Chapter 6

Electrical Systems and Electromechanical Systems

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6.1 INTRODUCTION

The majority of engineering systems now have at least one electrical subsystem. This may be a power supply, sensor, motor, controller, or an acoustic device such as a speaker. So an understanding of electrical systems is essential to understanding the behavior of many systems.

6.2 ELECTRICAL ELEMENTS

Current and Voltage *Current* and *voltage* are the primary variables used to describe a circuit's behavior. *Current* is the flow of electrons. It is the time rate of change of electrons passing through a defined area, such as the cross-section of a wire. Because electrons are negatively charged, the positive direction of current flow is opposite to that of electron flow. The mathematical description of the relationship between the number of electrons (called charge q) and current i is

$$i = \frac{dq}{dt} \qquad \text{or} \qquad q(t) = \int i \, \mathrm{d}t$$

The unit of charge is the *coulomb* (C) and the unit of current is ampere (A), which is one coulomb per second.

Energy is required to move a charge between two points in a circuit. The work per unit charge required to do this is called *voltage.* The unit of voltage is *volt* (V), which is defined to be joule per coulomb. The voltage difference between two points in a circuit is a measure of the energy required to move charge from one point to the other.

Active and Passive Elements. Circuit elements may be classified as *active* or *passive*.

- ***Passive Element***: an element that contains no energy sources (i.e. the element needs power from another source to operate); these include resistors, capacitors and inductors
- ***Active Element***: an element that acts as an energy source; these include batteries, generators, solar cells, and op-amps.

Current Source and Voltage Source A *voltage source* is a device that causes a specified voltage to exist between two points in a circuit. The voltage may be time varying or time invariant (for a sufficiently long time). Figure 6-1(a) is a schematic diagram of a voltage source. Figure 6-1(b) shows a voltage source that has a constant value for an indefinite time. Often the voltage is denoted by E or V . A battery is an example of this type of voltage.

A *current source* causes a specified current to flow through a wire containing this source. Figure 6-1(c) is a schematic diagram of a current source

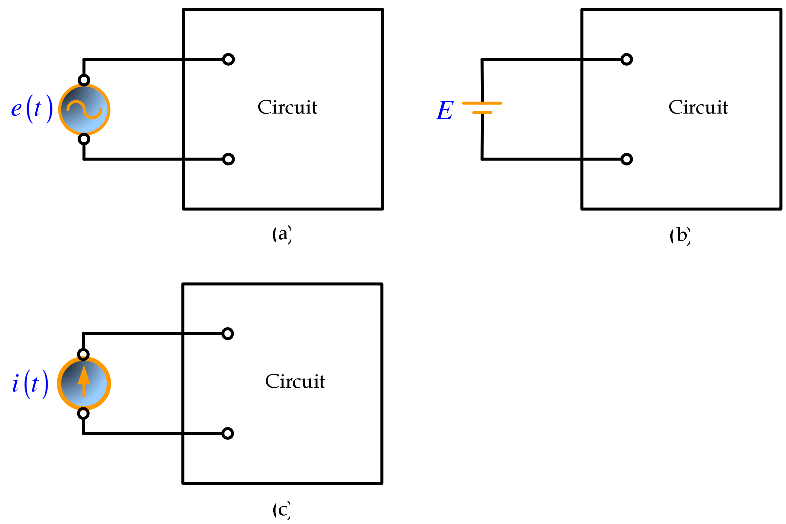


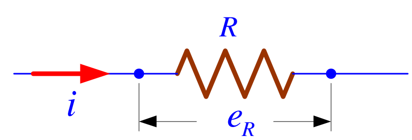
Figure 6.1 (a) Voltage source; (b) constant voltage source; (c) current source

Resistance elements. The resistance R of a linear resistor is given by

$$R = \frac{e_R}{i}$$

where e_R is the voltage across the resistor and i is the current through the resistor. The unit of resistance is the ohm (Ω), where

$$\text{ohm}=\frac{\text{volt}}{\text{ampere}}$$

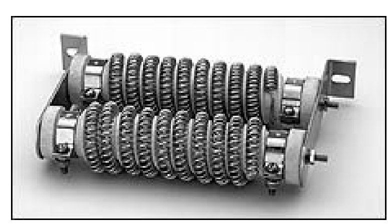


Resistances do not store electric energy in any form, but instead dissipate it as heat. Real resistors may not be linear and may also exhibit some capacitance and inductance effects.

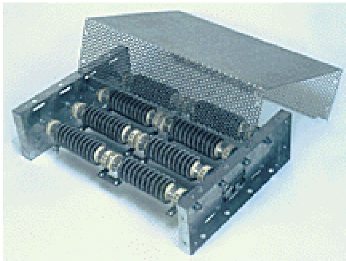
PRACTICAL EXAMPLES: Pictures of various types of real-world resistors are found below.

