LASER GENERATED THERMOELASTIC WAVES IN AN ANISOTROPIC INFINITE PLATE

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ABSTRACT. An analysis of the propagation of thermoelastic waves in a homogenous, anisotropic, thermally conducting plate has been presented in the context of the generalized (L-S) theory of thermoelasticity. Three different methods are used in this analysis: two of them are exact and the third is a semi-analytic finite element method (SAFE). In our exact analysis, two different approaches are used. The first one, which is applicable to transversely isotropic plate, is based on introducing displacement potential functions, whereas in the second approach, which is applicable to any triclinic material, we rewrite the governing equations and boundary conditions in a matrix form. Finally, in the SAFE method, the plate is discretized along its thickness using N parallel, homogeneous layers, which are perfectly bonded together. Frequency spectrums are obtained using the three methods and are shown to agree well with each other. Numerical calculations have been presented for a silicon nitride (Si_3N_4) plate. However, the methods can be used for other materials as well.

INTRODUCTION

Most materials undergo appreciable changes of volume when subjected to variations of the temperature. If thermal expansions or contractions are not freely admitted, temperature variations give rise to thermal stresses. Conversely, a change of volume is accompanied by a change of the temperature. When a given element is compressed or dilated, these volume changes are accompanied by heating and cooling, respectively. The study of the influence of the temperature of an elastic solid upon the distribution of stress and strain, and of the inverse effect of the deformation upon the temperature distribution is the subject of thermoelasticity [1].

Little work has been reported on thermoelastic waves in anisotropic plates. The main focus of our work will be focused on the laser-generated waves in thermoelastic anisotropic plate. As mentioned above, the technique of laser-generated waves has potential application to noncontact and nondestructive evaluation and characterization of sheet materials in industry. It was demonstrated that the thickness of and moduli of thin plates can be measured experimentally using laser-generated waves [2]. In the first part of our work, dispersion relations for thermoelastic anisotropic plate will be studied. Then, transient responses of a plate heated by a laser pulse will be analyzed. The study is carried out in the context of Lord-Shulman (L-S) theory of thermoelasticity using single relaxation time.

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THEORETICAL FORMULATIONS

We consider an infinite homogeneous transversely isotropic thermally conducting elastic plate at a uniform temperature T_0 in the undisturbed state having a thickness H see Figure (1). The motion is assumed to take place in three dimensions (x, y, z). The displacements in the x, y, and z directions are u, v and w, respectively.

Exact Analysis

In the absence of body forces and internal heat source, the generalized (L-S) thermoelasticity governing equations are:

$$c_{11}u_{,xx} + c_{12}v_{,xy} + c_{12}w_{,xz} + c_{55}\left(u_{,yy} + v_{,xy}\right) + c_{55}\left(u_{,zz} + w_{,xz}\right) - \beta_{xx}T_{,x} = \rho\ddot{u} \tag{1}$$

$$c_{55}\left(u_{,xy} + v_{,xx}\right) + c_{12}u_{,xy} + c_{22}v_{,yy} + c_{23}w_{,yz} + c_{44}\left(v_{,zz} + w_{,yz}\right) - \beta_{yy}T_{,y} = \rho\ddot{v} \tag{2}$$

$$c_{55}\left(u_{,xz} + w_{,xx}\right) + c_{44}\left(v_{,yz} + w_{,yy}\right) + c_{12}u_{,xz} + c_{23}v_{,yz} + c_{22}w_{,zz} - \beta_{yy}T_{,z} = \rho\ddot{w} \tag{3}$$

$$\begin{aligned} K_{xx}T_{,xx} + K_{yy}T_{,yy} + K_{yy}T_{,zz} - \rho C_E\left(\dot{T} + \tau_0\ddot{T}\right) &= T_0\left[\beta_{xx}\left(\dot{u}_{,x} + \tau_0\ddot{u}_{,x}\right)\right. \\ &\left.+\beta_{yy}\left(\dot{v}_{,y} + \tau_0\ddot{v}_{,y}\right) + \beta_{yy}\left(\dot{w}_{,z} + \tau_0\ddot{w}_{,z}\right)\right] \end{aligned} \tag{4}$$

Note that Eq. (4) reduces to the classical coupled thermoelasticity equation for heat conduction if $\tau_0 = 0$.

