# 2
# Relativity II

**_Chapter Outline_**



In this chapter we extend the theory of special relativity to classical mechanics, that is, we give relativistically correct expressions for momentum, Newton's second law, and the famous equivalence of mass and energy. The final section, on general relativity, deals with the physics of accelerating reference frames and Einstein's theory of gravitation.

## 2.1 RELATIVISTIC MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

The conservation of linear momentum states that when two bodies collide, the total momentum remains constant, assuming the bodies are isolated (that is, they interact only with each other). Suppose the collision is described in a reference frame S in which momentum is conserved. If the velocities of the colliding bodies are calculated in a second inertial frame S′ using the Lorentz transformation, and the classical definition of momentum $\mathbf{p} = m\mathbf{u}$ applied, one finds that momentum *is not* conserved in the second reference frame. However, because the laws of physics are the same in all inertial frames, momentum must be conserved in all frames if it is conserved in any one. This application of the principle of relativity demands that we modify the classical definition of momentum.

To see how the classical form $\mathbf{p} = m\mathbf{u}$ fails and to determine the correct relativistic definition of $\mathbf{p}$, consider the case of an inelastic collision

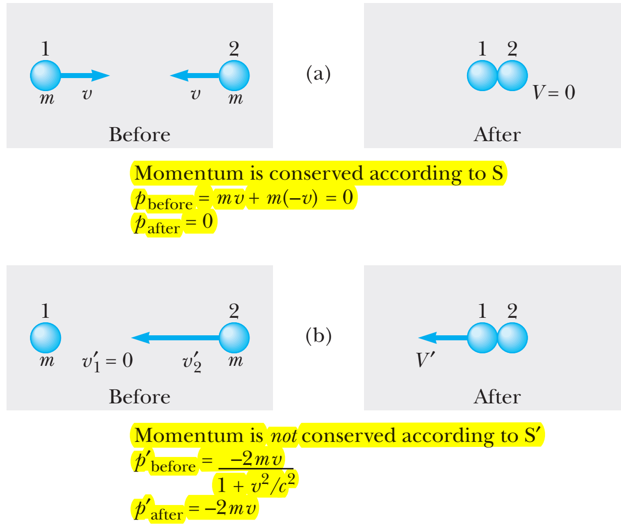


**Figure 2.1** (a) An inelastic collision between two equal clay lumps as seen by an observer in frame S. (b) The same collision viewed from a frame S′ that is moving to the right with speed $v$ with respect to S.

between two particles of equal mass. Figure 2.1a shows such a collision for two identical particles approaching each other at speed $v$ as observed in an inertial reference frame S. Using the classical form for momentum, $\mathbf{p} = m\mathbf{u}$ (we use the symbol $\mathbf{u}$ for particle velocity rather than $\mathbf{v}$, which is reserved for the relative velocity of two reference frames), the observer in S finds momentum is conserved as shown in Figure 2.1a. Suppose we now view things from an inertial frame S′ moving to the right with speed $v$ relative to S. In S′ the new speeds are $v'_1$, $v'_2$ and $V'$ (see Fig. 2.1b). If we use the Lorentz velocity transformation

$$u'_x = \frac{u_x - v}{1 - (u_x v/c^2)}$$

to find $v'_1$, $v'_2$ and $V'$, and the classical form for momentum, $\mathbf{p} = m\mathbf{u}$, will momentum be conserved according to the observer in S′? To answer this question we first calculate the velocities of the particles in S′ in terms of the given velocities in S.

$$v'_1 = \frac{v_1 - v}{1 - (v_1 v/c^2)} = \frac{v - v}{1 - (v^2/c^2)} = 0$$

$$v'_2 = \frac{v_2 - v}{1 - (v_2 v/c^2)} = \frac{-v - v}{1 - [(-v)(v)/c^2]} = \frac{-2v}{1 + (v^2/c^2)}$$

$$V' = \frac{V - v}{1 - (Vv/c^2)} = \frac{0 - v}{1 - [(0)v/c^2]} = -v$$

Checking for momentum conservation in S′, we have

$$p'_{\text{before}} = mv'_1 + mv'_2 = m(0) + m\left[\frac{-2v}{1 + (v^2/c^2)}\right] = \frac{-2mv}{1 + (v^2/c^2)}$$

$$p'_{\text{after}} = 2mV' = -2mv$$

Thus, *in the frame S′*, the momentum before the collision is not equal to the momentum after the collision, and *momentum is not conserved.*

It can be shown (see Example 2.6) that momentum is conserved in both S and S′, (and indeed in all inertial frames), if we redefine momentum as

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - (u^2/c^2)}} \tag{2.1}$$

**Definition of relativistic momentum**

where $\mathbf{u}$ is the velocity of the particle and *m is the proper mass,* that is, the mass measured by an observer at rest with respect to the mass.[1] Note that when $u$ is much less than $c$, the denominator of Equation 2.1 approaches unity and $\mathbf{p}$ approaches $m\mathbf{u}$. Therefore, the relativistic equation for $\mathbf{p}$ reduces to the classical expression when $u$ is small compared with $c$. Because it is a simpler expression, Equation 2.1 is often written $\mathbf{p} = \gamma m\mathbf{u}$, where $\gamma = 1/\sqrt{1 - (u^2/c^2)}$. Note that this $\gamma$ has the same functional form as the $\gamma$ in the Lorentz transformation, but here $\gamma$ contains $u$, the particle speed, while in the Lorentz transformation, $\gamma$ contains $v$, the relative speed of the two frames.

