

## SCALING

Scaling is a very general and well-known method for:

- i- Describing the response of a system to a disturbance.
- ii- For presenting theoretical predictions and experimental data for different values of the parameters on a single curve.

For physical systems, it is most readily studied in terms of dimensional analysis and the construction of dimensionless parameters.

As we approach the critical point, the distance over which fluctuations are correlated approaches infinity and all effects of the finite lattice spacing are wiped out. There are no natural length scales left. Thus we might expect that in the neighborhood of the critical point, as we change the distance from the critical point, we do not change the form of the free energy but only its scale. The idea of scaling underlies all critical exponent calculations. To understand scaling, we must first introduce the concept of a homogeneous function. See for example: L. E. Reichl, **A Modern Course in Statistical Physics** (U. Texas, 1980).

### Homogeneous function

A function is homogeneous if for all values of

$$F(\lambda x) = g(\lambda) F(x)$$

**Example:** For the homogeneous function  $F = \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} + \frac{x}{y^2 z^2}$ , find its order.

**Answer:** Use the substitution,  $x' = \lambda x, y' = \lambda y, z' = \lambda z$ , one gets:

$$F = \lambda^p \left( \frac{1}{x'^3} + \frac{1}{y'^3} + \frac{1}{z'^3} + \frac{x'}{y'^2 z'^2} \right) = g(\lambda) f, \quad p = -3$$

Now, what is the general form of  $g(\lambda)$ ? The form can be found as follows:

1- First we note that:

$$\begin{aligned} F(\lambda \mu x) &= g(\lambda) g(\mu) F(x) = g(\lambda \mu) F(x) \\ \Rightarrow g(\lambda \mu) &= g(\lambda) g(\mu) \end{aligned}$$

2- Take the derivatives  $\frac{\partial}{\partial \mu}$ , for both sides of the above equation, we find

$$\lambda g'(\lambda \mu) = g(\lambda) g'(\mu)$$

3- Make the following tricks: Put  $\mu = 1$ ,  $g'(1) = p$ , we have

$$\lambda g'(\lambda) = g(\lambda) \underbrace{g'(1)}_p$$

4- Simple integration will give:

$$\int \frac{dg(\lambda)}{g(\lambda)} = p \int \frac{d\lambda}{\lambda} \Rightarrow \ln g(\lambda) = p \ln \lambda \Rightarrow g(\lambda) = \lambda^p$$

5- Finally, we have:

$$F(\lambda x) = \lambda^p F(x) \Rightarrow F(x) = \lambda^{-p} F(\lambda x)$$

and  $F(x)$  is said to be a homogeneous function of degree  $p$ . A homogeneous function has a very special form.

In the last equation, if we let  $\lambda = x^{-1}$ , we obtain  $F(x) = x^p F(1)$ . Thus the homogeneous function  $F(x)$  has power law dependence on its arguments.

Let us now consider a generalized homogeneous function of two variables  $f(x, y)$ . Such a function can always be written in the form:

$$f(\lambda^p x, \lambda^q y) = \lambda f(x, y)$$

Then we have:

$$f(x, y) = \lambda^{-1} f(\lambda^p x, \lambda^q y)$$

and is characterized by two parameters, p and q. It is convenient to write  $f(x, y)$  in another form. We let

$$\lambda y^q = 1 \Rightarrow \lambda = y^{-1/q}. \text{ Then}$$

$$f(x, y) = y^{1/q} f\left(\frac{x}{y^{p/q}}, 1\right)$$

and we see that the generalized homogeneous function  $f(x, y)$ , depends on  $x$  and  $y$  only through the ratio  $x / y^{p/q}$  aside from a multiplicative factor. We can now apply these ideas to thermodynamic quantities near the critical point.

**But, why are we doing this trick?** The idea is simple, which is separating the singular part of the given

function. For example, consider the function  $f(x) \approx \frac{1}{x^2} + a$ , with the first term is singular at  $x \rightarrow 0$ . To

separate the singular part, let us change the function to a homogeneous by using:  $x = \ell y$ , then

$$f(x) \approx \frac{1}{\ell^2 x^2} + \text{reglar part}. \text{ For the singular part, let us change the variable } \frac{1}{\ell^2 x^2} = 1 \Rightarrow \ell^2 = x^{-2}, \text{ then}$$

$$f(x) \approx \frac{1}{\ell^2 x^2} + \text{reglar part} = x^{-2} f(1) + \text{reglar part}$$

### Relations between Thermodynamic Critical Indices

Critical indices, which were introduced in previous notes, define the behavior of the thermodynamic parameters near the critical point. A list of such indices, with their definitions, is given in Table 4.1. While the definitions listed there are for a magnetic system, they can easily be formulated for other types of system. For instance, for the liquid–gas system, the magnetic field  $H$  is replaced by the pressure  $P$ , and the magnetic moment  $M$  by the specific volume. The different thermodynamic parameters are not independent, since there are some thermodynamic relations between them, and these lead to connections between different critical indices. We now consider two examples of such relations.