Wirewound Resistors

Wirewound Resistors in Parallel



Wirewound Resistors in Series and in Parallel

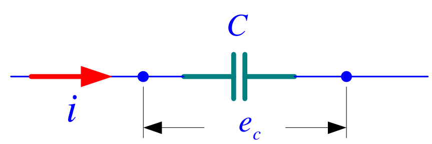


Capacitance Elements. Two conductors separated by a nonconducting medium form a capacitor, so two metallic plates separated by a very thin dielectric material form a capacitor. The capacitance C is a measure of the quantity of charge that can be stored for a given voltage across the plates. The capacitance C of a capacitor can thus be given by

$$C = \frac{q}{e_c}$$

where q is the quantity of charge stored and e_c is the voltage across the capacitor. The unit of capacitance is the farad (F), where

$$\mathrm{farad} = \frac{\text{ampere-second}}{\text{volt}} = \frac{\text{coulomb}}{\text{volt}}$$



Notice that, since $i = \mathrm{d}q/\mathrm{d}t$ and $e_c = q/C$, we have

$$i = C\frac{\mathrm{d}e_c}{\mathrm{d}t}$$

or

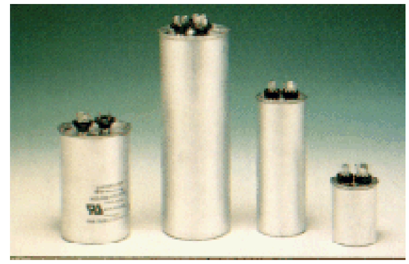
$$\mathrm{d}e_c = \frac{1}{C}i\,\mathrm{d}t$$

Therefore,

$$e_c = \frac{1}{C}\int_0^t i\ \mathrm{d}t + e_c(0)$$

Although a pure capacitor stores energy and can release all of it, real capacitors exhibit various losses. These energy losses are indicated by a *power factor*, which is the ratio of energy lost per cycle of ac voltage to the energy stored per cycle. Thus, a small-valued power factor is desirable.

PRACTICAL EXAMPLES: Pictures of various types of real-world capacitors are found below.



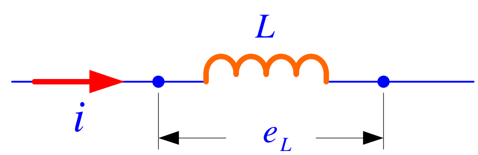
Inductance Elements. If a circuit lies in a time varying magnetic field, an electromotive force is induced in the circuit. The inductive effects can be classified as *self inductance* and *mutual inductance*.

Self inductance, or simply inductance, L is the proportionality constant between the induced voltage e_L volts and the rate of change of current (or change in current per second) $\mathrm{d}i/\mathrm{d}t$ amperes per second; that is,

$$L = \frac{e_L}{\mathrm{d}i/\mathrm{d}t}$$

The unit of inductance is the henry (H). An electrical circuit has an inductance of 1 henry when a rate of change of 1 ampere per second will induce an emf of 1 volt:

$$\text{henry} = \frac{\text{volt}}{\text{ampere/second}} = \frac{\text{weber}}{\text{ampere}}$$



The voltage e_L across the inductor L is given by

$$e_L = L\frac{\mathrm{d}i_L}{\mathrm{d}t}$$

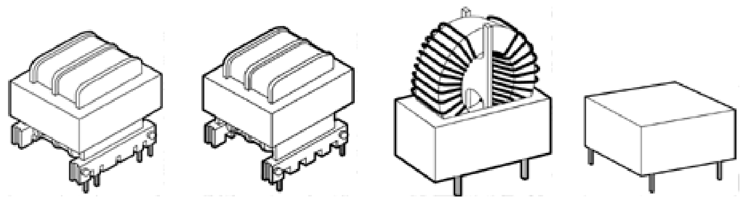
Where i_L is the current through the inductor. The current $i_L(t)$ can thus be given by

$$i_L(t) = \frac{1}{L}\int_0^t e_L \mathrm{d}t + i_L(0)$$

Because most inductors are coils of wire, they have considerable resistance. The energy loss due to the presence of resistance is indicated by the *quality factor* Q, which denotes the ratio of stored dissipated energy. A high value of Q generally means the inductor contains small resistance.

Mutual Inductance refers to the influence between inductors that results from interaction of their fields.

PRACTICAL EXAMPLES: Pictured below are several real-world examples of inductors.



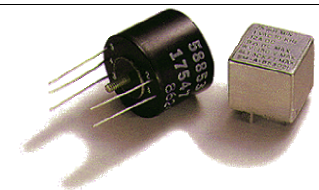


TABLE 6-1. Summary of elements involved in linear electrical systems

Element	Voltage-current	Current-voltage	Voltage-charge	Impedance, $Z(s)=V(s)/I(s)$
Capacitor	$v(t)=\frac{1}{c}\int_0^t i(\tau)d\tau$	$i(t)=C\frac{dv(t)}{dt}$	$v(t)=\frac{1}{c}q(t)$	$\frac{1}{Cs}$
Resistor	$v(t)=R\,i(t)$	$i(t)=\frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R
Inductor	$v(t)=L\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t)=L\frac{d^2q(t)}{dt^2}$	Ls

The following set of symbols and units are used: $v(t)$ = V (Volts), $i(t)$ = A (Amps), q(t) = Q (Coulombs), C = F (Farads), R = Ω (Ohms), L = H (Henries).

6.3 FUNDAMENTALS OF ELECTRICAL CIRCUITS

Ohm's Law. *Ohm's law* states that the current in circuit is proportional to the total electromotive force (emf) acting in the circuit and inversely proportional to the total resistance of the circuit. That is

$$i=\frac{e}{R}$$

were i is the current (amperes), e is the emf (volts), and R is the resistance (ohms).

