\cosh(2K)e^{I}\right]^N \tag{7.29}$$

with

$$I = \int_0^\infty \frac{d\phi}{2\pi} \ln\left\{\frac{1}{2}\left[1 + (1-\kappa^2\sin^2\phi)^{1/2}\right]\right\} \tag{7.30}$$

$$\kappa = \frac{2\sinh(2K)}{\cosh^2(2K)} . \tag{7.31}$$

and the system exhibits spontaneous magnetization below a critical temperature

$$T_c = 2.269\, J/k_B , \quad \text{or} \quad \beta_c J = 0.44069 , \quad \text{at which} \quad \sinh(2\beta_c J) = 1 . \tag{7.32}$$

For temperatures just below the critical point,

$$C = \frac{\partial \bar{E}}{\partial T}\bigg|_{B=0} \sim -\frac{8 k_B N}{\pi} (\beta J)^2 \ln|T - T_c| , \qquad \bar{M} \sim \text{const} \times N |T - T_c|^{1/8} ,$$

consistently with a second-order phase transition.

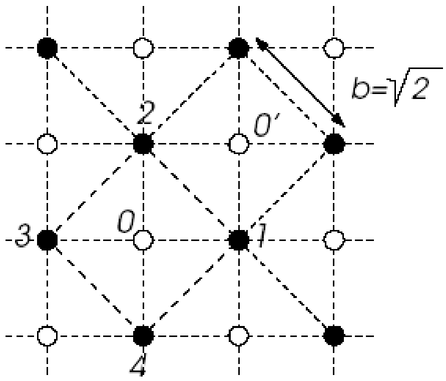

Fig.7.3: Decimation scheme for two-dimensional Ising model: Every second site is integrated yielding an effective coupling among all surrounding spins.

Now we turn to the renormalization group treatment. There are various decimation schemes we could imagine. In analogy to the one-dimensional case we divide the square lattice into two sublattices as shown in Fig. 7.3: The white sites are integrated out. We take the plaquette indicated in the figure. Spins 1, 2, 3, and 4 encircle spin 0. The latter, spin 0, couples through nearest neighbor interaction to the former four spins. Take:

$$H_0 = -J\ s_0(s_1 + s_2 + s_3 + s_4)$$

Thus, our decimation works as follows:

$$\begin{aligned} Z &= \cdots \sum_{s_1,s_2,s_3,s_4} \sum_{s_0} e^{K s_0 (s_1+s_2+s_3+s_4)} \cdots \\ &= \cdots \sum_{s_1,s_2,s_3,s_4} \left[e^{K(s_1+s_2+s_3+s_4)} + e^{-K(s_1+s_2+s_3+s_4)} \right] \cdots . \end{aligned} \tag{7.33}$$

We reformulate the partition function for the remaining spin degrees of freedom

$$Z = \cdots \sum_{s_1,s_2,s_3,s_4} e^{K_0'+K_1'(s_1s_2+s_2s_3+s_3s_4+s_4s_1)+K_2'(s_1s_3+s_2s_4)+K_3's_1s_2s_3s_4} \cdots . \tag{7.34}$$

Going through the space of spin configurations we find new effective interactions between the four surrounding spins with the relation[2]

$$\begin{aligned} K_0' &= \frac{1}{8}\ln\left\{\cosh^4(2K)\cosh(4K)\right\} + \ln 2 \\ K_1' &= \frac{1}{8}\ln\{\cosh(4K)\} \\ K_2' &= \frac{1}{8}\ln\{\cosh(4K)\} \\ K_3' &= \frac{1}{8}\ln\{\cosh(4K)\} - \frac{1}{2}\ln\{\cosh(2K)\} \end{aligned} \tag{7.38}$$

where

0- $K_0^{'}$ connected with the reduced free energy,

1- $K_1^{'}$ denote nearest spin-spin interaction ,

2- $K_2^{'}$ denote next-nearest-neighbor spin-spin interaction, and

3- $K_3^{'}$ gives a four-spin interaction on the plaquette.

