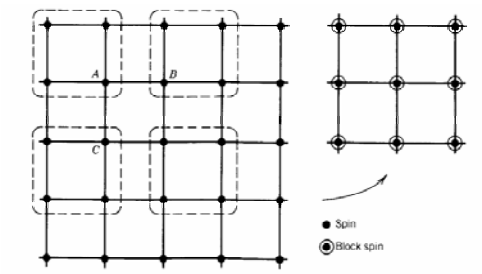
Summary:

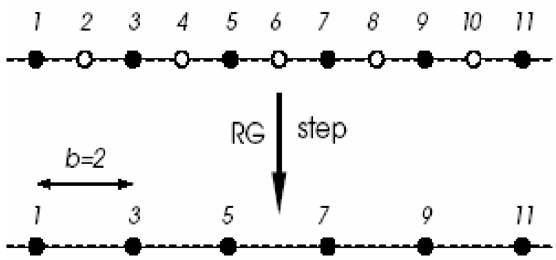
What is the renormalization group?

Ans: The renormalization group consists of analytic and computational schemes to integrate systematically over degrees of freedom in a system near a critical point. After integration, the control parameters, for example temperature and magnetic field in a magnetic system, are rescaled to restore the system Hamiltonian to its original form. The behavior of the control parameters under this rescaling enables calculation of the critical behavior of the model.

How can we do the renormalization?



Block-spin transformation.



Decimation scheme

In real-space we use:

1- block-spin transformation, or
2- decimation (Coarse graining*)

In k-space:

1- Integrating out large momenta (short wavelength).

* **Coarse Graining**:- Replacement of microscopic variables by average variables on an expanded length scale (with an upper wave number cutoff Λ).

The scaling factor is defined as:

$$p = \frac{N}{N'} = b^d$$

where N is the number of old sites, N' is the number of new sites and d is the number of spatial dimensions. If the transformation done n times, we will have:

$$p = b^{nd}$$

In 1D, $b = 2$ and the lattice spacing is increased as,

$$a' = b^d a$$

Other lengths which are measured in units of the lattice spacing are reduced by a factor b . For example, the new correlation length is

$$\xi' = \xi / b^d$$

The remaining spins on the decimated lattice interact with their new nearest neighbors through the renormalized coupling constants K' and are subject to other renormalized fields, such as h'.

For the free energy F :

$$F' = b^d F$$

And the partition function:

$$Z(K, N) = Z(K', N/b^d)$$

Thermodynamic behavior of the d = 1 Ising model

Now we are in a position to analyze the behavior of the RG equations. If we have a phase transition, it should exist only at h = 0, which we focus on. To look for a phase transition, we look for the equation:

$$K = \frac{1}{2}\ln[\cosh(2K)]$$

at a fixed point

$$K^* = \frac{1}{2}\ln[\cosh(2K^*)]$$

Unfortunately, this equation has no "nontrivial" fixed point at finite K^*, since the function $f(x) = \frac{1}{2}\ln[\cosh(2x)] \leq x$, for any $0 < x < \infty$. In fact, precisely because of this condition, the coupling constant decreases under iteration, i.e. $K \to K^* = 0$.

Examples:

```
1- Solve[K - (1 / 2) Log[ Cosh[2 K]] == 0, K]  // N

   {{K → 0.}}
2- Solve[K - (3 / 8) Log[ Cosh[4 K]] == 0, K]  // N

   {{K → 0.}, {K → 0.506981},
    {K → -0.103544 + 1.00843 i}, {K → -0.103544 - 1.00843 i}}
```

According Kadanoff's argument:

1- stable fixed points describe phases, while
2- the unstable fixed point represents the critical point where $\xi \to \infty$.

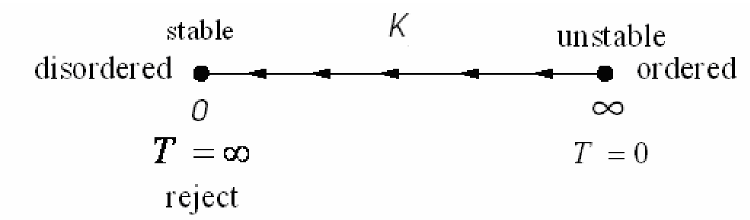
Unfortunately, in Ising 1D example, we have one stable fixed point at K = 0, corresponding to $T = \infty$, i.e. the high-temperature (paramagnetic) phase. We have also found that this fixed point is reached no matter which initial (bare) value of K we chose. We conclude that no phase transition exists at any finite temperature for Ising 1D model.