Table 4.1 Critical indices for the specific heat ( $\alpha$ ), order parameter ( $\beta$ ), susceptibility ( $\gamma$ ), magnetic field ( $\delta$ ), correlation length ( $\nu$ ) and correlation function ( $\eta$ ). The indices  $\alpha, \gamma, \nu$  and  $\alpha', \gamma', \nu'$  refer to  $T > T_c$  and  $T < T_c$ , respectively. The results for the mean field and 2D Ising models are exact, while those for the 3D Ising model are approximate.

Exponent	Definition	Condition	Mean field	2D Ising	3D Ising
$\alpha$	$C \sim \left  \frac{T-T_c}{T_c} \right ^{-\alpha(-\alpha')}$	$H = 0$	0 ( <i>jump</i> )	0 (ln)	$\sim 0.11$
$\beta$	$M \sim \left( \frac{T_c-T}{T_c} \right)^\beta$	$H = 0; T < T_c$	0.5	0.125	$\sim 0.32$
$\gamma$	$\chi \sim \left  \frac{T-T_c}{T_c} \right ^{-\gamma(-\gamma')}$	$H = 0$	1	1.75	$\sim 1.24$
$\delta$	$H \sim  M ^\delta$	$T = T_c$	3	15	$\sim 4.82$
$\nu$	$\xi \sim \left  \frac{T-T_c}{T_c} \right ^{-\nu(-\nu')}$	$H = 0$	0.5	1	$\sim 0.63$
$\eta$	$g(r) \sim \frac{1}{r^{d-2+\eta}}$	$T = T_c$	0	0.25	$\sim 0.03$

Heat Capacity:  $C \sim |t|^{-\alpha}$ ,

Magnetisation:  $m \sim |t|^\beta$ ,

Susceptibility:  $\chi \sim |t|^{-\gamma}$ ,

Correlation Length:  $\xi \sim |t|^{-\nu}$ ,

Correlation function:  $g(r) \sim r^{d-2+\eta}$

As  $T = T_c$  and  $B \rightarrow 0$  one finds

$$m \sim B^{1/\delta}$$

## Widom's scaling law (Huang)

If we assume that the singular part of the thermodynamic potential scales, then we can find a relation between various critical exponents. If there are conjugate-variable pairs in the theory other than  $M$  and  $H$ , say  $\varphi_i$  and  $J_i$ , then the scaling of the free energy may be generalized to the following:

$$f_s \approx |t|^{2-\alpha} F_{f\pm} \left( \frac{H}{|t|^{\Delta}}, \frac{J_1}{|t|^{\Delta_1}}, \frac{J_2}{|t|^{\Delta_2}}, \dots \right) \quad (\text{A})$$

which is known as Widom's scaling form. The fields  $H, J_1, \dots$  are called "scaling fields". The associated exponents  $\Delta_1, \Delta_2, \dots$  called "crossover exponents," control the relative importance of the fields near  $t = 0$ .

If  $\Delta_i < 0$ , the field  $J_i$  is said to be "irrelevant". The dependence on  $J_i$  drops out near  $t = 0$ . Example:

$$f_s \propto t^2$$

If  $\Delta_i > 0$ , the field  $J_i$  is "relevant"; Example:  $f_s \propto \frac{1}{t}$

If  $\Delta_i = 0$  we would have a "marginal" case.

We should bear in mind that the scaling form above specifically refers to the neighborhood of a particular critical point. A system may have more than one critical point, and a form like (A) is supposed to hold near each of them, with different sets of crossover exponents.

From that, one can explain:

- (i) Relevant and irrelevant fields.
- (ii) Scaling law could be derived.
- (iii) Data collapse.

The scaling hypothesis assumes that only properties which are given in dimensionless units remain invariant under change of length scale of the system. Dimensionless properties show self-similarities.

