A BOCHNER TECHNIQUE FOR HARMONIC MORPHISMS

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Abstract. We establish a Weitzenböck formula for harmonic morphisms between Riemannian manifolds and show that under suitable curvature conditions, such a map is totally geodesic. As an application of the Weitzenböck formula we obtain some non-existence results of a global nature for harmonic morphisms and totally geodesic horizontally conformal maps between compact Riemannian manifolds. In particular, it is shown that the only harmonic morphisms from a Riemannian symmetric space of compact type to a compact Riemann surface of genus ≥ 1 are the constant maps.

1. Introduction

A smooth map $\phi\colon M \to N$ between Riemannian manifolds is called a harmonic morphism if it preserves germs of harmonic functions, i.e., if f is a real valued harmonic function on an open set $V \subseteq N$ then the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V) \subseteq M$. Due to a characterization obtained by B. Fuglede [16] and T. Ishihara [22], harmonic morphisms are precisely the harmonic maps which are horizontally (weakly) conformal.

The purpose of this paper is to develop a Bochner technique for harmonic morphisms, extending the work of Eells-Sampson [15], to

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study harmonic morphisms between suitably curved Riemannian manifolds, for instance, to investigate harmonic morphisms from compact Lie groups or symmetric spaces of compact type.

We divide this section in two parts giving an introduction to harmonic morphisms and explaining the Bochner technique for harmonic maps.

Notation. Throughout this paper we adopt the following conventions. By $(M^m, \langle\cdot,\cdot\rangle^M)$ we shall mean a smooth connected, oriented, complete Riemannian manifold, without boundary, of dimension m endowed with a Riemannian metric $\langle\cdot,\cdot\rangle^M$. We also suppose that all the maps are smooth. We denote by ∇^M and R^M, respectively, the connection on the manifold M and the curvature of this connection where

$$R^M(X,Y) = -\nabla^M_X\nabla^M_Y + \nabla^M_Y\nabla^M_X + \nabla^M_{[X,Y]}.$$

For a map $\phi\colon M \to N$ we shall denote by ∇ the connection on the bundle $T^*M \otimes \ \phi^{-1}TN$. By Δ we shall mean the Hodge-deRham Laplacian defined as $\Delta = dd^* + d^*d$, so that the Laplacian on a function f defined on M is given as

$$\Delta f = -\mathbf{trace}\nabla^M df,$$

i.e. the negative of the usual Laplacian on functions. Generally, we shall be following the notations of [12] for differential operators on manifolds.

1.1. **Harmonic morphisms.** Recall that a map $\phi\colon M^m \to N^n$ is *harmonic* if and only if its tension field $\tau(\phi) = \mathbf{trace}\nabla d\phi$ vanishes. The reader is referred to [11], [12] and [13] for a detailed account of harmonic maps.

Definition 1.1. A map $\phi\colon M^m \to N^n$ between Riemannian manifolds is called a *harmonic morphism* if $f\circ\phi$ is a real valued harmonic function on $\phi^{-1}(V) \subseteq M$ for every real valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ non-empty.

It can be easily seen that the composition of harmonic morphisms is a harmonic morphism. Let $C_\phi = \{x \in M \mid \text{rank } d\phi_x = 0\}$ be the critical set of $\phi\colon M^m \to N^n$. It is shown in [16] that, for a non-constant map, the set $M \backslash C_\phi$ is open and dense in M. For each $x \in (M \backslash C_\phi)$, the vertical space $T_x^V M$ at x is defined by $T_x^V M = \text{Ker} d\phi_x$. The horizontal space $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$ so that $T_x M = T_x^V M \oplus T_x^H M$.

Definition 1.2. A map $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^m, \langle\cdot,\cdot\rangle^N)$ is called *horizontally (weakly) conformal* if for each $x \in (M^m \setminus C_\phi)$, the restriction of $d\phi_x$ to $T_x^H M$ (i.e. $d\phi_x : T_x^H M \to T_{\phi(x)} N$) is conformal and surjective, that is, there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\langle d\phi(X), d\phi(Y)\rangle^N = \lambda^2 \langle X, Y\rangle^M \quad \forall X, Y \in T^H M.$$

By setting $\lambda = 0$ on C_ϕ, we can extend λ to M such that $\lambda^2 : M^m \to \mathbb{R}$ is a smooth function; λ is called the *dilation* of ϕ.

The following curvature, defined for a horizontally (weakly) conformal map, plays an important role in the subsequent sections.