First Approach

The displacement can be written in terms of three potential functions as in Buchwald [3] in the form

$$u = \frac{\partial\Theta}{\partial x} \tag{5}$$

$$v = \frac{\partial\Phi}{\partial y} + \frac{\partial\Psi}{\partial z} \tag{6}$$

$$w = \frac{\partial\Phi}{\partial z} - \frac{\partial\Psi}{\partial y} \tag{7}$$

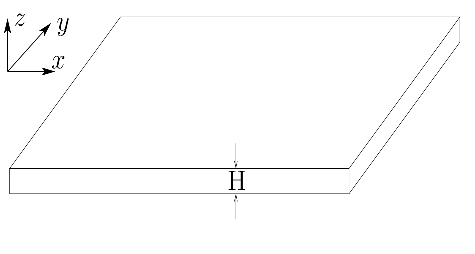

FIGURE 1. Geometry of the problem.

Eliminating the displacements from the equations of motion produces

$$\left[c_2\frac{\partial^2}{\partial x^2}+c_5\nabla^2-\frac{\partial^2}{\partial t^2}\right]\nabla^2\Psi=0 \tag{8}$$

$$\delta\frac{\partial^2}{\partial x^2}\nabla^2\Phi+\left[c_2\nabla^2+\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right]\frac{\partial^2\Theta}{\partial x^2}-\frac{\partial^2 T}{\partial x^2}=0 \tag{9}$$

$$\delta\frac{\partial^2}{\partial x^2}\nabla^2\Theta+\left[c_3\nabla^2+c_2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right]\nabla^2\Phi-\overline{\beta}\nabla^2 T=0 \tag{10}$$

$$\varepsilon\left[\frac{\partial^2}{\partial x^2}\left(\dot{\Theta}+\tau_0\ddot{\Theta}\right)+\overline{\beta}\nabla^2\left(\dot{\Phi}+\tau_0\ddot{\Phi}\right)\right]-\frac{\partial^2 T}{\partial x^2}-\overline{K}\nabla^2 T+\left(\dot{T}+\tau_0\ddot{T}\right)=0 \tag{11}$$

The above equations are in nondimensional form with

$$c_1=\frac{c_{12}}{c_{11}},\quad c_2=\frac{c_{55}}{c_{11}},\quad c_3=\frac{c_{22}}{c_{11}},\quad c_4=\frac{c_{23}}{c_{11}},$$
$$c_5=\frac{c_{44}}{c_{11}},\quad \delta=c_1+c_2,\quad \overline{\beta}=\frac{\beta_{yy}}{\beta_{xx}},\quad \overline{K}=\frac{K_{yy}}{K_{xx}},$$
$$\tau_0^*=\frac{v_x^2}{k_x}\tau_0,\quad \varepsilon=\frac{\beta_{xx}^2 T_0}{\rho^2 C_E v_x^2}.\quad \nabla^2=\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}$$

where $v_x=\sqrt{c_{11}/\rho}$ is the velocity of compressional waves, $k_x=K_{xx}/\rho C_E$ is the thermal diffusivity in the x-direction, and ε is the thermoelastic coupling constant. Generally, this constant is small for most materials [4].

It can be noted that the first equation is decoupled from others. Since we are interested in propagating waves in the plane of x,y, potentials and temperature are assumed to be harmonic along these directions. By substitution of assumed harmonic potentials into Eqs. (8-11), and solving for the potentials, we get the displacement and temperature and hence, eventually stresses and temperature gradient.

The boundary conditions are that stresses and temperature gradient on the surfaces of the plate should vanish. Hence, we demand that,

$$T_{,z}=\sigma_{zz}=\sigma_{zx}=\sigma_{zy}=\mathbf{0} \tag{12}$$

Using boundary conditions (12) in the resulting stresses and temperature gradient yields eight equations involving a generalized column vector. In order for the eight boundary conditions to be satisfied simultaneously, the determinant of the resulting algebraic system of equations must vanish. This gives an equation for the frequency of the guided wave for a given wavenumber which is the dispersion relation.