The **relativistic form of Newton's second law** is given by the expression

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u}) \tag{2.2}$$

where $\mathbf{p}$ is given by Equation 2.1. This expression is reasonable because it preserves classical mechanics in the limit of low velocities and requires **the momentum of an isolated system (F = 0) to be conserved relativistically as well as classically.** It is left as a problem (Problem 3) to show that the relativistic acceleration $a$ of a particle *decreases* under the action of a constant force applied in the direction of $\mathbf{u}$, as

$$a = \frac{F}{m}(1 - u^2/c^2)^{3/2}$$

From this formula we see that as the velocity approaches $c$, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed equal to or greater than $c$.

[1] In this book we shall always take $m$ to be constant with respect to speed, and we call $m$ the speed invariant mass, or proper mass. Some physicists refer to the mass in Equation 2.1 as the rest mass, $m_0$, and call the term $m_0/\sqrt{1 - (u^2/c^2)}$ the relativistic mass. Using this description, the relativistic mass is imagined to increase with increasing speed. We exclusively use the invariant mass $m$ because we think it is a clearer concept and that the introduction of relativistic mass leads to no deeper physical understanding.

**EXAMPLE 2.1 Momentum of an Electron**

An electron, which has a mass of $9.11 \times 10^{-31}$ kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

**Solution** Using Equation 2.1 with $u = 0.750c$, we have

$$\begin{aligned} p &= \frac{mu}{\sqrt{1-(u^2/c^2)}} \\ &= \frac{(9.11 \times 10^{-31}\text{ kg})(0.750 \times 3.00 \times 10^8\text{ m/s})}{\sqrt{1-[(0.750c)^2/c^2]}} \\ &= 3.10 \times 10^{-22}\text{ kg}\cdot\text{m/s} \end{aligned}$$

The incorrect classical expression would give

$$\text{momentum} = mu = 2.05 \times 10^{-22}\text{ kg}\cdot\text{m/s}$$

Hence, for this case the correct relativistic result is 50% greater than the classical result!

**EXAMPLE 2.2 An Application of the Relativistic Form of F = dp/dt: The Measurement of the Momentum of a High-Speed Charged Particle**

Suppose a particle of mass $m$ and charge $q$ is injected with a relativistic velocity $\mathbf{u}$ into a region containing a magnetic field $\mathbf{B}$. The magnetic force $\mathbf{F}$ on the particle is given by $\mathbf{F} = q\mathbf{u} \times \mathbf{B}$. If $\mathbf{u}$ is perpendicular to $\mathbf{B}$, the force is radially inward, and the particle moves in a circle of radius $R$ with $|\mathbf{u}|$ constant. From Equation 2.2 we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u})$$

**Solution** Because the magnetic force is always perpendicular to the velocity, it does no work on the particle, and hence the speed, $u$, and $\gamma$ are both constant with time. Thus, the magnitude of the force on the particle is

$$F = \gamma m\left|\frac{d\mathbf{u}}{dt}\right| \tag{2.3}$$

Substituting $F = quB$ and $|d\mathbf{u}/dt| = u^2/R$ (the usual definition of centripetal acceleration) into Equation 2.3, we can solve for $p = \gamma mu$. We find

$$p = \gamma mu = qBR \tag{2.4}$$

Equation 2.4 shows that the momentum of a relativistic particle of known charge $q$ may be determined by measuring its radius of curvature $R$ in a known magnetic field, $\mathbf{B}$. This technique is routinely used to determine the momentum of subatomic particles from photographs of their tracks in space.

## 2.2 RELATIVISTIC ENERGY

We have seen that the definition of momentum and the laws of motion required generalization to make them compatible with the principle of relativity. This implies that the relativistic form of the kinetic energy must also be modified.

