## QUESTIONS

1. The probability density at certain points for a particle in a box is zero, as seen in Figure 6.9. Does this imply that the particle cannot move across these points? Explain.
2. Discuss the relation between the zero-point energy and the uncertainty principle.
3. Consider a square well with one finite wall and one infinite wall. Compare the energy and momentum of a particle trapped in this well to the energy and momentum of an identical particle trapped in an infinite well with the same width.
4. Explain why a wave packet moves with the group velocity rather than with the phase velocity.
5. According to Section 6.2, a free particle can be represented by any number of waveforms, depending on the values chosen for the coefficients $a(k)$. What is the source of this ambiguity, and how is it resolved?
6. Because the Schrödinger equation can be formulated in terms of operators as $[H]\Psi = [E]\Psi$, is it incorrect to conclude from this the operator equivalence $[H] = [E]$?
7. For a particle in a box, the squared momentum $p^2$ is a sharp observable, but the momentum itself is fuzzy. Explain how this can be so, and how it relates to the classical motion of such a particle.
8. A philosopher once said that "it is necessary for the very existence of science that the same conditions always produce the same results." In view of what has been said in this chapter, present an argument showing that this statement is false. How might the statement be reworded to make it true?

## PROBLEMS

### 6.1 The Born Interpretation

1. Of the functions graphed in Figure P6.1, which are candidates for the Schrödinger wavefunction of an actual physical system? For those that are not, state why they fail to qualify.
2. A particle is described by the wavefunction

$$\psi(x) = \begin{cases} A\cos\left(\dfrac{2\pi x}{L}\right) & \text{for } -\dfrac{L}{4} \le x \le \dfrac{L}{4} \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the normalization constant $A$. (b) What is the probability that the particle will be found between $x = 0$ and $x = L/8$ if a measurement of its position is made?

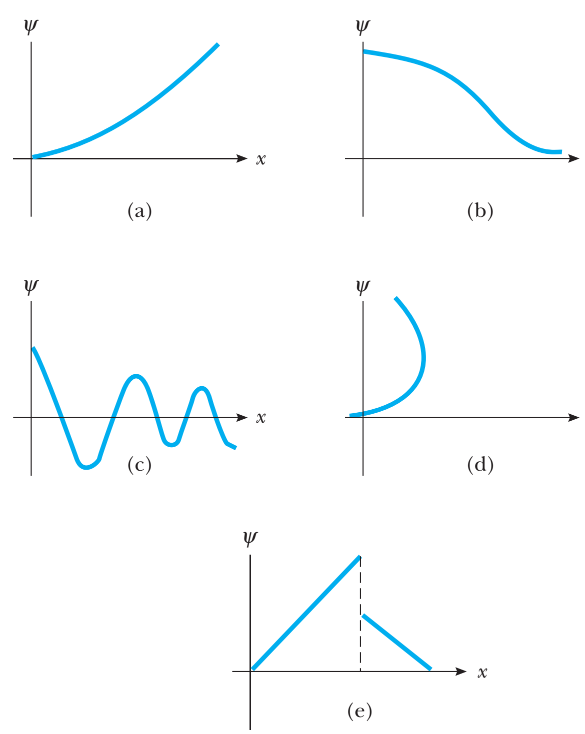


**Figure P6.1**

### 6.2 Wavefunction for a Free Particle

3. A free electron has a wavefunction

$$\psi(x) = A\sin(5 \times 10^{10}\, x)$$

where $x$ is measured in meters. Find (a) the electron's de Broglie wavelength, (b) the electron's momentum, and (c) the electron's energy in electron volts.

4. *Spreading of a Gaussian wave packet.* The Gaussian wave packet $\Psi(x, 0)$ of Example 6.3 is built out of plane waves according to the amplitude distribution function $a(k) = (C\alpha/\sqrt{\pi})\exp(-\alpha^2 k^2)$. Calculate $\Psi(x, t)$ for this packet and describe its evolution.

