Model of ferromagnet with Infinite (long) range interaction

☞ See W. D. McComb, "**renormalization method, A GUIDE FOR BEGINNERS**", (Oxford, 2004). Sec. 7.8

7.8 Validity of mean-field theory

Mean-field theory can be shown to be equivalent to an assumption that each spin interacts equally with every other spin in the lattice: this implies infinite interaction range in the limit $N \to \infty$.

7.8.1 The model (Kac's Model)

We wish to set up a model in which all spins interact with each other. For N spins, we have effectively $N^2/2$ pairs of spins, so in order to have the same overall energy as the Ising model the interaction of each pair must be proportional to $1/N$. Then the total energy behaves as

$$E \sim \frac{N^2}{2} \times \frac{1}{N} \sim N$$

which is correct.

In view of this, let us suppose that the N spins have an interaction energy $-J/N$ between each pair and, assuming zero external fields, write the Hamiltonian as

$$H = -\frac{2J}{N} \sum_{1 \le i \le j \le N} S_i S_j, \quad J > 0 \tag{5.67}$$

Here $S_i = \pm 1$ for all i. Our aim is to obtain the Helmholtz potential for one spin in the thermodynamic limit $N \to \infty$, where

$$G(T,H) = -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z_N \tag{5.68}$$

To do so, we have to do the following:

1-

$$H = -\frac{2J}{N} \sum_{1 \le i \le j \le N}^{N-1} S_i S_j = -\frac{J}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} S_i S_j + \frac{J}{N} \sum_{i=1}^{N} S_i^2 \tag{5.69}$$

Then the following points should be noted:

i. In the second equality we have abandoned the condition $i < j$ in the double sum therefore each off-diagonal term is now counted twice. Accordingly we drop the factor of two.
ii. The double sum now includes (erroneously) the diagonal terms, for which $i = j$, and so we cancel these by adding the last term.

2- Now each $S_i^2 = 1$, hence the last term in (5.69) is equal to $N \times J/N$ and so the Hamiltonian (5.67) becomes:

$$H = J - \frac{J}{N} \left(\sum_{i=1}^{N} S_i \right) \left(\sum_{j=1}^{N} S_j \right) = J - \frac{J}{N} \left(\sum_{i=1}^{N} S_i \right)^2 \tag{5.70}$$

3- Then the expression for the partition function reads:

$$Z_N = \sum_{S_1=-1}^{1} \cdots \sum_{S_N=-1}^{1} e^{\frac{-J}{k_B T}} e^{\frac{J}{N k_B T} \left(\sum_{i=1}^{N} S_i \right)^2} \quad . \tag{5.71}$$

4-

In order to perform summation over $S_1 \dots S_N$ in (5.71) let us make use of the so-called Stratonovich-Hubbard transformation, based on the equality:

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{b^2/4a} \qquad \mathrm{Re}(a) > 0 \qquad (5.72)$$

This transformation enables one to represent the function of binary type (term b^2 in the right-hand side) in the from of function depending on the first power of b. In our case we are going to trasform the expression:

$$e^{\frac{J}{Nk_BT}\left(\sum_{i=1}^{N} S_i\right)^2}. \qquad (5.73)$$

Comparing (5.73) and (5.72) we have:

$$\sum_{i=1}^{N} S_i = b, \quad \frac{J}{Nk_BT} = \frac{1}{4a} \qquad (5.74)$$

And finally the exponent (5.73) reads:

$$e^{\frac{J}{Nk_BT}\left(\sum_{i=1}^{N} S_i\right)^2} = \sqrt{\frac{Nk_BT}{4\pi J}} \int_{-\infty}^{+\infty} \mathrm{d}x e^{-\frac{Nk_BT}{4J}x^2 + \sum_{i=1}^{N} S_i x}. \qquad (5.75)$$