Note that the neighboring plaquettes contribute to the nearest-neighbor interaction, e.g. summing in Fig. 7.3 over $s_0^{'}$ on site $0'$, yields another interaction between s_1 and s_2. Therefore we have to modify the second equation in (7.38) by multiplying by a factor 2,

$$K_1^{'} = \frac{1}{4}\ln[\cosh(4K)] \tag{7.39}$$

[2]The renormalization group relations are obtained by looking at different spin configurations for

$$e^{K(s_1+s_2+s_3+s_4)} + e^{-K(s_1+s_2+s_3+s_4)} = e^{K_0'+K_1'(s_1s_2+s_2s_3+s_3s_4+s_4s_1)+K_2'(s_1s_3+s_2s_4)+K_3's_1s_2s_3s_4} . \tag{7.35}$$

We use now the configurations

$$(s_1,s_2,s_3,s_4) = (+,+,+,+),(+,+,+,-),(+,-,+,-),(+,+,-,-) \tag{7.36}$$

and obtain the equations,

$$\begin{aligned} e^{4K}+e^{-4K} &= e^{K_0'+4K_1'+2K_2'+K_3'} , \\ e^{2K}+e^{-2K} &= e^{K_0'-K_3'} , \\ 2 &= e^{K_0'-4K_1'+2K_2'+K_3'} , \\ 2 &= e^{K_0'-2K_2'+K_3'} , \end{aligned} \tag{7.37}$$

whose solution leads to (7.38).

Unlike in the Ising chain we end up here with a different coupling pattern than we started. More spins are coupled on a wider range. Repeating the decimation procedure would even further enlarge the interaction range and complexity. This is not a problem in principle. However, in order to have a more practical iterative procedure we have to make an approximation. We restrict ourselves to the nearest neighbor interactions which would give a well-defined iterative procedure. But simply ignoring the other couplings which additionally help to align the spins would lead to an oversimplified behavior and would actually give no phase transition. Thus we have to add the other couplings in some way to the nearest-neighbor coupling. It can be shown that the four-spin interaction is small and not important close to the transition point and we concentrate on $K_1^{'}$ and $K_2^{'}$ only. Let us define the effective nearest-neighbor in a way to give the same ground state energy as both couplings. Each site has four nearest- and four next-nearest neighbors, which yield the energy per site for full spin alignment

$$E_0 = 2NK_1' + 2NK_2' = 2NK' \tag{7.40}$$

We introduce an approximation in which we neglect $K_3^{'}$ and replace $K_2^{'}$ by a modified, effective nearest-neighbor coupling constant $K^{'}\left(K_1^{'}, K_2^{'}\right)$ that takes into account next-nearest neighbors; one gets new renormalized nearest-neighbor coupling

$$K^{'} = K_1^{'} + K_2^{'} = \frac{3}{8}\ln[\cosh(4K)] \tag{7.41}$$

which now can be iterated and has:

1- a stable (trivial-solution) fixed points at $K = 0$ and $K = \infty$, and
2- an (nontrivial) unstable fixed point at

$$K_c = 0.50698 \tag{7.42}$$

The (nontrivial) unstable fixed point corresponds now to a finite-temperature phase transition at $K_c = \beta_c J \Rightarrow k_B T_c = \dfrac{J}{K_c} = 1.97\, J$ which is lower than the mean field result $k_B T_c = 4\, J$ but relatively inaccurate compared to the exact result of $2.27\, J$.

```
Solve[K - (3 / 8) Log[ Cosh[4 K]] == 0, K]  // N

{{K → 0.}, {K → 0.506981},
 {K → -0.103544 + 1.00843 i}, {K → -0.103544 - 1.00843 i}}
```

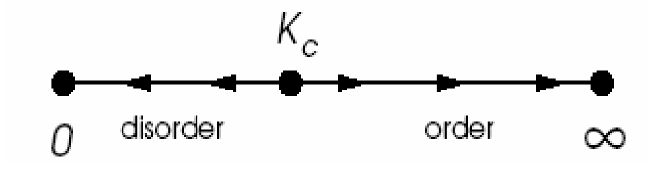

Fig.7.4: Renormalization group flow of coupling constant; the unstable fixed point K_c represents the critical point. On the left hand side, the flow leads to the stable fixed point $K = 0$ corresponding to the uncoupled spins: disorder phase (paramagnetic). The right hand side flows to the stable fixed point $K = \infty$, where system is ordered (ferromagnetic).

