PHYS 301 – Term 151

Quiz#6 - chapter 8

06/12/2015

Name: Key Id#:

An object of unit mass orbits in a central potential U(r). Its orbit is $r = ae^{-b\theta}$, where a and b are constant.

(a) Find the potential U(r).

(b) Find θ(t).

$m = 1$

a) $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{\ell^2} F(r)$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = +\frac{b}{a} e^{b\theta} \qquad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{b}{a} e^{b\theta}\right) = \frac{b^2}{a} e^{b\theta}$$

$$\frac{b^2}{a} e^{b\theta} + \frac{e^{b\theta}}{a} = -\frac{a^2 e^{-2b\theta}}{\ell^2} F(r)$$

$$\frac{(b^2+1)}{a} e^{b\theta} = -\frac{a^2}{\ell^2} e^{-2b\theta} F(r)$$

$$F(r) = -\frac{\ell^2(b^2+1)}{a^3} e^{3b\theta} = -\frac{\ell^2(b^2+1)}{a^3 e^{-3b\theta}} = -\frac{\ell^2(b^2+1)}{r^3}$$

$$U(r) = -\int F(r)\, dr = -\frac{\ell^2(b^2+1)}{2r^2} + C \qquad \left(C = 0 \quad U \xrightarrow[r\to\infty]{} 0\right)$$

$$\boxed{U(r) = -\frac{\ell^2(b^2+1)}{2r^2}}$$

b) $$\dot\theta = \frac{d\theta}{dt} = \frac{\ell}{r^2} = \frac{\ell}{a^2 e^{-2b\theta}} \Rightarrow \frac{\ell}{a^2} dt = e^{-2b\theta} d\theta$$

$$\frac{\ell}{a^2} t = -\frac{1}{2b} e^{-2b\theta} + C \Rightarrow -\frac{2\ell b t}{a^2} = e^{-2b\theta} + C$$

$$-2b\theta(t) = \ln\left(C - \frac{2\ell b}{a^2} t\right) \Rightarrow \boxed{\theta(t) = -\frac{1}{2b} \ln\left(C - \frac{2\ell b}{a^2} t\right)}$$