To derive the relativistic form of the work–energy theorem, let us start with the definition of work done by a force $F$ and make use of the definition of relativistic force, Equation 2.2. That is,

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \tag{2.5}$$

where we have assumed that the force and motion are along the $x$-axis. To perform this integration and find the work done on a particle or the relativistic kinetic energy as a function of the particle velocity $u$, we first evaluate $dp/dt$:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1-(u^2/c^2)}} = \frac{m\left(\dfrac{du}{dt}\right)}{[1-(u^2/c^2)]^{3/2}} \tag{2.6}$$

Substituting this expression for $dp/dt$ and $dx = u\,dt$ into Equation 2.5 gives

$$W = \int_{x_1}^{x_2} \frac{m\left(\dfrac{du}{dt}\right)u\,dt}{[1-(u^2/c^2)]^{3/2}} = m\int_0^u \frac{u\,du}{[1-(u^2/c^2)]^{3/2}}$$

where we have assumed that the particle is accelerated from rest to some final velocity $u$. Evaluating the integral, we find that

$$W = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2 \qquad (2.7)$$

Recall that the work–energy theorem states that the work done by all forces acting on a particle equals the change in kinetic energy of the particle. Because the initial kinetic energy is zero, we conclude that the work $W$ in Eq. 2.7 is equal to the relativistic kinetic energy $K$, that is,

$$K = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2 \qquad (2.8)$$

**Relativistic kinetic energy**

At low speeds, where $u/c \ll 1$, Equation 2.8 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1-x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \cdots$, for $x \ll 1$, where the higher-order powers of $x$ are ignored in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1-(u^2/c^2)}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

Substituting this into Equation 2.8 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 = \frac{1}{2}mu^2$$

which agrees with the classical result. A graph comparing the relativistic and nonrelativistic expressions for $u$ as a function of $K$ is given in Figure 2.2. Note that in the relativistic case, the particle speed never exceeds $c$, regard-

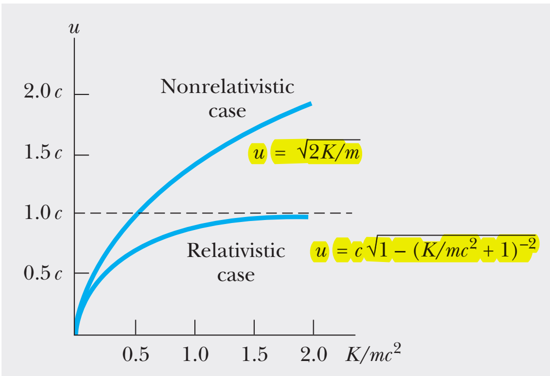


**Figure 2.2** A graph comparing the relativistic and nonrelativistic expressions for speed as a function of kinetic energy. In the relativistic case, $u$ is always *less* than $c$.

less of the kinetic energy, as is routinely confirmed in very high energy particle accelerator experiments. The two curves are in good agreement when $u \ll c$.

It is instructive to write the relativistic kinetic energy in the form

$$K = \gamma mc^2 - mc^2 \tag{2.9}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

The constant term $mc^2$, which is independent of the speed, is called the **rest energy** of the particle. The term $\gamma mc^2$, which depends on the particle speed, is therefore the sum of the kinetic and rest energies. We define $\gamma mc^2$ to be the **total energy** $E$, that is,

**Definition of total energy**

$$E = \gamma mc^2 = K + mc^2 \tag{2.10}$$

**Mass–energy equivalence**

**The expression $E = \gamma mc^2$ is Einstein's famous mass–energy equivalence equation, which shows that mass is a measure of the total energy in all forms.** Although we have been considering single particles for simplicity, Equation 2.10 applies to macroscopic objects as well. In this case it has the remarkable implication that any kind of energy added to a "brick" of matter—electric, magnetic, elastic, thermal, gravitational, chemical—actually increases the mass! Several end-of-chapter questions and problems explore this idea more fully. Another implication of Equation 2.10 is that a small mass corresponds to an enormous amount of energy because $c^2$ is a very large number. This concept has revolutionized the field of nuclear physics and is treated in detail in the next section.

In many situations, the momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy $E$ to the relativistic momentum $p$. This is accomplished using $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate $u$ (Problem 7). The result, after some algebra, is

**Energy–momentum relation**

$$E^2 = p^2c^2 + (mc^2)^2 \tag{2.11}$$

When the particle is at rest, $p = 0$, and so we see that $E = mc^2$. That is, the total energy equals the rest energy. For the case of particles that have zero mass, such as photons (massless, chargeless particles of light), we set $m = 0$ in Equation 2.11, and find

$$E = pc \tag{2.12}$$

This equation is an *exact* expression relating energy and momentum for photons, which always travel at the speed of light.

Finally, note that because the mass $m$ of a particle is independent of its motion, $m$ must have the same value in all reference frames. On the other hand, the total energy and momentum of a particle depend on the reference frame in which they are measured, because they both depend on velocity. Because $m$ is a constant, then according to Equation 2.11 the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is *invariant* under a Lorentz transformation.

When dealing with electrons or other subatomic particles, it is convenient to express their energy in **electron volts (eV)**, since the particles are usually given this energy by acceleration through a potential difference. The conversion factor is

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

For example, the mass of an electron is $9.11 \times 10^{-31}$ kg. Hence, the rest energy of the electron is

$$m_ec^2 = (9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 = 8.20 \times 10^{-14} \text{ J}$$

Converting this to electron volts, we have

$$m_ec^2 = (8.20 \times 10^{-14} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J}) = 0.511 \text{ MeV}$$

where 1 MeV = $10^6$ eV. Finally, note that because $m_ec^2 = 0.511$ MeV, the mass of the electron may be written $m_e = 0.511 \text{ MeV}/c^2$, accounting for the practice of measuring particle masses in units of MeV/$c^2$.