Series Circuit. The combined resistance of series-connected resistors is the sum of the separate resistances. Figure 6-2 shows a simple series circuit. The voltage between points A and B is

$$e=e_1+e_2+e_3$$

where

$$e_1=i\,R_1,\qquad e_2=i\,R_2,\qquad e_3=i\,R_3$$

Thus,

$$\frac{e}{i} = R_1 + R_2 + R_3$$

The combined resistance is given by

$$R = R_1 + R_2 + R_3$$

In general,

$$R = \sum_{i=1}^{n} R_i$$

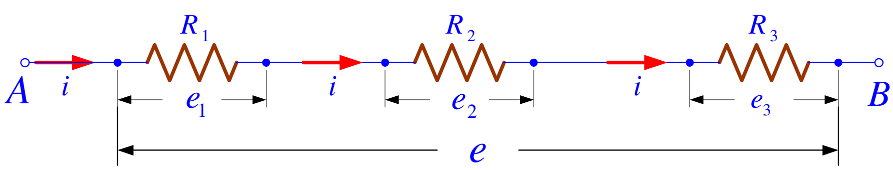


Figure 6-2 Series Circuit

Parallel Circuit. For the parallel circuit shown in figure 6-3,

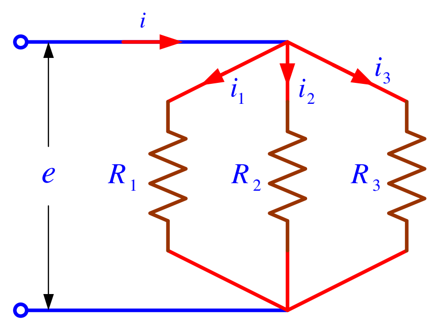


Figure 6-3 Parallel Circuit

$$i_1 = \frac{e}{R_1}, \qquad i_2 = \frac{e}{R_2}, \qquad i_3 = \frac{e}{R_3}$$

Since $i = i_1 + i_2 + i_3$, it follows that]

$$i = \frac{e}{R_1} + \frac{e}{R_2} + \frac{e}{R_3} = \frac{e}{R}$$

where R is the combined resistance. Hence,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

or

$$R = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}} = \frac{R_1R_2R_3}{R_1R_2+R_2R_3+R_3R_1}$$

In general

$$\frac{1}{R} = \sum_{i=1}^{n}\frac{1}{R_i}$$

Kirchhoff's Current Law (KCL) (Node Law). A *node* in an electrical circuit is a point where three or more wires are joined together. Kirchhoff's Current Law (KCL) states that

The algebraic sum of all currents entering and leaving a node is zero.

or

The algebraic sum of all currents entering a node is equal to the sum of all currents leaving the same node .

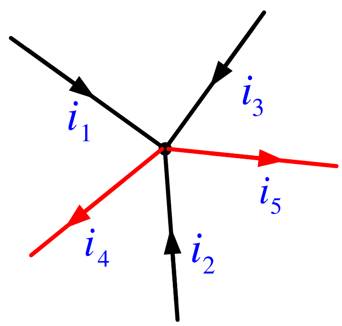


Figure 6-4 Node.

As applied to Figure 6-4, kirchhoff's current law states that

$$i_1 + i_2 + i_3 - i_4 - i_5 = 0$$

or

$$\underbrace{i_1 + i_2 + i_3}_{\text{Entering currents}} = \underbrace{i_4 + i_5}_{\text{Leaving currents}}$$

Kirchhoff's Voltage Law (KVL) (Loop Law). Kirchhoff's Voltage Law (KVL) states that at any given instant of time

The algebraic sum of the voltages around any loop in an electrical circuit is zero.

or

The sum of the voltage drops is equal to the sum of the voltage rises around a loop.

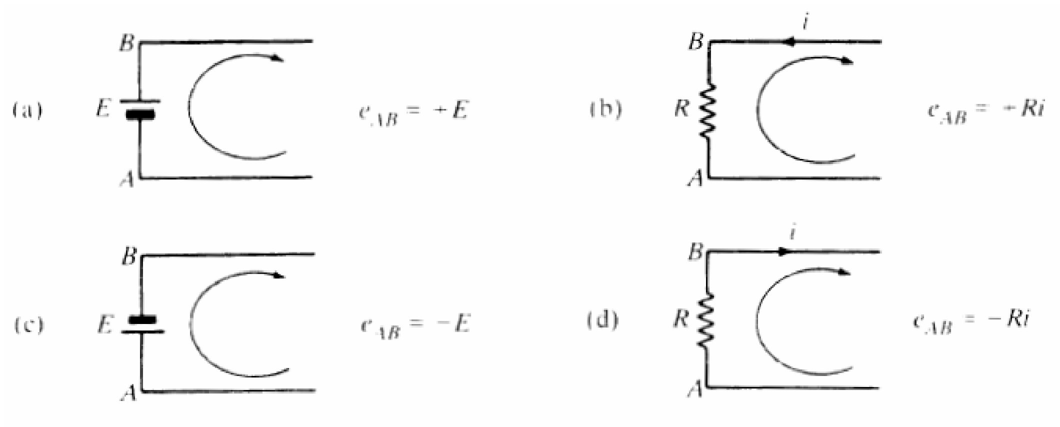


Figure 6-5 Diagrams showing voltage rises and voltage drops in circuits. (Note: Each circular arrows shows the direction one follows in analyzing the respective circuit)

A rise in voltage [which occurs in going through a source of electromotive force from the negative terminal to the positive terminal, as shown in Figure 6-5 (a), or in going through a resistance in opposition to the current flow, as shown in Figure 6-5 (b)] should be preceded by a plus sign.