It is now interesting to consider the exponents which we had discussed above. Thus we take into account that in our decimation scheme $b = \sqrt{2}$ and calculate

$$K' - K_c = \frac{3}{8}\left[\ln\{\cosh(4K)\} - \ln\{\cosh(4K_c)\}\right] = \frac{3}{8}(K - K_c)\,\frac{d}{dK}\ln\{\cosh(4K)\}\Big|_{K=K_c}$$
$$= (K - K_c)\frac{3}{2}\tanh(4K_c) = 1.45(K - K_c)$$

Note that:

$$K' = \frac{3}{8}\ln[\cosh(4K_c)]$$
$$\Rightarrow \quad dK' = \frac{3}{2}\tanh(4K)\Big|_{K=K_c} dK = \frac{3}{2}\tanh(4\times 0.507)dK = 1.5\times 0.966\, dK = 1.449\, dK \tag{7.43}$$

$$\Rightarrow \quad b^{y_1} = 2^{y_1/2} = 1.45 \quad \Rightarrow \quad y_1 = 2\frac{\ln 1.45}{\ln 2} = 1.07 \,.$$

From this result we obtain the critical exponents ν and α:

$$\nu = \frac{1}{y_1} = 0.93 \quad \text{and} \quad \alpha = 2 - \frac{d}{y_1} = 0.135 \,. \tag{7.44}$$

The exact result is $\nu_{exact} = 1$ (mean field $\nu_{mf} = 1/2$) and $\alpha_{exact} = 0$ meaning that the specific heat has a logarithmic singularity not describable by an exponent.

The decimation method used here is only one among many which could be imagined and have been used. Unfortunately, for practice reasons approximations had to be made so that the results are only of qualitative value. Nevertheless these results demonstrate that non-trivial properties appear in the critical region close to the second order phase transition.

Other decimations schemes can be used. Decimations in real space are only one type of method; know under the name of block spin method. Alternatively, also decimations schemes in momentum space can be used. Depending on the problem and the purpose different methods are more useful.

Higher and more terms may be worse results !!!!!!!!: (Vilfan)

As we noticed; after decimation we get a more complicated Hamiltonian:

$$H' = -\Big[\sum K_0' + K_1'\left(S_{i+x}S_{i+y} + S_{i+y}S_{i-x} + S_{i-x}S_{i-y} + S_{i-y}S_{i+x}\right) + K_2'\left(S_{i+x}S_{i-x} + S_{i+y}S_{i-y}\right) + K_3'\left(S_{i+x}S_{i+y}S_{i-y}S_{i-x}\right)\Big] \quad (1)$$

where K_1' is the new NN interaction, K_2' the new NNN (next nearest neighbors) interaction, Q_0 the four-spin interaction, and K_0' a constant. Only with the Hamiltonian of this form, it is possible to satisfy the condition $Z' = Z$ for arbitrary configurations of $S_{\pm x,y}$. Although we started with only NN interactions, decimation generated NNN interaction and even four-spin interactions. So, we have to **start from the beginning again with a more complicated Hamiltonian** which includes all these interactions. In the following we shall neglect K_3 and assume that K_2 is small. In this case we obtain the following approximate recursion relations:

$$K_1' \approx 2K_1^2 + K_2$$
$$K_2' \approx K_1^2 \quad (2)$$

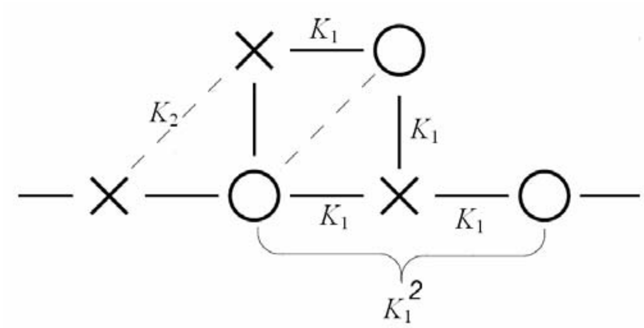

Figure 2: The renormalized interaction after decimation is $2K_1^2 + K_2$.

In fact, these relations could be guessed immediately, see Fig. 2. The new nearest neighbors are the previous NNN, therefore the term L in the first equation. Besides, they were connected by two pairs of consecutive bonds, this brings $2K_1^2$. In the second equation, the new next nearest neighbors (NNN) are connected by two consecutive NN bonds on the old lattice - each of them contributing a factor K_2.

The recursion relations (2) have:

1- two trivial fixed points,
   (a) $\left(K_1^*, K_2^*\right) = (0,0)$ corresponds to the infinite-temperature, paramagnetic fixed point,
   (b) $\left(K_1^*, K_2^*\right) = (\infty,\infty)$ corresponds to zero-temperature, ferromagnetic fixed point.