But is there a critical point anywhere? Yes there is! We do find an unstable fixed point at $K^* = +\infty$ (i.e. corresponding to $T = 0$). We can now examine the RG flows in the vicinity of this $T = 0$ fixed point, in order to determine the scaling of the correlation length in this regime. We concentrate on the RG equation for K in the regime of $K >> 1$. Here

$$K' = \frac{1}{2}\ln[\cosh(2K)] = \frac{1}{2}\ln[\frac{e^{2K} + e^{-2K}}{2}] = \frac{1}{2}\ln\frac{e^{2K}}{2}[1 + e^{-4K}]$$
$$< \frac{1}{2}\ln\frac{e^{2K}}{2} = K - \frac{1}{2}\ln 2$$

$$\Rightarrow \quad \delta K = K' - K = -\frac{1}{2}\ln 2$$

If we start at a "bare" value $K_o = J/T >> 1$ (i.e. $T << J$), K deceases under iterations, but it does that **very slowly**!



How does K depend on the renormalized length scale after many iterations? Well, after n iterations, the new scale is $b = 2^n$, or $\ln b = n \ln 2$. Under a single iteration, (assume $n >> 1$), the change of $\ln b$ is $d(\ln b) = \ln 2$, And the change of the coupling constant is $dK = -\frac{\ln 2}{2}$

Using these results, the β-function is $\beta(K) = \frac{dK}{d \ln b} = -\frac{1}{2}$.

Note that the β-function in this particular case is not linear in the coupling constant as at a standard (finite K^*) fixed point. As a result, we do not obtain power law, but rather exponential behavior, as follows. A similar situation is found, for example, at the Berezinskii-Kosterlitz-Thouless transition and its descendents (Caldeira-Legett problem, the Kondo problem). Quite generally, such behavior is often found at the **lower critical dimension**; the *d* = 1 Ising model indeed falls into that category.

To obtain the explicit scale dependence $K(b)$ of the coupling constant we integrate the above differential equation and find:

$$K(b) = K_0 - \tfrac{1}{2} \ln b$$

But how can we calculate the correlation length? Well, we use the Kadanoff scaling expression (we focus at *h* = 0)

$$\xi(K) = b\,\xi\left(K(b)\right).$$

Under renormalization ($b >> 0$) $\Rightarrow$ $K(b) \to 0$, which corresponds to $T \to \infty$, where $\xi \approx 1$. In this regime $b \approx e^{2K_0} = e^{2J/T}$ and we find

$$\xi \approx e^{2J/T}$$

The extreme simplicity of the *d* = 1 example resulted from the fact that each site had only two nearest neighbors, and that decimation over any one given spin (site) simply generated an effective interaction between second neighbors, but the new effective Hamiltonian retained the same form. In higher dimensions this is not longer true. Here, a simple calculation shows that new multiple-spin interactions terms are generated, and the scale-invariance is ruined! Is the decimation approach now completely useless? Well...yes and no. In a rigorous sense it cannot be used, since the applicability of the RG ideas of Kadanoff require preserving the scale invariance.

4.3.1 A Trivial Example: The d = 1 Ising Model

In the renormalization-group methods the temperature changes under successive transformations, therefore it is convenient to work with the *reduced Hamiltonian*, we divide $\mathcal{H}$ by $k_B T$. The reduced Hamiltonian of an Ising Hamiltonian with NN exchange interaction in an external field is

$$\mathcal{H} = -K\sum_i S_i S_{i+1} - h\sum_i S_i, \qquad K = \frac{J}{k_B T} \quad h = \frac{H}{k_B T}. \tag{4.65}$$

Recursion relations: If we want to make several successive renormalization group (RG) transformations, we have to watch for the following:

1- the structure of $\mathcal{H}$ must not change after each transformation to $\mathcal{H}'$, also
2- the partition function *does not* change

The above conditions are necessary because this ensures that the physical properties of the system are not changed.