A drop in voltage [which occurs in going through a source of electromotive force from the positive to the negative terminal, as shown in Figure 6-5 (c), or in going through a resistance in the direction of the current flow, as shown in Figure 6-5 (d)] should be preceded by a minus sign.

Figure 6-6 shows a circuit that consists of a battery and an external resistance.

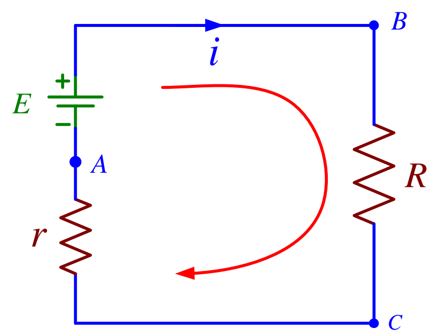


Figure 6-6 Electrical Circuit.

Here E is the electromotive force, r is the internal resistance of the battery, R is the external resistance and i is the current. Following the loop in the clockwise direction $(A \to B \to C \to D)$, we have

$$\vec{e}_{AB} + \vec{e}_{BC} + \vec{e}_{CA} = \vec{0}$$

or

$$E - iR - ir = 0$$

From which it follows that

$$i = \frac{E}{R + r}$$

6.4 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

The first step in analyzing circuit problems is to obtain mathematical models for the circuits. (Although the terms *circuit* and *network* are sometimes used interchangeably, *network* implies a more complicated interconnection than *circuit*.) A mathematical model may consist of algebraic equations, differential equations, integrodifferential equations, and similar ones. Such a model may be obtained by applying one or both of Kirchhoff's laws to a given circuit. The variables of interest in the circuit analysis are voltages and currents at various points along the circuit.

In this section, we first present the mathematical modeling of electrical circuits and obtain solutions of simple circuit problems. Then we review the concept of complex impedances, followed by derivations of mathematical models of electrical circuits.

Example 6–1

Consider the circuit shown in Figure 6–11. Assume that the switch S is open for $t < 0$ and closed at $t = 0$. Obtain a mathematical model for the circuit and obtain an equation for the current $i(t)$.

By arbitrarily choosing the direction of the current around the loop as shown in the figure, we obtain

$$E - L\frac{di}{dt} - Ri = 0$$

or

$$L\frac{di}{dt} + Ri = E \tag{6–3}$$

This is a mathematical model for the given circuit. Note that at the instant switch S is closed the current $i(0)$ is zero, because the current in the inductor cannot change from zero to a finite value instantaneously. Thus, $i(0) = 0$.

Let us solve Equation (6–3) for the current $i(t)$. Taking the Laplace transforms of both sides, we obtain

$$L[sI(s) - i(0)] + RI(s) = \frac{E}{s}$$

Noting that $i(0) = 0$, we have

$$(Ls + R)I(s) = \frac{E}{s}$$

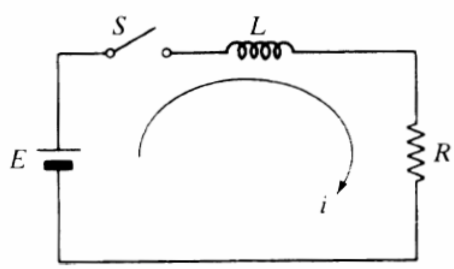


Figure 6–11 Electrical circuit.

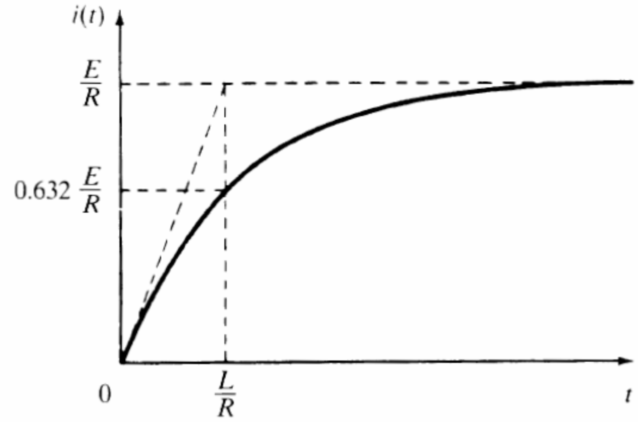


Figure 6–12 Plot of $i(t)$ versus t for the circuit shown in Figure 6–11 when switch S is closed at $t = 0$.

or

$$I(s) = \frac{E}{s(Ls + R)} = \frac{E}{R}\left[\frac{1}{s} - \frac{1}{s + (R/L)}\right]$$

The inverse Laplace transform of this last equation gives

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t}] \tag{6–4}$$

A typical plot of $i(t)$ versus t appears in Figure 6–12.

Example 6–2

Consider again the circuit shown in Figure 6–11. Assume that switch S is open for $t < 0$, it is closed at $t = 0$, and is open again at $t = t_1 > 0$. Obtain a mathematical model for the system, and find the current $i(t)$ for $t \geq 0$.