### 6.3 Wavefunctions in the Presence of Forces

5. In a region of space, a particle with zero energy has a wavefunction

$$\psi(x) = Axe^{-x^2/L^2}$$

(a) Find the potential energy $U$ as a function of $x$.
(b) Make a sketch of $U(x)$ versus $x$.

6. The wavefunction of a particle is given by

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

where $A$, $B$, and $k$ are constants. Show that $\psi$ is a solution of the Schrödinger equation (Eq. 6.13), assuming the particle is free ($U = 0$), and find the corresponding energy $E$ of the particle.

## 6.4 The Particle in a Box

7. Show that allowing the state $n = 0$ for a particle in a one-dimensional box violates the uncertainty principle, $\Delta x\, \Delta p \geq \hbar/2$.
8. A bead of mass 5.00 g slides freely on a wire 20.0 cm long. Treating this system as a particle in a one-dimensional box, calculate the value of $n$ corresponding to the state of the bead if it is moving at a speed of 0.100 nm per year (that is, apparently at rest).
9. The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by a square well. Imagine a proton confined in an infinite square well of length $10^{-5}$ nm, a typical nuclear diameter. Calculate the wavelength and energy associated with the photon that is emitted when the proton undergoes a transition from the first excited state ($n = 2$) to the ground state ($n = 1$). In what region of the electromagnetic spectrum does this wavelength belong?
10. An electron is contained in a one-dimensional box of width 0.100 nm. (a) Draw an energy-level diagram for the electron for levels up to $n = 4$. (b) Find the wavelengths of *all* photons that can be emitted by the electron in making transitions that would eventually get it from the $n = 4$ state to the $n = 1$ state.
11. Consider a particle moving in a one-dimensional box with walls at $x = -L/2$ and $x = L/2$. (a) Write the wavefunctions and probability densities for the states $n = 1$, $n = 2$, and $n = 3$. (b) Sketch the wavefunctions and probability densities. (*Hint:* Make an analogy to the case of a particle in a box with walls at $x = 0$ and $x = L$.)
12. A ruby laser emits light of wavelength 694.3 nm. If this light is due to transitions from the $n = 2$ state to the $n = 1$ state of an electron in a box, find the width of the box.
13. A proton is confined to moving in a one-dimensional box of width 0.200 nm. (a) Find the lowest possible energy of the proton. (b) What is the lowest possible energy of an electron confined to the same box? (c) How do you account for the large difference in your results for (a) and (b)?
14. A particle of mass $m$ is placed in a one-dimensional box of length $L$. The box is so small that the particle's motion is *relativistic,* so that $E = p^2/2m$ is *not valid.* (a) Derive an expression for the energy levels of the particle using the relativistic energy–momentum relation and the quantization of momentum that derives from confinement. (b) If the particle is an electron in a box of length $L = 1.00 \times 10^{-12}$ m, find its lowest possible kinetic energy. By what percent is the nonrelativistic formula for the energy in error?
15. Consider a "crystal" consisting of two nuclei and two electrons, as shown in Figure P6.15. (a) Taking into account all the pairs of interactions, find the potential energy of the system as a function of $d$. (b) Assuming the electrons to be restricted to a one-dimensional box of length $3d$, find the minimum kinetic energy of the two electrons. (c) Find the value of $d$ for which the total energy is a *minimum.* (d) Compare this value of $d$ with the spacing of atoms in lithium, which has a density of 0.53 g/cm$^3$ and an atomic weight of 7. (This type of calculation can be used to estimate the densities of crystals and certain stars.)