The partition function of the one-dimensional Ising model in an external field is written in the form:

$$\begin{aligned} Z(N,K,h) &= \sum_{\{S\}} \mathrm{e}^{-\mathcal{H}} = \sum_{\{S\}} \mathrm{e}^{K\sum_i S_i S_{i+1} + h\sum_i S_i} \\ &= \sum_{\{S\}} \mathrm{e}^{K(S_1S_2+S_2S_3)+hS_2+\frac{h}{2}(S_1+S_3)} \\ &\qquad \times \mathrm{e}^{K(S_3S_4+S_4S_5)+hS_4+\frac{h}{2}(S_3+S_5)} \times \cdots, \end{aligned} \tag{4.66}$$

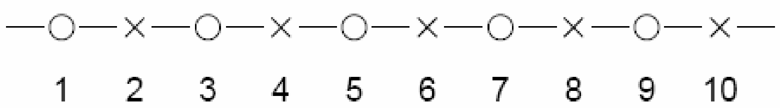


Figure 4.14: Decimation on the d = 1 Ising model.

We divide the lattice into the sublattices × and ○, denote the spins on the × sublattice by σ_i (Fig. 4.14),

$$Z = \sum_{\{S\}}\sum_{\{\sigma\}} \mathrm{e}^{\sigma_2[K(S_1+S_3)+h]+\frac{h}{2}(S_1+S_3)} \mathrm{e}^{\sigma_4[K(S_3+S_5)+h]+\frac{h}{2}(S_3+S_5)} \cdots, \tag{4.67}$$

and carry out a partial trace, i.e., we sum over $2^{N/2}$ configurations of the $\{\sigma\}$ spins sitting on the × sublattice:

$$Z(N,K,h) = \sum_{\{S\}} \left[\mathrm{e}^{(K+\frac{h}{2})(S_1+S_3)+h} + \mathrm{e}^{(-K+\frac{h}{2})(S_1+S_3)-h}\right] \times \cdots. \tag{4.68}$$

The entire partition function must not change and $\mathcal{H}$ must keep its structure, therefore we have the condition

$$Z(N,K,h) = \mathrm{e}^{Ng(K,h)} Z(\frac{N}{2},K',h') = \mathrm{e}^{Ng}\sum_{\{S\}} \mathrm{e}^{-\mathcal{H}'} \tag{4.69}$$

Where

$$\mathcal{H}' = -K'\sum_{i\in\mathrm{odd}} S_i S_{i+2} - h'\sum_{i\in\mathrm{odd}} S_i \tag{4.70}$$

and g is a constant. This will ensure that the physical properties will not change under successive iterations. The two conditions (4.70) and (4.69) imply that

$$e^{(K+\frac{h}{2})(S_i+S_{i+2})+h}+e^{(-K+\frac{h}{2})(S_i+S_{i+2})-h}=e^{K'S_iS_{i+2}+\frac{h'}{2}(S_i+S_{i+2})+2g}. \quad (4.71)$$

The last equation has to be hold for any $S_i, S_{i+2}=\pm 1$:

$$\begin{aligned} S_i=S_{i+2}=+1 &\quad\Rightarrow\quad e^{2K+2h}+e^{-2K}=e^{K'+h'+2g}, \\ S_i=S_{i+2}=-1 &\quad\Rightarrow\quad e^{-2K}+e^{2K-2h}=e^{K'-h'+2g}, \\ S_i=-S_{i+2} &\quad\Rightarrow\quad e^{+h}+e^{-h}=e^{-K'+2g}, \end{aligned} \quad (4.72)$$

From which we get

$$\begin{aligned} e^{4K'} &= \frac{\left(e^{2K+2h}+e^{-2K}\right)\left(e^{2K-2h}+e^{-2K}\right)}{\left(e^{h}+e^{-h}\right)^2} \\ e^{2h'} &= \frac{e^{2K+2h}+e^{-2K}}{e^{2K-2h}+e^{-2K}} \\ e^{8g} &= \left(e^{2K+2h}+e^{-2K}\right)\left(e^{2K-2h}+e^{-2K}\right)\left(e^{h}+e^{-h}\right)^2 \end{aligned} \quad (4.73)$$

or

$$\begin{aligned} K' &= \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \\ h' &= h+\frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \\ g &= \frac{1}{8}\ln\left[16\cosh(2K+h)\cosh(2K-h)\cosh^2 h\right]. \end{aligned} \quad (4.74)$$

Equations (4.73) or (4.74) are the *recursion relations* and determine the fixed points and the flow diagram of the system.
In each iteration, the number of degrees of freedom is reduced by one half; the new Hamiltonian $\mathcal{H}$ has only one half of the previous spins,

$$N'=b/N, \quad b=2 \quad (4.75)$$

and the lattice spacing is increased as,

$$a'=b\,a \quad (4.76)$$

Other lengths which are measured in units of the lattice spacing are reduced by a factor b. For example, the new correlation length is

$$\xi'=\xi/b \quad (4.77)$$

The remaining spins on the decimated lattice interact with their new nearest neighbors through the renormalized coupling constants K' and are subject to renormalized fields h'.