The equation for the circuit is

$$L\frac{di}{dt} + Ri = E \qquad i(0) = 0 \qquad t_1 > t \geq 0 \tag{6–5}$$

From Equation (6–4), the solution of Equation (6–5) is

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t}] \qquad t_1 > t \geq 0 \tag{6–6}$$

At $t = t_1$, the switch is opened. The equation for the circuit for $t \geq t_1$ is

$$L\frac{di}{dt} + Ri = 0 \qquad t \geq t_1 \tag{6–7}$$

where the initial condition at $t = t_1$ is given by

$$i(t_1) = \frac{E}{R}[1 - e^{-(R/L)t_1}] \tag{6–8}$$

(Note that the instantaneous value of the current at the switching instant $t = t_1$ serves as the initial condition for the transient response for $t \geq t_1$.) Equations (6–5), (6–7), and (6–8) constitute a mathematical model for the system.

Now we shall obtain the solution of Equation (6–7) with the initial condition given by Equation (6–8). The Laplace transform of Equation (6–7), with $t = t_1$ the initial time, gives

$$L[sI(s) - i(t_1)] + RI(s) = 0$$

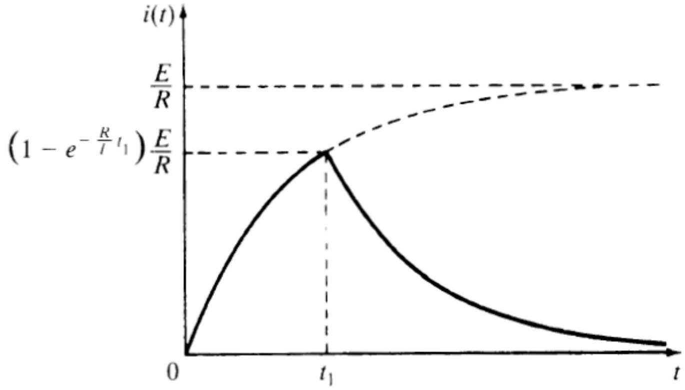


Figure 6–13 Plot of $i(t)$ versus t for the circuit shown in Figure 6–12 when switch S is closed at $t = 0$ and opened at $t = t_1$.

or

$$(Ls + R)I(s) = Li(t_1)$$

Hence,

$$I(s) = \frac{Li(t_1)}{Ls + R} = \frac{E}{R}[1 - e^{-(R/L)t_1}]\frac{1}{s + (R/L)} \qquad (6\text{–}9)$$

The inverse Laplace transform of Equation (6–9) gives

$$i(t) = \frac{E}{R}[1 - e^{-(R/L)t_1}]e^{-(R/L)(t-t_1)} \qquad t \geq t_1 \qquad (6\text{–}10)$$

Consequently, from Equations (6–6) and (6–10), the current $i(t)$ for $t \geq 0$ can be written

$$\begin{aligned} i(t) &= \frac{E}{R}[1 - e^{-(R/L)t}] && t_1 > t \geq 0 \\ &= \frac{E}{R}[1 - e^{-(R/L)t_1}]e^{-(R/L)(t-t_1)} && t \geq t_1 \end{aligned}$$

A typical plot of $i(t)$ versus t for this case is given in Figure 6–13.

Example 6–3

Consider the electrical circuit shown in Figure 6–14. The circuit consists of a resistance R(in ohms) and a capacitance C (in farads). Obtain the transfer function $E_o(s)/E_i(s)$. Also, obtain a state-space representation of the system.

Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$Ri + \frac{1}{C}\int i\,dt = e_i \qquad (6\text{–}11)$$

$$\frac{1}{C}\int i\,dt = e_o \qquad (6\text{–}12)$$

The transfer-function model of the circuit can be obtained as follows: Taking the Laplace transform of Equations (6–11) and (6–12), assuming zero initial conditions, we get

$$RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

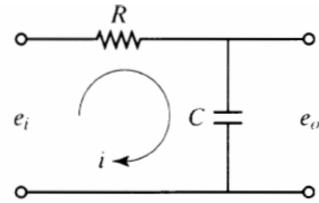


Figure 6–14 *RC* circuit.

Assuming that the input is e_i and the output is e_o, the transfer function of the system is

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C}\frac{1}{s}I(s)}{\left(R + \frac{1}{C}\frac{1}{s}\right)I(s)} = \frac{1}{RCs + 1} \tag{6–13}$$

This system is a first-order system.

A state-space model of the system may be obtained as follows: First, note that, from Equation (6–13), the differential equation for the circuit is

$$RC\dot{e}_o + e_o = e_i$$

If we define the state variable

$$x = e_o$$

and the input and output variables

$$u = e_i, \qquad y = e_o = x$$

then we obtain

$$\dot{x} = -\frac{1}{RC}x + \frac{1}{RC}u$$
$$y = x$$

These two equations give a state-space representation of the system.

Example 6–4

Consider the electrical circuit shown in Figure 6–15. The circuit consists of an inductance L (in henrys), a resistance R (in ohms), and a capacitance C (in farads). Obtain the transfer function $E_o(s)/E_i(s)$. Also, obtain a state-space representation of the system.

Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{6–14}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{6–15}$$

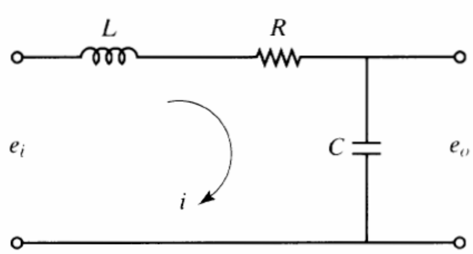


Figure 6–15 Electrical circuit.