**Example:** Derive Rushbrooke equality relation:  $\alpha + 2\beta + \gamma = 2$ ,

**Answer:** Start with the free energy per unit volume or at least its singular part  $f_s(t, h)$ :

$$f_s(t, h) = |t|^{2-\alpha} F_f \left( \frac{h}{|t|^{\Delta}} \right)$$

Using

$$M \approx \frac{\partial f_s}{\partial h} = |t|^{2-\alpha-\Delta} F_{fh} \left( \frac{h}{|t|^{\Delta}} \right),$$

and compare with  $M \approx |t|^{\beta}$  as  $h \rightarrow 0 \Rightarrow \beta = 2 - \alpha - \Delta$

Using

$$\chi \approx \frac{\partial^2 f_s}{\partial h^2} \sim |t|^{2-\alpha-2\Delta} F_{fhh} \left( \frac{h}{|t|^{\Delta}} \right)$$

and compare with  $\chi \approx |t|^{-\gamma}$  as  $h \rightarrow 0 \Rightarrow -\gamma = 2 - \alpha - 2\Delta$

Eliminating  $\Delta$ , we obtain the desired results.

Consider the Helmholtz potential in the form:

$$f_s(\lambda^p \tau, \lambda^q h, \dots) = \lambda f_s(\tau, h, \dots), \quad \tau = \frac{T - T_c}{T_c}$$

where  $h$  is the external field conjugate to  $m$ .

**Q:** How can we relate the critical exponents,  $(\alpha, \beta, \gamma, \dots)$ , to  $p$  and  $q$ ?

**Answer:** Let us now consider a generalized homogeneous function of two variables  $f(\tau, h)$ . Such a function can always be written in the form:

$$f(\lambda^p \tau, \lambda^q h) = \lambda f(\tau, h)$$

Then we have:  $f(\tau, h) = \lambda^{-1} f(\lambda^p \tau, \lambda^q h)$ .

Put  $\lambda = h^{-1/q} \Rightarrow f(\tau, h) = h^{1/q} f(\tau h^{-p/q}, 1)$ ;

or put  $\lambda = \tau^{-1/p} \Rightarrow f(\tau, h) = |\tau|^{1/p} f(1, h |\tau|^{-q/p})$

1- for the magnetization  $m \sim |t|^\beta$ :

$$m = - \left( \frac{\partial f}{\partial h} \right)_{h=0} = - \left( \frac{\partial}{\partial h} |\tau|^{1/p} f(1, h |\tau|^{-q/p}) \right)_{h=0} \Rightarrow m = - |\tau|^{1/p} |\tau|^{-q/p} f(1, 0)$$

$$\Rightarrow \beta = (1 - q) / p$$

2- for the susceptibility  $\chi \sim |t|^{-\gamma}$ :

$$\chi_T \sim \left( \frac{\partial m}{\partial h} \right)_{h=0} = \left( \frac{\partial^2}{\partial h^2} |\tau|^{1/p} f(1, h |\tau|^{-q/p}) \right)_{h=0}; \quad \chi_T(\tau, 0) = |\tau|^{(1-2q)/p} f(1, 0)$$

$$\Rightarrow \gamma = (2q - 1) / p$$

3- for the specific heat  $c_V(\tau, 0) = |\tau|^{-\alpha}$ :

$$c_V = -T \left( \frac{\partial^2}{\partial T^2} \left\{ |\tau|^{1/p} f(1, h |\tau|^{-q/p}) \right\} \right)_{h=0} \propto \lambda^{\frac{1}{p}-2} f(1, 0) \Rightarrow c_V(\tau, 0) = |\tau|^{(1-2p)/p} f(1, 0)$$

$$\Rightarrow \alpha = 2 - \frac{1}{p}$$

H.W. For  $m \sim B^{1/\delta}$  prove that  $\delta = q/(1-q)$

### Hyper-scaling

$$\text{Fisher: } \gamma = \nu(2 - \eta),$$

$$\text{Rushbrooke: } \alpha + 2\beta + \gamma = 2,$$

$$\text{Widom: } \gamma = \beta(\delta - 1),$$

$$\text{Josephson: } \nu d = 2 - \alpha$$

The equality will be given as:

$$p = 1/(2 - \alpha), \quad q = 1 - \beta/(2 - \alpha),$$

$$\gamma = (1 - 2q) / p \Rightarrow \alpha + 2\beta + \gamma = 2$$

$$\delta = q/(1 - q) \Rightarrow \alpha + \beta(1 + \delta) = 2$$