Definition 1.3. Let $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a horizontally (weakly) conformal map. We define $\mathbf{Scal^M}|_{\mathbf{H}}$ by

$$\mathbf{Scal^M}|_{\mathbf{H}} = 0 \quad \textit{at a critical point},$$

and

$$\mathbf{Scal^M}|_{\mathbf{H}} = \sum_{s=1}^{n} \mathbf{Ricci^M}(e_s, e_s) \quad \textit{at a regular point } x$$

where $(e_s)_{s=1}^n$, $(e_s)_{s=n+1}^m$ are orthonormal bases of $T_x^H M$ and $T_x^V M$ respectively, so that $(e_s)_{s=1}^m$ is an orthonormal basis of $T_x M = T_x^V M \oplus T_x^H M$.

Harmonic morphisms can be characterized as follows [16], [22]:

Theorem 1.4. *A map* $\phi\colon (M^m, \langle\cdot,\cdot\rangle^M) \to (N^m, \langle\cdot,\cdot\rangle^N)$ *is a harmonic morphism if and only if it is a harmonic and horizontally conformal map.*

In the study of harmonic morphisms the significance of dimensions of the manifolds under consideration is clear from the following.

Proposition 1.5. [16] *Let* M^m, N^n *be Riemannian manifolds.*

1. *If* $m < n$ *then every harmonic morphism* $\phi\colon M^m \to N^n$ *is constant.*
2. *If* $m = n = 2$, *the harmonic morphisms* $\phi\colon M^m \to N^n$ *are just the weakly conformal maps.*
3. *If* $m = n \geq 3$, *then the harmonic morphisms* $\phi\colon M^m \to N^n$ *are conformal mappings with constant dilation.*

Unless otherwise stated we shall assume that $m > n$ for a harmonic morphism $\phi\colon M^m \to N^n$. We refer the reader to [16], [2], [33] for an introduction and basic properties of harmonic morphisms. Many examples of harmonic morphisms can be found in [2], [4], [5], [6], [7], [17], [18], [19], [20], [21] and [34].

1.2. **Bochner technique for harmonic maps.** The Bochner technique is a method devised by S. Bochner in [9], to obtain vanishing theorems under appropriate curvature conditions on compact Riemannian manifolds. This can be best described by the following steps.

- Develop an identity relating Laplacians on sections of a vector bundle. Such an identity naturally involves the curvature of the bundle and is commonly known as a Weitzenböck formula.
- Impose suitable curvature restrictions to apply a maximum principle, in order to obtain the required vanishing results.

In this section we shall focus our attention on the applications of the Bochner technique in the theory of harmonic maps. A good account of applications of the Bochner technique in differential geometry, in general, may be found in [35].

The first employment of this technique to study harmonic maps was in the well-known paper of J. Eells and J. H. Sampson [15], where the following Weitzenböck formula was obtained.

Proposition 1.6 (WF for harmonic maps). *Let M^m, N^n be Riemannian manifolds. If $\phi : (M^m, \langle\cdot,\cdot\rangle^M)\to(N^n, \langle\cdot,\cdot\rangle^N)$ is harmonic, then*

$$-\mathbf{trace}\nabla^2 d\phi = \sum_{s=1}^{m}\mathbf{R^N}(d\phi{\cdot}e_s, d\phi)d\phi{\cdot}e_s - d\phi{\cdot}\mathbf{Ricci^M} \tag{1.1}$$

and

(1.2)

$$\frac{1}{2}\Delta\|d\phi\|^2 = -\|\nabla d\phi\|^2 + \sum_{s,t=1}^{m}\langle\mathbf{R^N}(d\phi{\cdot}e_s, d\phi{\cdot}e_t)d\phi{\cdot}e_s, d\phi{\cdot}e_t\rangle^N - \sum_{s=1}^{m}\langle d\phi{\cdot}\mathbf{Ricci^M}e_s, d\phi{\cdot}e_s\rangle^M$$

where $(e_s)_{s=1}^m$ is an orthonormal basis at the point under consideration on M.

As a consequence it was shown that

Theorem 1.7. *Let M^m be a compact Riemannian manifold. Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M)\to(N^n, \langle\cdot,\cdot\rangle^N)$ be harmonic and suppose* $\mathbf{Ricci^M}\geq 0$ *and* $\mathbf{Riem^N}\leq 0$*. Then*

1. *ϕ is totally geodesic.*
2. *If $\mathbf{Ricci^M} > 0$ at some point, then ϕ is constant.*
3. *If $\mathbf{Riem^N} < 0$, then ϕ is either constant or of rank one, in which case its image is a closed geodesic.*

The above scheme was extended in [28] to show that a harmonic map of finite energy from a complete non-compact manifold of non-negative

Ricci curvature to a compact manifold of non-positive sectional curvature is constant. Some further applications of Proposition 1.6 were given in [29] in studying harmonic maps from a compact Riemannian manifold M with $\mathbf{Ricci^M} \geq A > 0$ to a compact Riemannian manifold N with $\mathbf{Riem^N} \leq B > 0$.