The transfer-function model of the circuit can be obtained as follows: Taking the Laplace transforms of Equations (6–14) and (6–15), assuming zero initial conditions, we get

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

Then the transfer function $E_o(s)/E_i(s)$ becomes

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{6–16}$$

A state-space model of the system may be obtained as follows: First, note that, from Equation (6–16), the differential equation for the system is

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then, by defining state variables

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

and

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascade Elements. Consider the system shown in Figure 6.18. Assume e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially. Let us find transfer function $E_o(s)/E_i(s)$.

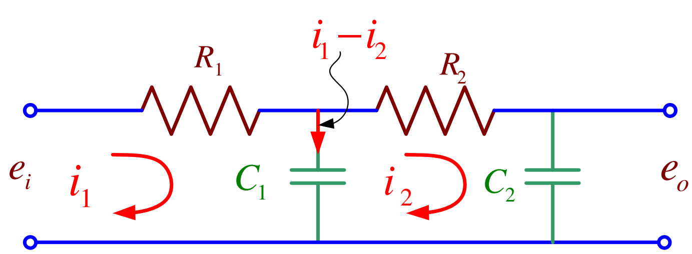


Figure 6-18 Electrical system

The equations of this system are:

Loop1 $$R_1 i_1 + \frac{1}{C_1}\int (i_1 - i_2)\, dt = e_i \tag{6-17}$$

Loop2 $$\frac{1}{C_1}\int(i_1-i_2)\,dt+R_2i_2+\frac{1}{C_2}\int i_2dt=0 \qquad (6\text{-}18)$$

Outer Loop $$\frac{1}{C_2}\int i_2dt=e_o \qquad (6\text{-}19)$$

Taking LT of the above equations, assuming zero I. C's, we obtain

$$R_1I_1(s)+\frac{1}{C_1s}\left[I_1(s)-I_2(s)\right]=E_i(s) \qquad (6\text{-}20)$$

$$\frac{1}{C_1s}\left[I_1(s)-I_2(s)\right]+R_2I_2(s)+\frac{1}{C_2s}I_2(s)=0 \qquad (6\text{-}21)$$

$$\frac{1}{C_2s}I_2(s)=E_o(s) \qquad (6\text{-}22)$$

From Equation (6-20)

$$R_1I_1(s)+\frac{1}{C_1s}I_1(s)-\frac{1}{C_1s}I_2(s)=E_i(s)$$

$$I_1(s)=\frac{E_i(s)+\dfrac{1}{C_1s}I_2(s)}{\dfrac{R_1C_1s+1}{C_1s}}=\frac{C_1sE_i(s)+I_2(s)}{R_1C_1s+1}$$

Substitute $I_1(s)$ into Equation (6-21)

$$\frac{E_i(s)}{E_o(s)}=\frac{1}{R_1C_1R_2C_2s^2+(R_1C_1+R_2C_2+R_1C_2)\,s+1}$$

$$=\frac{1/R_1C_1R_2C_2}{s^2+\dfrac{(R_1C_1+R_2C_2+R_1C_2)}{R_1C_1R_2C_2}s+\dfrac{1}{R_1C_1R_2C_2}}$$

which represents a transfer function of a second order system. The characteristic polynomial (denominator) of the above transfer function can be compared to that of a second order system $s^2+2\zeta\omega_ns+\omega_n^2$. Therefore, one can write

$$\omega_n^2=\frac{1}{R_1C_1R_2C_2} \qquad \text{and} \qquad 2\zeta\omega_n=\frac{(R_1C_1+R_2C_2+R_1C_2)}{R_1C_1R_2C_2}$$

or $$\zeta=\frac{(R_1C_1+R_2C_2+R_1C_2)}{2\omega_n(R_1C_1R_2C_2)}=\frac{(R_1C_1+R_2C_2+R_1C_2)}{2\sqrt{R_1C_1R_2C_2}}$$

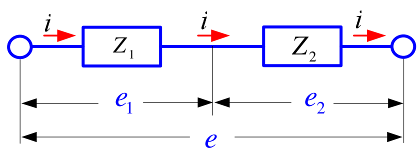
Complex Impedance. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations.

Table 6-1 gives the complex impedance of the basics passive elements such as resistance R, an inductance L, and a capacitance C. Figure 6-19 shows the complex impedances Z_1 and Z_2 in a series circuit while Figure 6-19 shows the transfer function between the output and input voltage. Remember that the impedance is valid only if the initial conditions involved are all zeros.