Fixed points. At a fixed point the parameters (the coupling constants – in our case *K* and *h*) do not change under successive decimations. This means that the system at a fixed point stays at this point in the parameter space. With the new variables

$$\begin{aligned} x &= e^{-4K}, \quad y = e^{-2h}, \quad z = e^{-8g} \quad 0 \le (x, y, z) \le 1; \\ x' &= e^{-4K'}, \quad y' = e^{-2h'}, \quad z' = e^{-8g'} \quad 0 \le (x', y', z') \le 1 \end{aligned} \tag{4.78}$$

the recursion relations become:

$$\begin{aligned} x' &= x \frac{(1+y)^2}{(x+y)(1+xy)} \\ y' &= y \frac{x+y}{1+xy} \\ z' &= z^2 xy^2 \frac{1}{(x+y)(1+xy)(1+y)^2} \end{aligned} \tag{4.79}$$

The recursion relations for x and y do not depend on *z*. Physically, this means that the singular behavior of the free energy does not depend on a shift in the energy scale. Therefore, we first investigate the flow diagram and fixed points in the (x, y) plane, see Fig. 9.2.

1- From the first equation, of 4.79, setting $x' = x \quad \Rightarrow \quad x = 1$ (corresponding to an infinite-temperature), i.e. $x = 1$ is a fixed point of this system for arbitrary y. We denote the fixed points by a *. There is a line of fixed points at $x^* = 1$ for $0 \le y^* \le 1$,. Along this line, the spins are completely disordered and $\xi \to 0$.

2- From the second equation, of 4.79, setting $x = 0$ (corresponding to an zero-temperature), $\Rightarrow \quad y' = y^2$ which has two fixed points, one at $y^* = 1$ and one at $y^* = 0$.

The fixed point at $\left(x^*, y^*\right) = (0,1)$ (this means $T = 0$ and *H* = 0) corresponds to the critical point where $\xi \to \infty$. This is correct because we know that the *d* = 1 Ising model has $T_c = 0$. This fixed point is unstable and any point in its vicinity will flow under RG transformations away from it towards the fixed points on the line at $x^* = 1$. The critical behavior of the *d* = 1 Ising model is governed by the least stable fixed point at $\left(x^*, y^*\right) = (0,1)$. The point at $\left(x^*, y^*\right) = (0,0)$ is another fixed point which corresponds to $T = 0$ and $H = \infty$. At this fixed point, the spins are completely ordered.