Substitution of (5.75) into (5.71) leads to the following expression for the partition function:

$$\begin{aligned} Z_N &= \sum_{S_1=-1}^{1} \cdots \sum_{S_N=-1}^{1} \sqrt{\frac{Nk_BT}{4\pi J}} e^{\frac{-J}{k_BT}} \int_{-\infty}^{+\infty} \mathrm{d}x e^{-\frac{Nk_BT}{4J}x^2 + \sum_{i=1}^{N} S_i x} \\ &= \sqrt{\frac{Nk_BT}{4\pi J}} e^{\frac{-J}{k_BT}} \int_{-\infty}^{+\infty} \mathrm{d}x e^{-\frac{Nk_BT}{4J}x^2} \left[2\cosh x\right]^N \\ &= \sqrt{\frac{Nk_BT}{4\pi J}} e^{\frac{-J}{k_BT}} 2^N \int_{-\infty}^{+\infty} \mathrm{d}x \left[e^{-\frac{k_BT}{4J}x^2} \cosh x\right]^N, \end{aligned} \qquad (5.76)$$

In (5.76) we used: $\sum_{S_i=\pm1}^{N} e^{\sum_{i=1}^{N} xS_i} = \sum_{S_i=\pm1}^{N} \prod_{i=1}^{N} e^{xS_i} = \left[2\cosh x\right]^N$.

5- Next, we will use the identity $y^N = e^{N\ln(y)}$, then the partition function reads: which can be represented as

$$Z_N = \sqrt{\frac{Nk_BT}{4\pi J}} e^{\frac{-J}{k_BT}} 2^N \int_{-\infty}^{+\infty} \mathrm{d}x e^{N\left\{-\frac{k_BT}{4J}x^2 + \ln(\cosh x)\right\}}, \qquad (5.77)$$

6- Considering that for large N the main contribution from the integral in (5.77) comes from the region of x where $f(x)$ has a maximum we can evaluate it on base of the steepest decent (Saddle-point) method in the form:

$$\int_{-\infty}^{+\infty} \mathrm{d}x e^{Nf(x)} = e^{Nf(x_0)} \sqrt{\frac{2\pi}{N\partial^2 f(x)/\partial x^2|_{x=x_0}}}, \qquad (5.78)$$

where x_0 means the maximum of function $f(x)$. Applying (5.78) for calculation of the integral in (5.77) we have for the partition function:

$$Z_N = \sqrt{\frac{N k_B T}{4\pi J}} e^{\frac{-J}{k_B T}} 2^N \sqrt{\frac{2\pi}{\partial^2 f(x)/\partial x^2|_{x=x_0}}} e^{N f(x_0)}, \qquad (5.79)$$

with

$$f(x) = -\frac{k_B T}{4J} x^2 + \ln\cosh x. \qquad (5.80)$$

For the Helmholtz potential for one spin we get:

$$\bar{G}(T,H) = -k_B T \lim_{N\to\infty} N^{-1} \ln Z_N = -k_B T \Big\{ \ln 2 + f(x_0) \Big\}. \qquad (5.81)$$

In order to find the maximum of function $f(x)$ putting $f'(x) = 0$ we get:

$$\frac{x k_B T}{2J} = \tanh x. \qquad (5.82)$$

Comparing (5.82) with the equation

$$\sigma = \tanh\left(\frac{g\mu_B H}{2k_B T} + \frac{\sigma}{t}\right).$$

for $H = 0$, we see that it is the same as those obtained in the frame of mean molecular field approximation. The behavior of the equation

$$f(x,t) = -tx^2 + \ln[\cosh(x)]$$

is shown in the following figure. We see that the maximum value of it for $t > 1/2$ correspond to $x = 0$, whereas for $t < 1/2$ it is given by $x \neq 0$. For the critical temperature one has:

$$T_c = \frac{2J}{k_B}$$

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In[1]:= f = - t x^2 + Log[Cosh[x]]
Out[1]=
        -t x^2 + Log[Cosh[x]]

In[2]:= f1 = f /. {t → 0.25}; f2 = f /. {t → 0.5}; f3 = f /. {t → 1};

In[3]:= Plot[{f1, f2, f3}, {x, -4, 4}, Frame → True, PlotRange → {-2, 0.5},
         PlotStyle → {GrayLevel[0], Dashing[0.01], Dashing[0.03]},
         PlotLegends → {"a=0.25", "a=0.5", "a=1"}]

Out[3]=
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