Chapter 5

- Born interpretation of the wave function Ψ

$$P(x)\,dx = |\Psi(x,t)|^2 dx$$

is the probability that a particle will be found in an interval dx about the point x

- Normalization condition

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$$

and $\int_a^b |\Psi(x,t)|^2 dx$ is the probability of findinding the particle in the interval $a \le x \le b$.

• The wavefunction for a "free particle" is a " plane wave "

$$\psi(x,t) = e^{i(kx - \omega t)}$$

where $k = \frac{p}{\hbar}$ (p: momentum of the particle)

and $\omega = \frac{E}{\hbar}$ (E: kinetic energy of the particle

⇒ This particle is NOT LOCALIZED however it momentum ($p = \hbar k$) and energy ($E = \hbar\omega$) are KNOWN PRECISELY.

• If the particle is initially LOCALIZED, then the wavefunction must be a WAVE PACKET

$$\psi(x,t) = \int_{-\infty}^{+\infty} \underset{\substack{\uparrow \\ \text{amplitude}}}{a(k)}\, e^{i(kx - \omega t)}\, dk$$

In the presence of a force Schrodinger equation is: $\left(F = -\frac{dU}{dx} \right)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} + U(x)\,\psi(x,t) = i\hbar\frac{\partial \psi(x,t)}{\partial t}$$

Stationary states are $\psi(x,t) = \psi(x)\, e^{-i\omega t}$

time independent wave function

Schrodinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\,\psi(x) = E\,\psi(x)$$

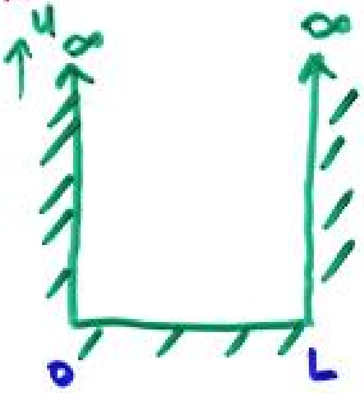
This is the time independent schrodinger equation.

Note that $|\psi(x,t)|^2 = |\psi(x)|^2 \left(e^{i\omega t} \cdot e^{-i\omega t} \right)$

$$= |\psi(x)|^2$$

$\Rightarrow$ For stationary states the probabilities calculated from $\psi(x,t)$ are time independent!

Example #1 : Particle in a box



$U(x) = 0 \qquad 0 \leq x \leq L$

$U(x) = \infty \qquad x < 0$
$\qquad\qquad\qquad x > L$

- Outside the box; $\psi(x) = 0$

- Inside the box; $U(x) = 0$

$$\frac{d^2\psi(x)}{dx^2} + \underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi(x) = 0$$

$$\Downarrow$$

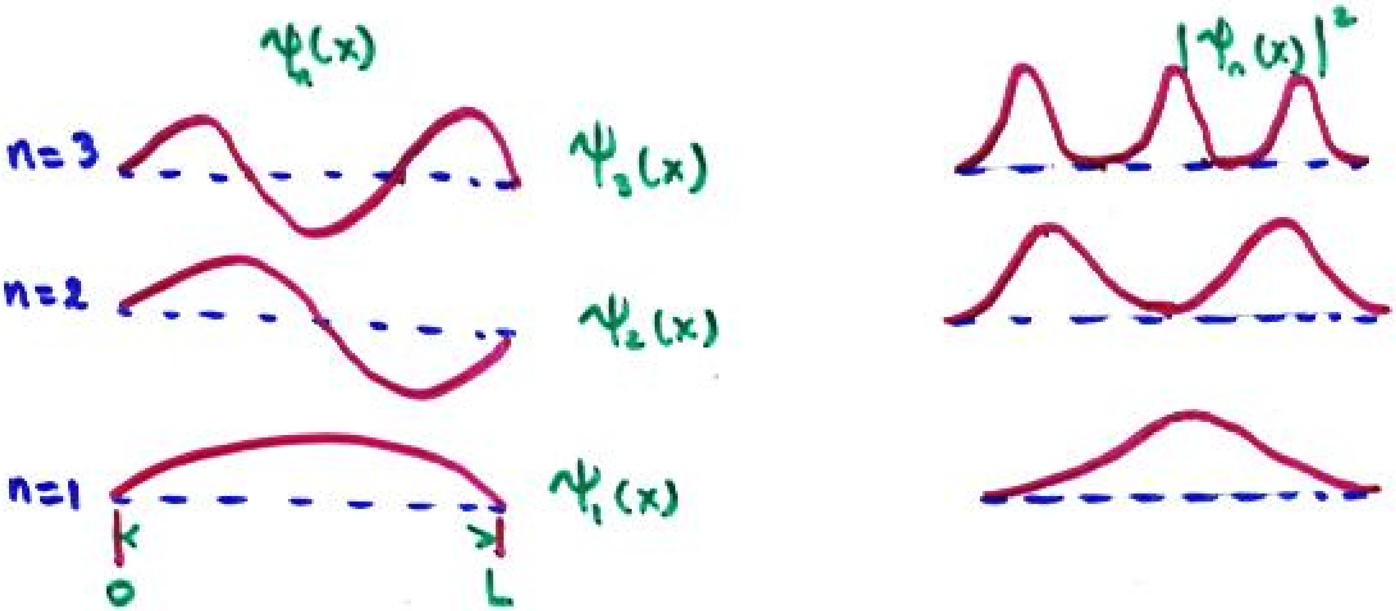
$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi}{L}x\right)} \qquad n = 1, 2, 3, \ldots$$