**EXAMPLE 2.3 The Energy of a Speedy Electron**

An electron has a speed $u = 0.850c$. Find its total energy and kinetic energy in electron volts.

**Solution** Using the fact that the rest energy of the electron is 0.511 MeV together with $E = \gamma mc^2$ gives

$$E = \frac{m_ec^2}{\sqrt{1 - (u^2/c^2)}} = \frac{0.511 \text{ MeV}}{\sqrt{1 - [(0.85c)^2/c^2]}}$$
$$= 1.90(0.511 \text{ MeV}) = 0.970 \text{ MeV}$$

The kinetic energy is obtained by subtracting the rest energy from the total energy:

$$K = E - m_ec^2 = 0.970 \text{ MeV} - 0.511 \text{ MeV} = 0.459 \text{ MeV}$$

**EXAMPLE 2.4 The Energy of a Speedy Proton**

The total energy of a proton is three times its rest energy.

(a) Find the proton's rest energy in electron volts.

**Solution**

$$\begin{aligned}\text{rest energy} &= m_pc^2 \\ &= (1.67 \times 10^{-27} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2 \\ &= (1.50 \times 10^{-10} \text{ J})(1 \text{ eV}/1.60 \times 10^{-19} \text{ J}) \\ &= 938 \text{ MeV}\end{aligned}$$

(b) With what speed is the proton moving?

**Solution** Because the total energy $E$ is three times the rest energy, $E = \gamma mc^2$ gives

$$E = 3m_pc^2 = \frac{m_pc^2}{\sqrt{1 - (u^2/c^2)}}$$

$$3 = \frac{1}{\sqrt{1 - (u^2/c^2)}}$$

Solving for $u$ gives

$$\left(1 - \frac{u^2}{c^2}\right) = \frac{1}{9} \quad \text{or} \quad \frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = 2.83 \times 10^8 \text{ m/s}$$

(c) Determine the kinetic energy of the proton in electron volts.

**Solution**

$$K = E - m_pc^2 = 3m_pc^2 - m_pc^2 = 2m_pc^2$$

Because $m_pc^2 = 938$ MeV, $K = 1876$ MeV.

(d) What is the proton's momentum?

**Solution** We can use Equation 2.11 to calculate the momentum with $E = 3m_pc^2$:

$$E^2 = p^2c^2 + (m_pc^2)^2 = (3m_pc^2)^2$$
$$p^2c^2 = 9(m_pc^2)^2 - (m_pc^2)^2 = 8(m_pc^2)^2$$
$$p = \sqrt{8}\,\frac{m_pc^2}{c} = \sqrt{8}\,\frac{(938 \text{ MeV})}{c} = 2650\,\frac{\text{MeV}}{c}$$

Note that the unit of momentum is left as MeV/$c$ for convenience.

## 2.3 MASS AS A MEASURE OF ENERGY

The equation $E = \gamma mc^2$ as applied to a particle suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy through its mass. The clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions in which both the conversion of mass into energy and the conversion of energy into mass take place. Because of this convertibility from the currency of mass into the currency of energy, we can no longer accept the separate classical laws of the conservation of mass and the conservation of energy; we must instead speak of a single unified law, **the conservation of mass–energy.** Simply put, this law requires that **the sum of the mass–energy of a system of particles before interaction must equal the sum of the mass–energy of the system after interaction where the mass–energy of the ith particle is defined as the total relativistic energy**

**Conservation of mass–energy**

$$E_i = \frac{m_i c^2}{\sqrt{1-(u_i^2/c^2)}}$$

To understand the conservation of mass–energy and to see how the relativistic laws possess more symmetry and wider scope than the classical laws of momentum and energy conservation, we consider the simple inelastic collision treated earlier.

As one can see in Figure 2.1a, *classically* momentum is conserved but kinetic energy is not because the total kinetic energy before collision equals $mu^2$ and the total kinetic energy after is zero (we have replaced the $v$ shown in Figure 2.1 with $u$). Now consider the same two colliding clay lumps using the relativistic mass–energy conservation law. If the mass of each lump is $m$, and the mass of the composite object is $M$, we must have

$$E_{\text{before}} = E_{\text{after}}$$

$$\frac{mc^2}{\sqrt{1-(u^2/c^2)}} + \frac{mc^2}{\sqrt{1-(u^2/c^2)}} = Mc^2$$

or

$$M = \frac{2m}{\sqrt{1-(u^2/c^2)}} \qquad (2.13)$$

Because $\sqrt{1-(u^2/c^2)} < 1$, the composite mass $M$ is greater than the sum of the two individual masses! What's more, it is easy to show that the mass increase of the composite lump, $\Delta M = M - 2m$, is equal to the sum of the incident kinetic energies of the colliding lumps ($2K$) divided by $c^2$:

$$\Delta M = \frac{2K}{c^2} = \frac{2}{c^2}\left(\frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2\right) \qquad (2.14)$$