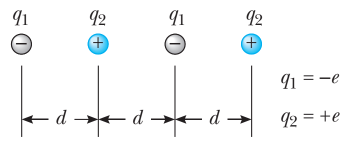


**Figure P6.15**

16. An electron is trapped in an infinitely deep potential well 0.300 nm in width. (a) If the electron is in its ground state, what is the probability of finding it within 0.100 nm of the left-hand wall? (b) Repeat (a) for an electron in the 99th excited state ($n = 100$). (c) Are your answers consistent with the correspondence principle?
17. An electron is trapped at a defect in a crystal. The defect may be modeled as a one-dimensional, rigid-walled box of width 1.00 nm. (a) Sketch the wavefunctions and probability densities for the $n = 1$ and $n = 2$ states. (b) For the $n = 1$ state, find the probability of finding the electron between $x_1 = 0.15$ nm and $x_2 = 0.35$ nm, where $x = 0$ is the left side of the box. (c) Repeat (b) for the $n = 2$ state. (d) Calculate the energies in electron volts of the $n = 1$ and $n = 2$ states.
18. Find the points of maximum and minimum probability density for the $n$th state of a particle in a one-dimensional box. Check your result for the $n = 2$ state.
19. A 1.00-g marble is constrained to roll inside a tube of length $L = 1.00$ cm. The tube is capped at both ends. Modeling this as a one-dimensional infinite square well, find the value of the quantum number $n$ if the marble is initially given an energy of 1.00 mJ. Calculate the *excitation energy* required to promote the marble to the next available energy state.

## 6.5 The Finite Square Well

20. Consider a particle with energy $E$ bound to a *finite* square well of height $U$ and width $2L$ situated on $-L \leq x \leq +L$. Because the potential energy is symmetric about the midpoint of the well, the stationary state waves will be either symmetric or antisymmetric about this point. (a) Show that for $E < U$, the conditions for smooth joining of the interior and exterior waves lead to the following equation for the allowed energies of the symmetric waves:

$$k \tan kL = \alpha \qquad \text{(symmetric case)}$$

where $\alpha = \sqrt{(2m/\hbar^2)(U - E)}$ and $k = \sqrt{2mE/\hbar^2}$ is the wavenumber of oscillation in the interior. (b) Show that the energy condition found in (a) can be rewritten as

$$k \sec kL = \frac{\sqrt{2mU}}{\hbar}$$

Apply the result in this form to an electron trapped at a defect site in a crystal, modeling the defect as a square well of height 5 eV and width 0.2 nm. Solve the equation numerically to find the ground-state energy for the electron, accurate to $\pm 0.001$ eV.

21. Sketch the wavefunction $\psi(x)$ and the probability density $|\psi(x)|^2$ for the $n = 4$ state of a particle in a *finite* potential well.

22. The potential energy of a proton confined to an atomic nucleus can be modeled as a square well of width $1.00 \times 10^{-5}$ nm and height 26.0 MeV. Determine the energy of the proton in the ground state and first excited state for this case, using the Java applet available at our companion Website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 6.22). Refer to Exercise 3 of Example 6.8 for details. Calculate the wavelength of the photon emitted when the proton undergoes a transition from the first excited state to the ground state, and compare your result with that found using the infinite-well model of Problem 9.

23. Consider a square well having an infinite wall at $x = 0$ and a wall of height $U$ at $x = L$ (Fig. P6.23). For the case $E < U$, obtain solutions to the Schrödinger equation inside the well ($0 \le x \le L$) and in the region beyond ($x > L$) that satisfy the appropriate boundary conditions at $x = 0$ and $x = \infty$. Enforce the proper matching conditions at $x = L$ to find an equation for the allowed energies of this system. Are there conditions for which no solution is possible? Explain.

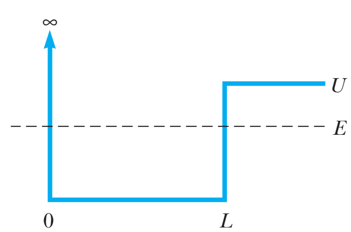


**Figure P6.23**

## 6.6 The Quantum Oscillator

24. The wavefunction

$$\psi(x) = Cxe^{-\alpha x^2}$$

also describes a state of the quantum oscillator, provided the constant $\alpha$ is chosen appropriately. (a) Using Schrödinger's equation, obtain an expression for $\alpha$ in terms of the oscillator mass $m$ and the classical frequency of vibration $\omega$. What is the energy of this state? (b) Normalize this wave. (*Hint:* See the integral of Problem 32.)