Second Approach

For an infinite plate one can apply Fourier transformation in space and time domains to the governing equations to get the following eigenvalue problem,

$$[\mathrm{A}]\,S_{,z}=[\mathrm{B}]\,S, \tag{13}$$

where $S_{,z}=dS/dz$ and $S\left(z\right)=\begin{bmatrix}\hat{u} & \hat{v} & \hat{w} & \hat{T} & \hat{\sigma}_{zx} & \hat{\sigma}_{yz} & \hat{\sigma}_{zz} & \hat{T}_{,z}\end{bmatrix}^T$. The general solution can be obtained by determining the eigenvalues and the eigenvectors of Eq. (13).Here S is the displacement-temperature-traction vector and matrices A and B are defined as,

$$\mathrm{A}=\begin{bmatrix} . & . & . & -1 & . & . & . & . \\ . & . & -c_3 & . & . & . & . & . \\ . & -c_5 & . & . & . & . & . & . \\ -c_2 & . & . & . & . & . & . & . \\ . & . & ic_1k & . & 1 & . & . & . \\ . & . & ic_4\xi & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & \overline{\beta}\varepsilon\tau & . & . & . & . & \overline{K} \end{bmatrix} \tag{14}$$

$$\mathrm{B}=\begin{bmatrix} . & . & . & . & . & . & . & -1 \\ ic_1k & ic_4\xi & . & -\overline{\beta} & . & . & -1 & . \\ . & . & ic_5\xi & . & . & -1 & . & . \\ . & . & ic_2k & . & -1 & . & . & . \\ k^2+c_2\xi^2-\omega^2 & k\xi\left(c_1+c_2\right) & . & ik & . & . & . & . \\ k\xi\left(c_1+c_2\right) & c_2k^2+c_3\xi^2-\omega^2 & . & i\overline{\beta}\xi & . & . & . & . \\ . & . & -\omega^2 & . & -ik & -i\xi & . & . \\ -i\varepsilon k\tau & -i\overline{\beta}\varepsilon\tau & . & k^2+\overline{K}\xi^2-\tau & . & . & . & . \end{bmatrix} \tag{15}$$

where $\tau=i\omega+\tau_0\omega^2$.

The general solution to Eq. (13) can be written as

$$S\left(z\right)=\left[\mathrm{QE}\right]\left\{C\right\}. \tag{16}$$

where Q is the resulting eigenvectors matrix from Eq. (13), C are generalized coefficients, and E is a diagonal matrix,

$$\mathrm{E}=Diag\left[e^{is_1z},e^{is_2z},e^{is_3z},e^{is_4z},e^{is_1(H-z)},e^{is_2(H-z)},e^{is_3(H-z)},e^{is_4(H-z)}\right] \tag{17}$$

where $is_p,(p=1,2,3,4)$ are the resulting eigenvalues of the characteristics equation (13), with $Im\left(s_p\right)\geq 0$. The first four elements of Eq. (17) represent the wave propagation along the positive $z-$axis, while the last four elements represent the wave propagation along the negative $z-$axis.

The boundary conditions are that tractions and temperature gradient in the z-direction on the surfaces of the plate should vanish. Applying these boundary conditions yields the dispersion relation for the infinite plate.

Finite Element Method

The plate is divided into N parallel, homogeneous, and anisotropic layers, which are perfectly bonded together. A global rectangular coordinate system (X,Y,Z) is adopted such that X and Y axes lie in the mid-plane of the plate, and $Z-$axis parallel to the thickness direction of the plate. To analyze the guided wave propagation in such an infinite plate, we discretize the thickness of the plate using three-noded bar elements, each of which has associated with it a local coordinate axes (x,y,z), which are parallel to the global coordinate axes.

Two sets of conventional FEM shape functions are introduced to approximate the displacement, and temperature fields on the element level.

$$\mathbf{u}\left(x,y,z,t\right) = N_1^e\left(z\right)\mathbf{u}^e\left(x,y,t\right) \tag{18}$$

$$\mathbf{T}\left(x,y,z,t\right) = N_2^e\left(z\right)\mathbf{T}^e\left(x,y,t\right) \tag{19}$$

where Nodal displacements and temperatures are stored in the two vectors $\mathbf{u}^e$ and θ^e, respectively. Using generalized linear thermoelasticity the strain tensor and temperature vector is derived from the kinematic equations,

$$\epsilon = \mathbf{D}_1\mathbf{u}^e_{,x} + \mathbf{D}_2\mathbf{u}^e_{,y} + \mathbf{D}_3\mathbf{u}^e \quad (20)$$

$$\mathbf{T}' = \mathbf{B}_1\mathbf{T}^e_{,x} + \mathbf{B}_2\mathbf{T}^e_{,y} + \mathbf{B}_3\mathbf{T}^e \quad (21)$$

where $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{D}_1, \mathbf{D}_2$, and $\mathbf{D}_3$ are matrices containing the shape functions and their derivatives.