Thus, we have an example of the conversion of kinetic energy to mass, and the satisfying result that in relativistic mechanics, kinetic energy is not lost in an inelastic collision but shows up as an increase in the mass of the final composite object. In fact, the deeper symmetry of relativity theory shows that *both relativistic mass–energy and momentum are always conserved in a collision,* whereas classical methods show that momentum is conserved but kinetic energy is not unless the collision is perfectly elastic. Indeed, as we show in Example 2.6, relativistic momentum and energy are inextricably linked because momentum conservation only holds in all inertial frames if mass–energy conservation also holds.

**EXAMPLE 2.5**

(a) Calculate the mass increase for a completely inelastic head-on collision of two 5.0-kg balls each moving toward the other at 1000 mi/h (the speed of a fast jet plane). (b) Explain why measurements on macroscopic objects reinforce the relativistically *incorrect* beliefs that mass is conserved ($M = 2m$) and that kinetic energy is lost in an inelastic collision.

**Solution** (a) $u = 1000$ mi/h $= 450$ m/s, so

$$\frac{u}{c} = \frac{4.5 \times 10^2 \text{ m/s}}{3.0 \times 10^8 \text{ m/s}} = 1.5 \times 10^{-6}$$

Because $u^2/c^2 \ll 1$, substituting

$$\frac{1}{\sqrt{1-(u^2/c^2)}} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

in Equation 2.14 gives

$$\Delta M = 2m\left(\frac{1}{\sqrt{1-(u^2/c^2)}} - 1\right)$$

$$\approx 2m\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right) \approx \frac{mu^2}{c^2}$$

$$= (5.0 \text{ kg})(1.5 \times 10^{-6})^2 = 1.1 \times 10^{-11} \text{ kg}$$

(b) Because the mass increase of $1.1 \times 10^{-11}$ kg is an unmeasurably minute fraction of $2m$ (10 kg), it is quite natural to believe that the mass remains constant when macroscopic objects suffer an inelastic collision. On the other hand, the change in kinetic energy from $mu^2$ to 0 is so large ($10^6$ J) that it is readily measured to be lost in an inelastic collision of macroscopic objects.

***Exercise 1*** Prove that $\Delta M = 2\Delta K/c^2$ for a completely inelastic collision, as stated.

**EXAMPLE 2.6**

Show that use of the relativistic definition of momentum

$$p = \frac{mu}{\sqrt{1-(u^2/c^2)}}$$

leads to momentum conservation in both S and S′ for the inelastic collision shown in Figure 2.1.

**Solution** In frame S:

$$p_{\text{before}} = \gamma mv + \gamma m(-v) = 0$$

$$p_{\text{after}} = \gamma MV = (\gamma M)(0) = 0$$

Hence, momentum is conserved in S. Note that we have used $M$ as the mass of the two combined masses after the collision and allowed for the possibility in relativity that $M$ is not necessarily equal to $2m$.

In frame S′:

$$p'_{\text{before}} = \gamma m v'_1 + \gamma m v'_2 = \frac{(m)(0)}{\sqrt{1-(0)^2/c^2}}$$

$$+ \frac{m}{\{\sqrt{1-[-2v/1+(v^2/c^2)]^2}\}(1/c^2)} \times \left(\frac{-2v}{1+v^2/c^2}\right)$$

After some algebra, we find

$$\frac{m}{\{\sqrt{1-[2v/1+(v^2/c^2)]^2}\}(1/c^2)} = \frac{m(1+v^2/c^2)}{(1-v^2/c^2)}$$

and we obtain

$$p'_{\text{before}} = \frac{m(1+v^2/c^2)}{(1-v^2/c^2)}\left(\frac{-2v}{1+v^2/c^2}\right) = \frac{-2mv}{(1-v^2/c^2)}$$

$$p'_{\text{after}} = \gamma M V' = \frac{M(-v)}{\sqrt{1-[(-v)^2/c^2]}} = \frac{-Mv}{\sqrt{1-v^2/c^2}}$$

To show that momentum is conserved in S′, we use the fact that $M$ is not simply equal to $2m$ in relativity. As shown, the combined mass, $M$, formed from the collision of two particles, each of mass $m$ moving toward each other with speed $v$, is greater than $2m$. This occurs because of the equivalence of mass and energy, that is, the kinetic energy of the incident particles shows up in relativity theory as a tiny increase in mass, which can actually be measured as thermal energy. Thus, from Equation 2.13, which results from imposing the conservation of mass–energy, we have

$$M = \frac{2m}{\sqrt{1-(v^2/c^2)}}$$

Substituting this result for $M$ into $p'_{\text{after}}$, we obtain

$$p'_{\text{after}} = \frac{2m}{\sqrt{1-(v^2/c^2)}}\frac{-v}{\sqrt{1-(v^2/c^2)}}$$

$$= \frac{-2mv}{1-(v^2/c^2)} = p'_{\text{before}}$$

Hence, momentum is conserved in both S and S′, provided that we use the correct relativistic definition of momentum, $p = \gamma mu$, and assume the conservation of mass–energy.