25. Show that the oscillator energies in Equation 6.29 correspond to the classical amplitudes

$$A_n = \sqrt{\frac{(2n+1)\hbar}{m\omega}}$$

26. Obtain an expression for the probability density $P_c(x)$ of a *classical* oscillator with mass $m$, frequency $\omega$, and amplitude $A$. (*Hint:* See Problem 28 for the calculation of classical probabilities.)

27. *Coherent states.* Use the Java applet available at our companion website (http://info.brookscole.com/mp3e QMTools Simulations → Problem 6.27) to explore the time development of a Gaussian waveform confined to the oscillator well. The default settings for the initial wave describe a Gaussian centered in the well with an adjustable width set by the value of the parameter $a$. Describe the time evolution of this wavefunction. Is it what you expected? Account for your observations. Now displace the initial waveform off of center by increasing the parameter $d$ from zero to $d = 1$. Again describe the time evolution of the resulting wavefunction. What is remarkable about this case? Such wavefunctions, called *coherent states*, are important in the quantum theory of radiation.

## 6.7 Expectation Values

28. *Classical probabilities.* (a) Show that the classical probability density describing a particle in an infinite square well of dimension $L$ is $P_c(x) = 1/L$. (*Hint:* The classical probability for finding a particle in $dx$—$P_c(x)\,dx$—is proportional to the *time* the particle spends in this interval.) (b) Using $P_c(x)$, determine the *classical* averages $\langle x \rangle$ and $\langle x^2 \rangle$ for a particle confined to the well, and compare with the quantum results found in Example 6.15. Discuss your findings in light of the correspondence principle.

29. An electron is described by the wavefunction

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ Ce^{-x}(1 - e^{-x}) & \text{for } x > 0 \end{cases}$$

where $x$ is in nanometers and $C$ is a constant. (a) Find the value of $C$ that normalizes $\psi$. (b) Where is the electron most likely to be found; that is, for what value of $x$ is the probability for finding the electron largest? (c) Calculate $\langle x \rangle$ for this electron and compare your result with its most likely position. Comment on any differences you find.

30. For any eigenfunction $\psi_n$ of the infinite square well, show that $\langle x \rangle = L/2$ and that

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$$

where $L$ is the well dimension.

31. An electron has a wavefunction

$$\psi(x) = Ce^{-|x|/x_0}$$

where $x_0$ is a constant and $C = 1/\sqrt{x_0}$ for normalization (see Example 6.1). For this case, obtain expressions for $\langle x \rangle$ and $\Delta x$ in terms of $x_0$. Also calculate the probability that the electron will be found within a standard deviation of its average position, that is, in the range $\langle x \rangle - \Delta x$ to $\langle x \rangle + \Delta x$, and show that this is independent of $x_0$.

32. Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\Delta x$ for a quantum oscillator in its ground state. *Hint:* Use the integral formula

$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \qquad a > 0$$

33. (a) What value do you expect for $\langle p \rangle$ for the quantum oscillator? Support your answer with a symmetry argument rather than a calculation. (b) Energy principles for the quantum oscillator can be used to relate $\langle p^2 \rangle$ to $\langle x^2 \rangle$. Use this relation, along with the value of $\langle x^2 \rangle$ from Problem 32, to find $\langle p^2 \rangle$ for the oscillator ground state. (c) Evaluate $\Delta p$, using the results of (a) and (b).

34. From the results of Problems 32 and 33, evaluate $\Delta x\, \Delta p$ for the quantum oscillator in its ground state. Is the result consistent with the uncertainty principle? (Note that your computation verifies the minimum uncertainty product; furthermore, the harmonic oscillator ground state is the *only* quantum state for which this minimum uncertainty is realized.)