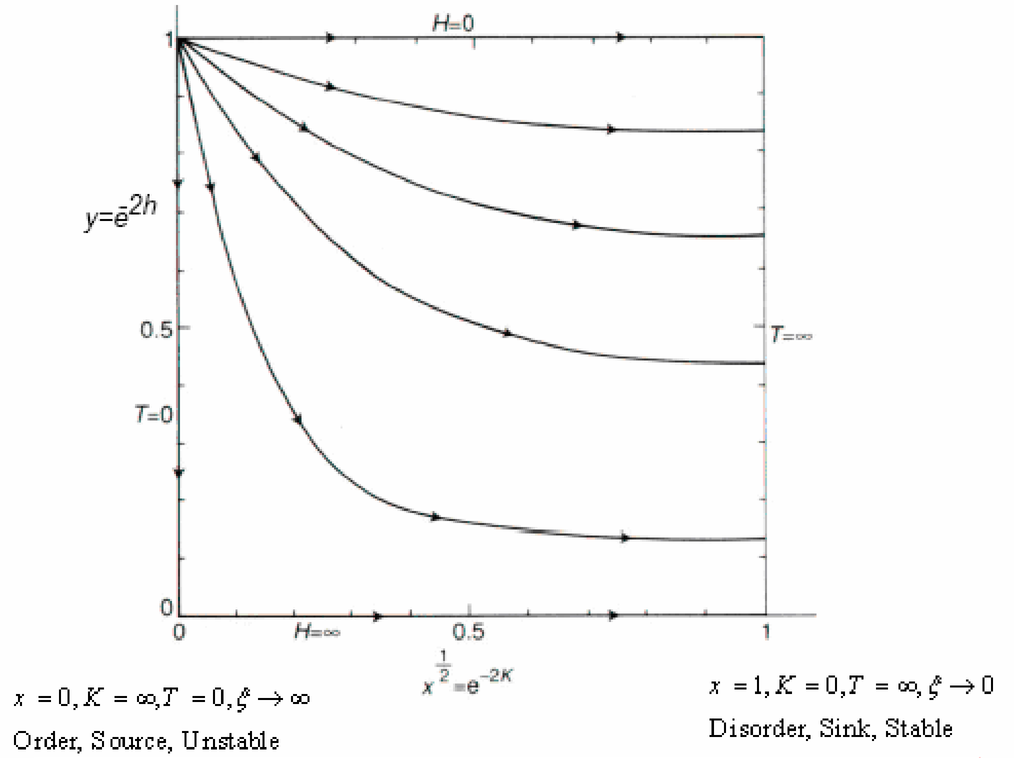


Fig. 9.2. Trajectories in parameter space for the recursion relations arising from the decimation of a one-dimensional Ising chain by a scale factor $b = 2$. There are fixed points at $x^* = 0$, $y^* = 1$ corresponding to criticality, at $x^* = y^* = 0$ corresponding to a fully aligned configuration, and at $x^* = 1$, $0 \le y^* \le 1$ corresponding to an infinite temperature sink. After Nelson, D. R. and Fisher, M. E. (1975). *Annals of Physics*, **91**, 226.

Eigenvalues and Eigenfunctions

$$x = e^{-4K}, \quad y = e^{-2h}, \quad z = e^{-8g} \quad 0 \le (x, y, z) \le 1;$$
$$x' = e^{-4K'}, \quad y' = e^{-2h'}, \quad z' = e^{-8g'} \quad 0 \le (x', y', z') \le 1 \tag{4.78}$$

the recursion relations become:

$$x' = x \frac{(1+y)^2}{(x+y)(1+xy)}$$
$$y' = y \frac{x+y}{1+xy} \tag{4.79}$$
$$z' = z^2 xy^2 \frac{1}{(x+y)(1+xy)(1+y)^2}$$

Example 1: For the d = 1 Ising model, $\mu_1 = x$ and $\mu_2 = y$. After linearization of the recursion relations (4.79) around the fixed point at $\vec{\mu}^* \equiv (0,1)$ we get:

As $h = 0 \Rightarrow y = 1$, then

$$x' \xrightarrow[y=1]{} x \frac{(1+1)^2}{(x+1)(1+x)} = \frac{4x}{(1+x)^2} \approx 4x$$
$$\Rightarrow \boxed{dx' = 4dx} \quad \Rightarrow \quad \lambda_x = 4$$

As $t = 0 \Rightarrow x = 0$, then

$$y' \xrightarrow[x=0]{} = y^2$$
$$\Rightarrow \boxed{dy' = 2y dy \Big|_{y=1} = 2dy} \quad \Rightarrow \quad \lambda_y = 2$$

Hence, we have the transformation matrix:

$$\begin{pmatrix} dx' \\ dy' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

From the equation $\lambda_i(b) = b^{y_i}$ where $b = 2$, we can have:

$$y_1 = \frac{\ln \lambda_x}{\ln b} = \frac{\ln 4}{\ln 2} = 2;$$
$$y_2 = \frac{\ln \lambda_y}{\ln b} = \frac{\ln 2}{\ln 2} = 1$$

and

$$\nu_1 = \frac{1}{y_x} = \frac{1}{2}$$
$$\nu d = 2 - \alpha \quad \Rightarrow \quad \alpha = 2 - \frac{d}{y_x} = \frac{3}{2}$$

The scaling factor is defined as:

$$p = \frac{N}{N'} = b^d$$

where N is the number of old sites, N' is the number of new sites and d is the number of spatial dimensions. If the transformation done n times, we will have:

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In 1D, $b = 2$ and the lattice spacing is increased as,

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And the partition function:

$$Z(K, N) = Z(K', N / b^d)$$