2- and one non-trivial fixed point at $\left(K_1^*, K_2^*\right) = \left(\frac{1}{3}, \frac{1}{9}\right)$, corresponds to the critical point of the system

Note: For no-trivial point, at the fixed points where we have $K_2^* \approx K_1^*$, which implies:

$$K_1^* \approx 2K_1^{*2} + K_1^{*2}$$

$$\Rightarrow K_1^*(1-3K_1^*) = 0 \Rightarrow K_1^* = \left(0, \frac{1}{3}\right)$$

Then the second equation implies:

$$K_2^* \approx K_1^{*2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Example: For the d = 2 Ising model, $\mu_1 = K_1$ and $\mu_2 = K_2$, and after linearization the equations $K_1' \approx 2K_1^2 + K_2, \quad K_2' \approx K_1^2$ become

$$\begin{aligned} \delta K_1' &= 4K_1^* \delta K_1 + \delta K_2; \\ \delta K_2' &= 2K_1^* \delta K_1 \end{aligned} \tag{4.103}$$

For the non-trivial fixed point at $\left(K_1^*, K_2^*\right) = \left(\frac{1}{3}, \frac{1}{9}\right)$, the RG transformation matrix is:

$$\mathcal{L} = \begin{vmatrix} 4K^* & 1 \\ 2K^* & 0 \end{vmatrix} = \begin{vmatrix} 4/3 & 1 \\ 2/3 & 0 \end{vmatrix} \tag{4.104}$$

The eigenvalues of $\mathcal{L}$ are:

$$\lambda_1 = \frac{2+\sqrt{10}}{3} \approx 1.72, \qquad \lambda_2 = \frac{2-\sqrt{10}}{3} \approx -0.387 \tag{4.105}$$

and the eigenvectors:

$$\begin{aligned} \vec{\nu}_1 &\propto \begin{pmatrix} 1+\sqrt{10}/2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2.58 \\ 1 \end{pmatrix} \\ \vec{\nu}_2 &\propto \begin{pmatrix} 1-\sqrt{10}/2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.58 \\ 1 \end{pmatrix} \end{aligned} \tag{4.106}$$

(Question: what does negative λ_2 mean physically, how does the system behave under successive RG transformations if λ is negative? **λ > 0 means relevant, and λ < 0 means irrelevant.**
) The corresponding critical exponents are:

$$\begin{aligned} x_1 &= 2\frac{\ln[(2+\sqrt{10})/3]}{\ln 2} \approx 1.57 \\ x_2 &= 2\frac{\ln|(2-\sqrt{10})/3|}{\ln 2} \approx -2.74, \end{aligned} \tag{4.107}$$

one is positive and the other one is negative.

$\nu = \frac{1}{x_1} = \frac{1}{1.57} = 0.64$ (it is worse), $\qquad \alpha = 2 - \frac{d}{x_1} = 2 - 2(0.62) = 0.76$ (again it is worse)

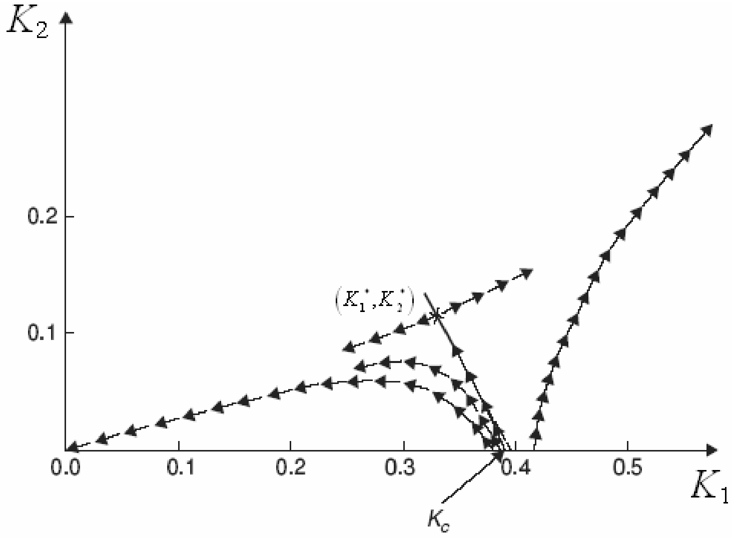

FIGURE A section of the critical curve for the two-dimensional Ising model near the nontrivial fixed point $\left(K_1^*, K_2^*\right) = \left(\frac{1}{3}, \frac{1}{9}\right)$. Points on the critical curve flow into the fixed point, while those off it flow away toward the trivial fixed point $\left(K_1^*, K_2^*\right) = (0, 0)$ or $\left(K_1^*, K_2^*\right) = (\infty, \infty)$

[8]In this reference one can also find a systematic method of constructing the scaling function $f_s(u_1, u_2, \ldots)$ from a knowledge of the regular function $f(K_0')$ of equation (14.3.4).