Considering the body force $\mathbf{f}^e$, the variational form of the thermoelasticity can be written in the following form

$$\int_{t_0}^{t_1}\int_V \left(\delta\epsilon^T\sigma - \delta\mathbf{T}'^T\mathbf{K}\mathbf{T}' - \delta\mathbf{T}'^T\left(q_i + \tau_0\dot{q}_i\right)\right)\mathrm{d}V\mathrm{d}t = \int_{t_0}^{t_1}\int_V \delta\mathbf{u}^T\left(\mathbf{f} - \rho\ddot{\mathbf{u}}\right)\mathrm{d}V\mathrm{d}t \quad (22)$$

Substituting the assumed displacement and temperature into the variational form Eq. (22) eventually leads to the following:

$$\mathbf{H}_1\ddot{\mathbf{V}} + \mathbf{H}_2\ddot{\mathbf{V}}_{,x} + \mathbf{H}_3\ddot{\mathbf{V}}_{,y} + \mathbf{H}_4\dot{\mathbf{V}}_{,x} + \mathbf{H}_5\dot{\mathbf{V}}_{,y} + \mathbf{H}_6\dot{\mathbf{V}} + \mathbf{H}_7\mathbf{V}_{,xx} + \mathbf{H}_8\mathbf{V}_{,xy} + \mathbf{H}_9\mathbf{V}_{,yy} + \mathbf{H}_{10}\mathbf{V}_{,x} + \mathbf{H}_{11}\mathbf{V}_{,y} + \mathbf{H}_{12}\mathbf{V} = \mathbf{0} \quad (23)$$

where the matrices $\mathbf{H}_i$ are global matrices and $\mathbf{V}$ is the column vector of assembled nodal displacements and temperatures.

For a wave propagating in the XY-plane, we take the Fourier transform with respect to x, y and t to get:

$$\left[\omega^2\mathbf{H}_1 + i\omega^2k_x\mathbf{H}_2 + i\omega^2k_y\mathbf{H}_3 - \omega k_x\mathbf{H}_4 - \omega k_y\mathbf{H}_5 + i\omega\mathbf{H}_6 + k_x^2\mathbf{H}_7 + k_xk_y\mathbf{H}_8 + k_y^2\mathbf{H}_9 - ik_x\mathbf{H}_{10} - ik_y\mathbf{H}_{11} - \mathbf{H}_{12}\right]\hat{\mathbf{V}} = \mathbf{0} \quad (24)$$

where ω is the circular frequency and k_x and k_y are the wavenumbers in the X and Y directions. For wave propagating in arbitrary direction in the xy plane making an angle θ with X-axis, i.e., $k_x = k\cos\theta$, and $k_y = k\sin\theta$, Eq. (24) is written as,

$$\left(k^2\overline{\mathbf{M}} + k\overline{\mathbf{C}} + \overline{\mathbf{K}}\right)\hat{\mathbf{V}} = 0 \quad (25)$$

where $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$, and $\overline{\mathbf{K}}$ are functions of H_i, ω and the propagation angle.

Solving the eigenvalue problem represented by Eq.(25) will determine the dispersion for guided thermoelastic waves in infinite plates.

Transient Response to Laser Pulse

Solving the homogeneous equation (Eq. 25) will yield the dispersion relation for elastic guided waves in infinite plates. In order to get the transient response to a laser pulse, one should rewrite Eq. (25) including the forcing term which is Q-switched laser pulse,

$$[\mathbf{A}]\boldsymbol{V} = k[\mathbf{B}]\boldsymbol{V} + \boldsymbol{P}, \quad (26)$$

where,

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathrm{K}_3 - \omega^2\mathrm{M} \\ \mathrm{K}_3 - \omega^2\mathrm{M} & \mathrm{K}_2 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathrm{K}_3 - \omega^2\mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{K}_1 \end{bmatrix},$$

and

$$\boldsymbol{V} = \left[\ \hat{\boldsymbol{U}} \quad k\hat{\boldsymbol{U}}\ \right]^T, \qquad \boldsymbol{P} = \left[\ \hat{\boldsymbol{F}} \quad \boldsymbol{0}\ \right]^T.$$