The general relationship is

$$E(s) = Z(s)\,I(s)$$

corresponds to Ohm's law for purely resistive circuits. (Notice that, like resistances, impedances can be combined in series and in parallel)



$$Z = Z_1 + Z_2 = \frac{E(s)}{I(s)}$$

Figure 6-19   Electrical circuit

Deriving Transfer Functions of Electrical Circuits Using Complex Impedances. The TF of an electrical circuit can be obtained as a ratio of complex impedances. For the circuit shown in Figure 6-20, assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the TF of this circuit can be obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)I(s)}{Z_1(s)I(s) + Z_2(s)I(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

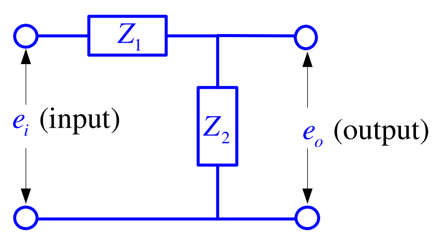


Figure 6-20   Electrical circuit

For the circuit shown in Figure 6-21,

$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

Hence, the transfer function $\frac{E_o(s)}{E_i(s)}$, is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

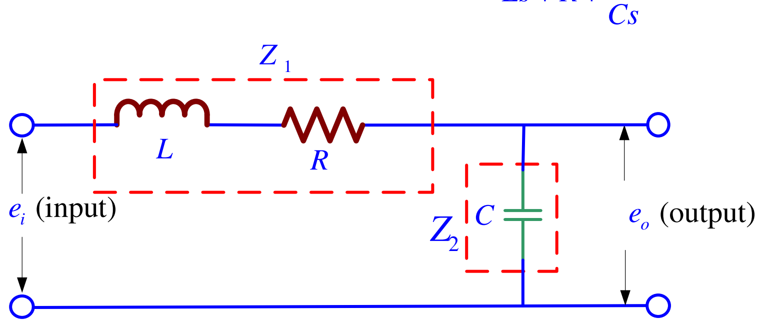


Figure 6-21 Electrical circuit

6.5 ANALOGOUS SYSTEMS

Systems that can be represented by the same mathematical model, but that are physically different, are called *analogous* systems. Thus analogous systems are described by the same differential or integrodifferential equations or transfer functions.

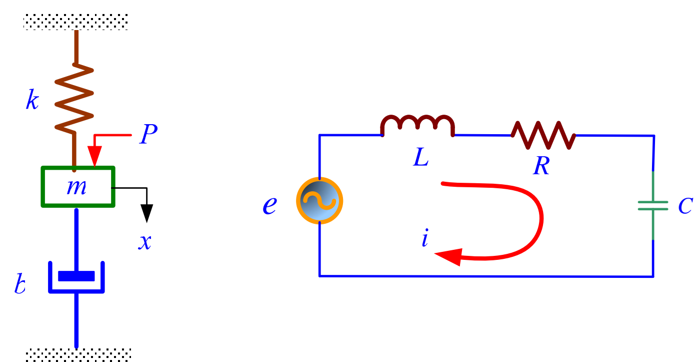
The concept of analogous is useful in practice, for the following reasons:

1. The solution of the equation describing one physical system can be directly applied to analogous systems in any other field.

2. Since one type of system may be easier to handle experimentally than another, instead of building and studying a mechanical system (or a hydraulic system, pneumatic system, or the like), we can build and study its electrical analog, for electrical or electronic system, in general, much easier to deal with experimentally.

Mechanical-Electrical Analogies Mechanical systems can be studied through their electrical analogs, which may be more easily constructed than models of the corresponding mechanical systems. There are two electrical analogies for mechanical systems: The Force-Voltage Analogy and The Force Current Analogy.

Force Voltage Analogy Consider the mechanical system of Figure 6-24(a) and the electrical system of Figure 6-24(b).



(a) (b)

Figure 6-24 Analogous mechanical and electrical systems.

The equation for the mechanical system is

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=p \qquad (6\text{-}24)$$

where x is the displacement of mass m, measured from equilibrium position. The equation for the electrical system is

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=e$$

In terms of electrical charge q, this last equation becomes

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e \qquad (6\text{-}25)$$

Comparing equations (6-24) and (6-25), we see that the differential equations for the two systems are of identical form. Thus, these two systems are analogous systems. The terms that occupy corresponding positions in the differential equations are called analogous quantities, a list of which appear in Table 6-2

TABLE 6-2 Force Voltage Analogy

Mechanical Systems	Electrical Systems
Force p (Torque T)	Voltage e
Mass m (Moment of inertia J)	Inductance L
Viscous-friction coefficient b	Resistance R
Spring constant k	Reciprocal of capacitance, $1/C$
Displacement x (angular displacement θ)	Charge q
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Current i

Force Current Analogy The student is advised to read this section from the textbook Page 272-273.

6.6 MATHEMATICAL MODELING OF ELCTROMECHANICAL SYSTEMS

To control the motion or speed of dc servomotors, we control the field current or armature current or we use a servodriver as motor-driver combination. There are many different types of servodrivers. Most are designed to control the speed of dc servomotors, which improves the efficiency of operating servomotors. Here we shall discuss only armature control of a dc servomotor and obtain its mathematical model in the form of a transfer function.

Armature control of dc servomotors. Consider the armature-controlled dc servomotor shown in Figure 6–27, where the field current is held constant. In this system,

R_a = armature resistance, Ω
L_a = armature inductance, H
i_a = armature current, A
i_f = field current, A
e_a = applied armature voltage, V
e_b = back emf, V
θ = angular displacement of the motor shaft, rad
T = torque developed by the motor, N-m
J = moment of inertia of the motor and load referred to the motor shaft, kg-m^2
b = viscous-friction coefficient of the motor and load referred to the motor shaft, N-m/rad/s

The torque T developed by the motor is proportional to the product of the armature current i_a and the air gap flux ψ, which in turn is proportional to the field current, or

$$\psi = K_f i_f$$

where K_f is a constant. The torque T can therefore be written as

$$T = K_f i_f K_1 i_a$$

where K_1 is a constant.