The absence of observable mass changes in inelastic collisions of macroscopic objects impels us to look for other areas to test this law, where particle velocities are higher, masses are more precisely known, and forces are stronger than electrical or mechanical forces. This leads us to consider nuclear reactions, because nuclear masses can be measured very precisely with a mass spectrometer, nuclear forces are much stronger than electrical forces, and decay products are often produced with extremely high velocities.

Perhaps the most direct confirmation of the conservation of mass–energy occurs in the decay of a heavy radioactive nucleus at rest into several lighter particles emitted with large kinetic energies. For such a nucleus

**Fission**

of mass $M$ undergoing *fission* into particles with masses $M_1$, $M_2$, and $M_3$ and having speeds $u_1$, $u_2$, and $u_3$, conservation of total relativistic energy requires

$$Mc^2 = \frac{M_1c^2}{\sqrt{1-(u_1^2/c^2)}} + \frac{M_2c^2}{\sqrt{1-(u_2^2/c^2)}} + \frac{M_3c^2}{\sqrt{1-(u_3^2/c^2)}} \qquad (2.15)$$

Because the square roots are all less than 1, $M > M_1 + M_2 + M_3$ and the loss of mass, $M - (M_1 + M_2 + M_3)$, appears as energy of motion of the products. This **disintegration energy** released per fission is often denoted by the symbol $Q$ and can be written for our case as

$$Q = [M - (M_1 + M_2 + M_3)]c^2 = \Delta mc^2 \qquad (2.16)$$

**EXAMPLE 2.7 A Fission Reaction**

An excited $^{236}_{92}$U nucleus decays at rest into $^{90}_{37}$Rb, $^{143}_{55}$Cs, and several neutrons, $^1_0$n. (a) By conserving charge and the total number of protons and neutrons, write a balanced reaction equation and determine the number of neutrons produced. (b) Calculate by how much the combined "offspring" mass is less than the "parent" mass. (c) Calculate the energy released per fission. (d) Calculate the energy released in kilowatt hours when 1 kg of uranium undergoes fission in a power plant that is 40% efficient.

**Solution** (a) In general, an element is represented by the symbol $^A_Z$X, where X is the symbol for the element, $A$ is the number of neutrons plus protons in the nucleus (mass number), and $Z$ is the number of protons in the nucleus (atomic number). Conserving charge and number of nucleons gives

$$^{236}_{92}\text{U} \longrightarrow {}^{90}_{37}\text{Rb} + {}^{143}_{55}\text{Cs} + 3\,^1_0\text{n}$$

So three neutrons are produced per fission.

(b) The masses of the decay particles are given in Appendix B in terms of atomic mass units, u, where 1 u = $1.660 \times 10^{-27}$ kg = 931.5 MeV/$c^2$.

$$\begin{aligned}\Delta m = M_\text{U} - (M_\text{Rb} + M_\text{Cs} + 3m_\text{n}) &= 236.045563\ \text{u} \\ &-(89.914811\ \text{u} + 142.927220\ \text{u} \\ &+ (3)(1.008665)\ \text{u}) \\ &= 0.177537\ \text{u} = 2.9471 \times 10^{-28}\ \text{kg}\end{aligned}$$

Therefore, the reaction products have a combined mass that is about $3.0 \times 10^{-28}$ kg less than the initial uranium mass.

(c) The energy given off per fission event is just $\Delta mc^2$. This is most easily calculated if $\Delta m$ is first converted to mass units of MeV/$c^2$. Because 1 u = 931.5 MeV/$c^2$,

$$\begin{aligned}\Delta m &= (0.177537\ \text{u})(931.5\ \text{MeV}/c^2) \\ &= 165.4\ \text{MeV}/c^2\end{aligned}$$

$$\begin{aligned}Q = \Delta mc^2 &= 165.4\,\frac{\text{MeV}}{c^2}\,c^2 = 165.4\ \text{MeV} \\ &= -165.4\ \text{MeV}\end{aligned}$$

(d) To find the energy released by the fission of 1 kg of uranium we need to calculate the number of nuclei, $N$, contained in 1 kg of $^{236}$U.

$$N = \frac{(6.02 \times 10^{23} \text{ nuclei/mol})}{(236 \text{ g/mol})}(1000 \text{ g})$$
$$= 2.55 \times 10^{24} \text{ nuclei}$$