Using modal expansion, one can expand the eigenvector $\boldsymbol{V}$ in terms of the right-handed eigenvector $\boldsymbol{\phi}_m$ as,

$$\boldsymbol{V} = \sum_{m=1}^{2M} \bar{V}_m \boldsymbol{\phi}_m. \tag{27}$$

For the sake of brevity, some steps are omitted to get finally,

$$\boldsymbol{V} = \sum_{m=1}^{2M} \frac{\boldsymbol{\psi}_m^T \boldsymbol{P}}{(k_m - k)B_m} \boldsymbol{\phi}_m. \tag{28}$$

In terms of the lower and upper eigenvectors, the derived displacement vector, in the wavenumber-frequency domain, is simplified as,

$$\hat{\boldsymbol{U}}(k, \omega) = \sum_{m=1}^{2M} \frac{k_m \boldsymbol{\psi}_{mu}^T \hat{\boldsymbol{F}}}{(k_m - k)B_m} \boldsymbol{\phi}_{mu}. \tag{29}$$

We evaluate the displacement in the space-frequency domain by using the inverse Fourier transform as,

$$\tilde{\boldsymbol{U}}(\xi; \omega) = \frac{1}{2\pi} \sum_{m=1}^{2M} \int_{-\infty}^{\infty} \frac{k_m \boldsymbol{\psi}_{mu}^T \hat{\boldsymbol{F}}}{(k_m - k)B_m} \boldsymbol{\phi}_{mu} e^{ik\xi} dk, \tag{30}$$

where, ξ is the wave propagation direction. Numerical evaluation of Eq. (30), having singularities at the poles ($k = k_m$), is carried out using Cauchy's residue theorem,

$$\tilde{\boldsymbol{U}}(\xi; \omega) = -i \sum_{m=1}^{M} \frac{k_m \boldsymbol{\psi}_{mu}^T \hat{\boldsymbol{F}}}{B_m} \boldsymbol{\phi}_{mu} e^{ik_m \xi}. \tag{31}$$

To obtain the time-domain response, inverse transform of Eq. (31) needs to be carried out numerically. The time domain displacement $\boldsymbol{U}(\xi, t)$ can be expressed as

$$\boldsymbol{U}(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\boldsymbol{U}}(\xi; \omega) F(\omega) e^{i\omega t} d\omega. \tag{32}$$

RESULTS AND DISCUSSION

With the view of illustrating the numerical results obtained by methods presented in the preceding sections, the material chosen for the plate is silicon nitride (Si_3N_4).

Numerical results are presented in the form of 3-D view of frequency spectrum. These are obtained by keeping ω real and letting k to be complex. In order to find the solutions of the characteristic equation (12) of the exact analysis, Muller's method is used to solve it as an analytic complex function. The relations between the frequency and the wavenumber expressed by the characteristic equation yield an infinite number of branches for an infinite number of elastic and thermal modes. Figure 2 shows a 3-D

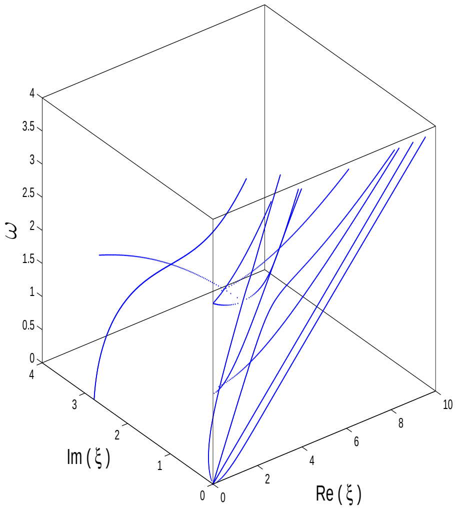

FIGURE 2. Frequency spectrum (exact).