also: $$\boxed{E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}} \qquad n = 1, 2, 3, \ldots$$

$\hookrightarrow$ Energy is quantized !!!

$E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$ is the <u>ground state</u>

$E_n = n^2 E_1$ are the <u>excited states</u>.



• For the harmonic oscillator $U(x) = \frac{1}{2} m \omega^2 x^2$

Schrodinger equation is

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(\frac{1}{2} m\omega^2 x^2 - E\right)\psi(x)$$

The ground state wavefunction is

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}} = C_0 e^{-\alpha x^2}$$

The ground state energy is

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

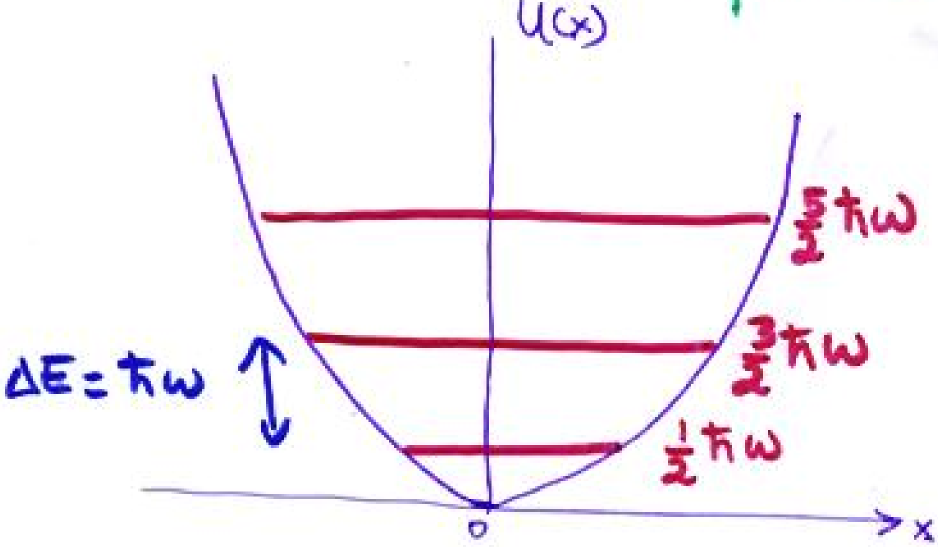
The first excited state wavefunction is

$$\psi_1(x) = C_0 x e^{-\alpha x^2}$$

and $\boxed{E_1 = \frac{3}{2}\hbar\omega}$

In general $\boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega}$ $n = 0, 1, 2, \ldots$

Total energy of the particle.



Note that $\Delta E = \hbar\omega = hf$ just like the energy of blackbody oscillators (Planck).

• The average position of a particle in quantum mechanics is called the expectation value $\langle x \rangle$.

$$\langle x \rangle = \int_{-\infty}^{+\infty} x \, |\psi|^2 dx$$

For any function $f(x)$

$$\langle f \rangle = \int_{-\infty}^{+\infty} f(x) \, |\psi|^2 dx$$

If $f(x) = x^2$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \, |\psi|^2 dx$$

The quantum uncertainty in particle position is

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

• In quantum mechanics there are two types of variables

- sharp such as energy
- fuzzy such as position

For sharp variables the quantum uncertainty = 0

An observable is any particle property that can be measured.

In Q.M. every observable $\longrightarrow$ operator

$\frac{d}{dx} f(x)$: take derivative

operand

operator

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \psi^* [Q] \psi \, dx$$

observable

operator

examples:

momentum $[p] = -i\hbar \frac{\partial}{\partial x}$

$[x^2] = x^2$

Kinetic energy $[K] = \frac{[p]^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$

Hamiltonian $[H] = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x)$

Energy $[E] = i\hbar \frac{\partial}{\partial t}$