The total energy produced, $E$, is

$$E = (\text{efficiency})NQ$$
$$= (0.40)(2.55 \times 10^{24} \text{ nuclei})(165 \text{ MeV/nucleus})$$
$$= 1.68 \times 10^{26} \text{ MeV}$$
$$= (1.68 \times 10^{26} \text{ MeV})(4.45 \times 10^{-20} \text{ kWh/MeV})$$
$$= 7.48 \times 10^{6} \text{ kWh}$$

***Exercise 2*** How long will this amount of energy keep a 100-W lightbulb burning?

***Answer*** ≈ 8500 years.

We have considered the simplest case showing the conversion of mass to energy and the release of this nuclear energy: the decay of a heavy unstable element into several lighter elements. However, the most common case is the one in which the mass of a composite particle is *less than* the sum of the particle masses composing it. By examining Appendix B, you can see that the mass of any nucleus is less than the sum of its component neutrons and protons by an amount $\Delta m$. This occurs because the nuclei are stable, *bound* systems of neutrons and protons (bound by strong attractive nuclear forces), and in order to disassociate them into separate nucleons an amount of energy $\Delta mc^2$ must be supplied to the nucleus. This energy or work required to pull a bound system apart, leaving its component parts free of attractive forces and at rest, is called the binding energy, *BE*. Thus, we describe the mass and energy of a bound system by the equation

$$Mc^2 + BE = \sum_{i=1}^{n} m_i c^2 \tag{2.17}$$

where $M$ is the bound system mass, the $m_i$'s are the free component particle masses, and $n$ is the number of component particles. Two general comments are in order about Equation 2.17. First, it applies quite generally to any type of system bound by attractive forces, whether gravitational, electrical (chemical), or nuclear. For example, the mass of a water molecule is less than the combined mass of two free hydrogen atoms and a free oxygen atom, although the mass difference cannot be directly measured in this case. (The mass difference can be measured in the nuclear case because the forces and the binding energy are so much greater.) Second, Equation 2.17 shows the possibility of liberating huge quantities of energy, *BE*, if one reads the equation from right to left; that is, one collides nuclear particles with a small but sufficient amount of kinetic energy to overcome proton repulsion and fuse the particles into new elements with less mass. Such a process is called *fusion*, one example of which is a reaction in which two deuterium nuclei combine to form a helium nucleus, releasing 23.9 MeV per fusion. (See Chapter 14 for more on fusion processes.) We can write this reaction schematically as follows: **Fusion**

$${}^{2}_{1}\text{H} + {}^{2}_{1}\text{H} \longrightarrow {}^{4}_{2}\text{He} + 23.9 \text{ MeV}$$

**EXAMPLE 2.8**

(a) How much lighter is a molecule of water than two hydrogen atoms and an oxygen atom? The binding energy of water is about 3 eV. (b) Find the fractional loss of mass per gram of water formed. (c) Find the total energy released (mainly as heat and light) when 1 gram of water is formed.

**Solution** (a) Equation 2.17 may be solved for the mass difference as follows:

$$\Delta m = (m_H + m_H + m_O) - M_{H_2O} = \frac{BE}{c^2} = \frac{3\text{ eV}}{c^2}$$

$$= \frac{(3.0\text{ eV})(1.6 \times 10^{-19}\text{ J/eV})}{(3.0 \times 10^8\text{ m/s})^2} = 5.3 \times 10^{-36}\text{ kg}$$

(b) To find the fractional loss of mass per molecule we divide $\Delta m$ by the mass of a water molecule, $M_{H_2O} = 18\text{u} = 3.0 \times 10^{-26}$ kg:

$$\frac{\Delta m}{M_{H_2O}} = \frac{5.3 \times 10^{-36}\text{ kg}}{3.0 \times 10^{-26}\text{ kg}} = 1.8 \times 10^{-10}$$

Because the fractional loss of mass per molecule is the same as the fractional loss per gram of water formed, $1.8 \times 10^{-10}$ g of mass would be lost for each gram of water formed. This is much too small a mass to be measured directly, and this calculation shows that nonconservation of mass does not generally show up as a measurable effect in chemical reactions.

(c) The energy released when 1 gram of $H_2O$ is formed is simply the change in mass when 1 gram of water is formed times $c^2$:

$$E = \Delta mc^2 = (1.8 \times 10^{-13}\text{ kg})(3.0 \times 10^8\text{ m/s})^2 \approx 16\text{ kJ}$$

This energy change, as opposed to the decrease in mass, is easily measured, providing another case similar to Example 2.5 in which mass changes are minute but energy changes, amplified by a factor of $c^2$, are easily measured.

## 2.4 CONSERVATION OF RELATIVISTIC MOMENTUM AND ENERGY

So far we have considered only cases of the conservation of mass–energy. By far, however, the most common and strongest confirmation of relativity theory comes from the daily application of relativistic momentum and energy conservation to elementary particle interactions. Often the measurement of momentum (from the path curvature in a magnetic field—see Example 2.2) and kinetic energy (from the distance a particle travels in a known substance before coming to rest) are enough when combined with conservation of momentum and mass–energy to determine fundamental particle properties of mass, charge, and mean lifetime.