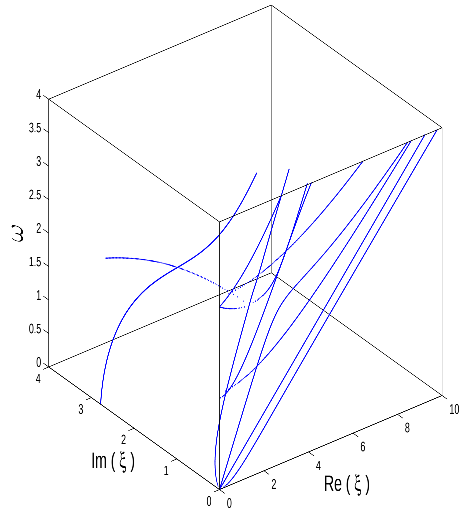

FIGURE 3. Frequency spectrum (SAFE).

view of the frequency spectrum using the exact approach. Elastic modes resemble those of isothermal case. Similar to the isothermal case, a complex branch is seen (Figure 2) originating from the minimum point on the second longitudinal mode. Moreover, note that 0th-order elastic modes propagate at all frequencies, but the higher order modes have cutoff frequencies below which they are evanescent. Since the propagation direction is along a principle direction, it is seen that the SH modes are uncoupled from the other two modes. This is evidenced by the intersection of the SH wave curves with those for the S and A waves.

The first thermal mode shows a similar behavior as the lowest elastic modes; however, it shows very high attenuation compared to elastic modes. Other thermal modes originate with higher imaginary values of wavenumbers and eventually approach the first thermal mode.

Frequency spectrum was also computed by the semi-analytic finite element method (SAFE) and graphically shown in Figure 3. Excellent agreement between FEM and analytic solution results is observed by comparing Figure 3 with Figure 2. Convergent results of FEM analysis can be attained by relatively small number of elements (10 elements) indicating that the FEM is a powerful and efficient technique for analyzing thermoelastic problems. Another advantage is the ease with which layered plates can be considered.

Figures 4 and 5 show the transient response to the laser pulse at two different locations. At early time, the mode S_0 arrives at the observation point, followed later by the dispersive A_0 mode.

CONCLUSIONS

Propagation of guided thermoelastic waves in a homogeneous, transversely isotropic, thermally conducting plate was investigated within the framework of the generalized theory of thermoelasticity proposed by Lord and Shulman. This theory includes a thermal relaxation time in the heat conduction equation in order to model the finite speed of the thermal wave. Three different methods were used to model the guided

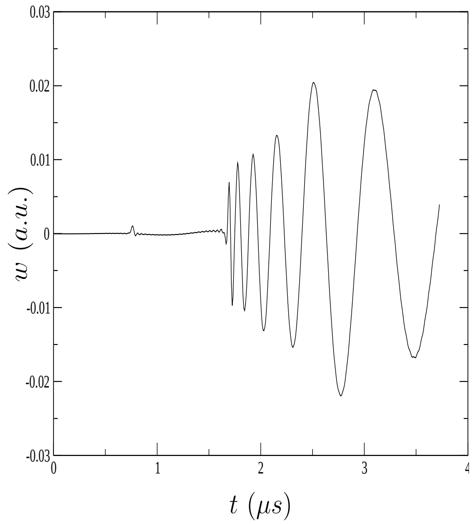

FIGURE 4. Transient response at $x = 100H$.

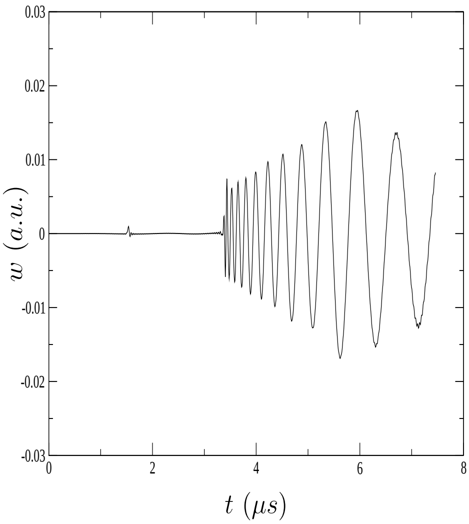

FIGURE 5. Transient response at $x = 200H$.

wave dispersion. These include: an exact analysis incorporating two different solution approaches and a semi-analytic finite element (SAFE) method. The results obtained by these methods were found to agree very well.

The results show that both elastic and thermal modes are attenuated, the thermal mode exhibit much larger attenuation than the elastic modes. The attenuation of the former is quite small. The results agree with previous observations by Hawwa and Nayfeh [5].

Finally, the transient responses due to a laser pulse were numerically calculated. The results show that the lowest antisymmetric mode dominates, demonstrating clearly it dispersive nature.

While this paper dealt with the modal dispersion of guided waves (exact and FEM), and the transient response of a plate due to a laser pulse using FEM approach, an exact analysis of transient response will be reported in a subsequent paper.

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