For a constant field current, the flux becomes constant and the torque becomes directly proportional to the armature current, so

$$T = K i_a$$

where K is a motor-torque constant. Notice that if the sign of the current i_a is reversed, the sign of the torque T will be reversed, which will result in a reversal of the direction of rotor rotation.

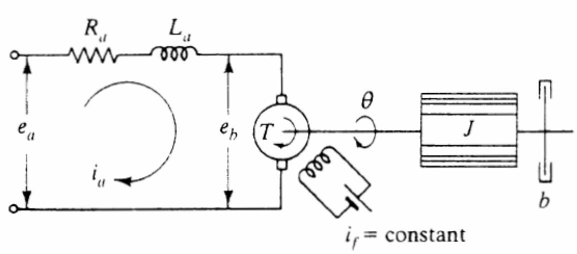


Figure 6–27 Armature-controlled dc servomotor.

When the armature is rotating, a voltage proportional to the product of the flux and angular velocity is induced in the armature. For a constant flux, the induced voltage e_b is directly proportional to the angular velocity $d\theta/dt$, or

$$e_b = K_b \frac{d\theta}{dt} \tag{6–30}$$

where e_b is the back emf and K_b is a back-emf constant.

The speed of an armature-controlled dc servomotor is controlled by the armature voltage e_a. The differential equation for the armature circuit is

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a \tag{6–31}$$

The armature current produces the torque that is applied to the inertia and friction; hence,

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = T = Ki_a \tag{6–32}$$

Assuming that all initial conditions are zero and taking the Laplace transforms of Equations (6–30), (6–31), and (6–32), we obtain the following equations:

$$K_b s\Theta(s) = E_b(s) \tag{6–33}$$

$$(L_a s + R_a)I_a(s) + E_b(s) = E_a(s) \tag{6–34}$$

$$(Js^2 + bs)\Theta(s) = T(s) = KI_a(s) \tag{6–35}$$

Considering $E_a(s)$ as the input and $\Theta(s)$ as the output and eliminating $I_a(s)$ and $E_b(s)$ from Equations (6–33), (6–34), and (6–35), we obtain the transfer function for the dc servomotor:

$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s[L_a Js^2 + (L_a b + R_a J)s + R_a b + KK_b]} \tag{6–36}$$

The inductance L_a in the armature circuit is usually small and may be neglected. If L_a is neglected, then the transfer function given by Equation (6–36) reduces to

$$\frac{\Theta(s)}{E_a(s)} = \frac{K}{s(R_a Js + R_a b + KK_b)} = \frac{\dfrac{K}{R_a J}}{s\left(s + \dfrac{R_a b + KK_b}{R_a J}\right)} \tag{6–37}$$

Notice that the term $(R_a b + KK_b)/(R_a J)$ in Equation (6–37) corresponds to the damping term. Thus, the back emf increases the effective damping of the system. Equation (6–37) may be rewritten as

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_m}{s(T_m s + 1)} \tag{6–38}$$

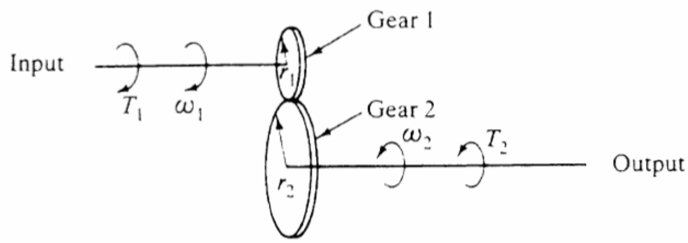


Figure 6–28 Gear train system.

where

$$K_m = K/(R_a b + KK_b) = \text{motor gain constant}$$
$$T_m = R_a J/(R_a b + KK_b) = \text{motor time constant}$$

Equation (6–38) is the transfer function of the dc servomotor when the armature voltage $e_a(t)$ is the input and the angular displacement $\theta(t)$ is the output. Since the transfer function involves the term $1/s$, this system possesses an integrating property. (Notice that the time constant T_m of the motor becomes smaller as the resistance R_a is reduced and the moment of inertia J is made smaller.)

Gear train. Gear trains are frequently used in mechanical systems to reduce speed, to magnify torque, or to obtain the most efficient power transfer by matching the driving member to the given load. Figure 6–28 illustrates a simple gear train system in which the gear train transmits motion and torque from the input member to the output member. If the radii of gear 1 and gear 2 are r_1 and r_2, respectively, and the numbers of teeth on gear 1 and gear 2 are n_1 and n_2, respectively, then

$$\frac{r_1}{r_2} = \frac{n_1}{n_2}$$

Because the surface speeds at the point of contact of the two gears must be identical, we have

$$r_1\omega_1 = r_2\omega_2$$

where ω_1 and ω_2 are the angular velocities of gear 1 and gear 2, respectively. Therefore,

$$\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{n_1}{n_2}$$

If we neglect friction loss, the gear train transmits the power unchanged. In other words, if the torque applied to the input shaft is T_1 and the torque transmitted to the output shaft is T_2, then

$$T_1\omega_1 = T_2\omega_2$$

Example 6–7

Consider the system shown in Figure 6–29. Here, a load is driven by a motor through the gear train. Assuming that the stiffness of the shafts of the gear train is infinite, that there is neither backlash nor elastic deformation, and that the number of teeth on each gear is