**EXAMPLE 2.9 Measuring the Mass of the $\pi^+$ Meson**

The $\pi^+$ meson (also called the *pion*) is a subatomic particle responsible for the strong nuclear force between protons and neutrons. It is observed to decay at rest into a $\mu^+$ meson (muon) and a neutrino,[2] denoted $v$. Because the neutrino has no charge and little mass (talk about elusive!), it leaves no track in a bubble chamber. (A bubble chamber is a large chamber filled with liquid hydrogen that shows the tracks of charged particles as a series of tiny bubbles.) However, the track of the charged muon is visible as it loses kinetic energy and comes to rest (Fig. 2.3). If the mass of the muon is known to be 106 MeV/$c^2$, and the kinetic energy, $K$, of the muon is measured to be 4.6 MeV from its track length, find the mass of the $\pi^+$.

**Solution** The decay equation is $\pi^+ \rightarrow \mu^+ + v$. Conserving energy gives

$$E_\pi = E_u + E_v$$

[2]Neutrino, from the Italian, means "little tiny neutral one." Following this practice, neutron should probably be neutrone (pronounced noo-trōn-eh) or "great big neutral one."

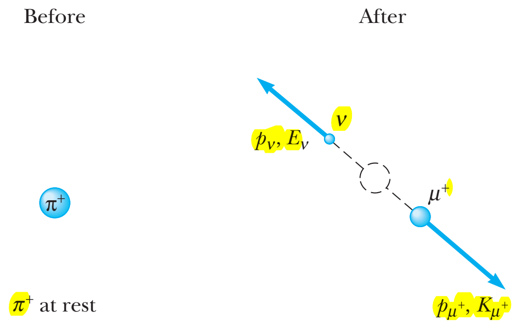


**Figure 2.3** (Example 2.9) Decay of the pion at rest into a neutrino and a muon.

Because the pion is at rest when it decays, and the neutrino has negligible mass,

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu^2 c^2)} + p_\nu c \qquad (2.18)$$

Conserving momentum in the decay yields $p_\mu = p_\nu$. Substituting the muon momentum for the neutrino momentum in Equation 2.18 gives the following expression for the rest energy of the pion in terms of the muon's mass and momentum:

$$m_\pi c^2 = \sqrt{(m_\mu c^2)^2 + (p_\mu^2 c^2)} + p_\mu c \qquad (2.19)$$

Finally, to obtain $p_\mu$ from the measured value of the muon's kinetic energy, $K_\mu$, we start with Equation 2.11, $E_\mu^2 = p_\mu^2 c^2 + (m_\mu c^2)^2$, and solve it for $p_\mu^2 c^2$:

$$p_\mu^2 c^2 = E_\mu^2 - (m_\mu c^2)^2 = (K_\mu + m_\mu c^2)^2 - (m_\mu c^2)^2$$
$$= K_\mu^2 + 2K_\mu m_\mu c^2$$

Substituting this expression for $p_\mu^2 c^2$ into Equation 2.19 yields the desired expression for the pion mass in terms of the muon's mass and kinetic energy:

$$m_\pi c^2 = \sqrt{(m_\mu^2 c^4 + K_\mu^2 + 2K_\mu m_\mu c^2} + \sqrt{K_\mu^2 + 2K_\mu m_\mu c^2} \qquad (2.20)$$

Finally, substituting $m_\mu c^2 = 106$ MeV and $K_\mu = 4.6$ MeV into Equation 2.20 gives

$$m_\pi c^2 = 111 \text{ MeV} + 31 \text{ MeV} \approx 1.4 \times 10^2 \text{ MeV}$$

Thus, the mass of the pion is

$$m_\pi = 140 \text{ MeV}/c^2$$

This result shows why this particle is called a meson; it has an intermediate mass (from the Greek word *mesos* meaning "middle") between the light electron (0.511 MeV/$c^2$) and the heavy proton (938 MeV/$c^2$).

## 2.5 GENERAL RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that represents a resistance to acceleration. To designate these two attributes, we use the subscripts $g$ and $i$ and write

$$\text{Gravitational property:} \qquad F_g = G\frac{m_g m_g'}{r^2}$$

$$\text{Inertial property:} \qquad \sum F = m_i a$$

The value for the gravitational constant $G$ was chosen to make the magnitudes of $m_g$ and $m_i$ numerically equal. Regardless of how $G$ is chosen, however, the strict proportionality of $m_g$ and $m_i$ has been established experimentally to an extremely high degree: a few parts in $10^{12}$. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses, and the resistance of a single mass to being accelerated. This question, which puzzled Newton and many other physicists over the years, was answered by Einstein in 1916 when he published his theory of gravitation, known as the *general theory of relativity*. Because it is a mathematically complex theory, we offer merely a hint of its elegance and insight.