### 6.8 Observables and Operators

35. Which of the following functions are eigenfunctions of the momentum operator $[p]$? For those that are eigenfunctions, what are the eigenvalues?
(a) $A \sin(kx)$ (c) $A \cos(kx) + iA \sin(kx)$
(b) $A \sin(kx) - A \cos(kx)$ (d) $Ae^{ik(x-a)}$

## ADDITIONAL PROBLEMS

36. *The quantum bouncer.* The bouncer is the quantum analog to the classical problem of a ball bouncing vertically (and elastically) on a level surface and is modeled by the potential energy shown in Figure P6.36. The coordinate normal to the surface is denoted by $x$, and the surface itself is located at $x = 0$. Above the surface, the potential energy for the bouncer is linear, representing the attractive force of a uniform field—in this case the gravity field near the Earth. Below the surface, the potential energy rises abruptly to a very large value consistent with the bouncer's inability to penetrate this region. Obtaining stationary states for the bouncer from the Schrödinger equation using analytical techniques requires knowledge of special functions. Numerical solution furnishes a simpler alternative and allows for effortless study of the bound-state waveforms, once they are found. The Java applet for the quantum bouncer can be found at http://info.brookscole.com/mp3e QMTools Simulations → Problem 6.36. Use the applet as described there to find the three lowest-lying states of a tennis ball (mass ≈ 50 g) bouncing on a hard floor. Count the number of nodes for each wavefunction to verify the general rule that the $n$th excited state exhibits exactly $n$ nodes. For each state, determine the most probable distance above the floor for the bouncing ball and compare with the maximum height reached in the classical case. (Classically, the ball is most likely to be found at the top of its flight, where its speed drops to zero—see Problem 28.)

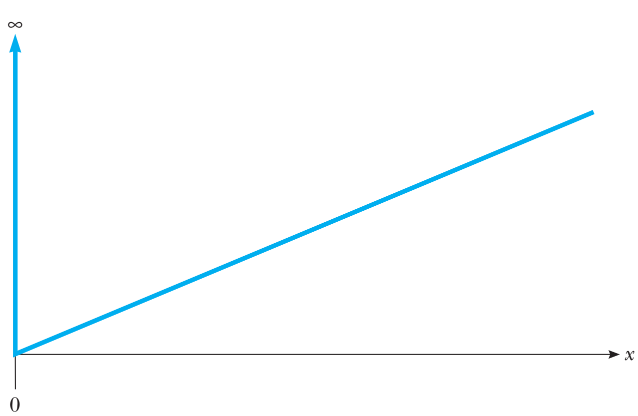


**Figure P6.36**

37. *Nonstationary states.* Consider a particle in an infinite square well described initially by a wave that is a superposition of the ground and first excited states of the well:

$$\Psi(x, 0) = C[\psi_1(x) + \psi_2(x)]$$

(a) Show that the value $C = 1/\sqrt{2}$ normalizes this wave, assuming $\psi_1$ and $\psi_2$ are themselves normalized. (b) Find $\psi(x, t)$ at any later time $t$. (c) Show that the superposition is *not* a stationary state, but that the average energy in this state is the arithmetic mean $(E_1 + E_2)/2$ of the ground- and first excited-state energies $E_1$ and $E_2$.

38. For the nonstationary state of Problem 37, show that the average particle position $\langle x \rangle$ oscillates with time as

$$\langle x \rangle = x_0 + A\cos(\Omega t)$$

where

$$x_0 = \frac{1}{2}\left(\int x|\psi_1|^2\, dx + \int x|\psi_2|^2\, dx\right)$$

$$A = \int x\psi_1^* \psi_2\, dx$$

and $\Omega = (E_2 - E_1)/\hbar$. Evaluate your results for the mean position $x_0$ and amplitude of oscillation $A$ for an electron in a well 1 nm wide. Calculate the time for the electron to shuttle back and forth in the well once. Calculate the same time classically for an electron with energy equal to the average, $(E_1 + E_2)/2$.