In the case of Kähler manifolds, H. C. Sealey [29] found a complex analogue of Proposition 1.6 to study holomorphic maps (which are harmonic) between Kähler manifolds. Another generalization of the Bochner technique was presented by Y. T. Siu in [30] where he obtained a Weitzenböck type formula, involving only the curvature of the image manifold, for harmonic maps between Kähler manifolds. This type of argument enabled him to study properties of harmonic mappings and to obtain rigidity results between compact Kähler manifolds, with curvature conditions on the image manifold only. A similar analysis was carried out by J. H. Sampson in [27] for harmonic maps from a compact Kähler manifold into a Riemannian manifold.

Siu's Weitzenböck type formula was generalized by K. Corlette in [10], where he presented a rigidity study of quaternionic hyperbolic space and the hyperbolic Cayley plane by making use of a Bochner type formula for twisted harmonic maps cf. [10, Theorem 3.1]. Jost-Yau in [23] established a Bochner type formula following [25] to investigate properties of harmonic maps from compact quotients of symmetric spaces of non-compact type. Finally in [26], the authors discovered a generalized Bochner formula to study rigidity and harmonic mappings. This generalization of the Bochner formula has the formulae of [30], [27] and [10] as special cases.

2. Bochner technique for harmonic morphisms

In order to extend the Bochner technique to the study of harmonic morphisms, we follow the work of J. Eells and J. H. Sampson [15]. Using the horizontal conformality condition we obtain the following analogue of Proposition 1.6, which gives a Weitzenböck formula for harmonic morphisms.

Proposition 2.1 (WF for harmonic morphisms). *Let M^m and N^n be Riemannian manifolds. Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a harmonic morphism with dilation λ. Then*

$$-\mathbf{trace}\nabla^2 d\phi = \lambda^2 \mathbf{Ricci^N} d\phi - d\phi\cdot\mathbf{Ricci^M} \tag{2.1}$$

and

$$\frac{n}{2}\Delta\lambda^2 = -\|\nabla d\phi\|^2 + \lambda^4 \mathbf{Scal^N} - \lambda^2 \mathbf{Scal^M}|_{\mathbf{H}} \tag{2.2}$$

where $\mathbf{Scal^M}|_{\mathbf{H}}$ is defined in Definition 1.3.

Proof. Let x be a regular point of M. The horizontal conformality condition implies that there exists an orthonormal basis $(e'_s)_{s=1}^n$ of $T_{\phi(x)}N$ such that

$$d\phi(e_s) = \lambda e'_s \quad for\ s = 1, \ldots, n \tag{2.3}$$

and

$$d\phi(e_s) = 0 \quad for\ s = n+1, \ldots, m. \tag{2.4}$$

Using this together with Theorem 1.4 and Proposition 1.6 we obtain

$$-\mathbf{trace}\nabla^2 d\phi = \lambda^2 \mathbf{Ricci^N} d\phi - d\phi\cdot\mathbf{Ricci^M}$$

and

$$\frac{n}{2}\Delta\lambda^2 = -\|\nabla d\phi\|^2 + \lambda^4 \mathbf{Scal^N} - \sum_{s=1}^{n} \langle d\phi\cdot\mathbf{Ricci^M} e_s, d\phi\cdot e_s\rangle^N$$

where

$$\mathbf{Scal^N} = \sum_{s,t=1}^{n} \langle \mathbf{R^N}(e'_s, e'_t)e'_s, e'_t\rangle^N.$$

The last term in the above equation can be simplified as follows.

$$\begin{aligned}
\langle d\phi \cdot \mathbf{Ricci^M} e_s, d\phi \cdot e_s\rangle^N &= \sum_{l=1}^{m} \langle \mathbf{Ricci^M} e_s, e_l\rangle^M \langle d\phi \cdot e_l, d\phi \cdot e_s\rangle^N \\
&= \lambda^2 \sum_{l=1}^{n} \mathbf{Ricci^M}(e_s, e_l)\langle e'_s, e'_l\rangle^N \\
&= \lambda^2 \mathbf{Ricci^M}(e_s, e_s).
\end{aligned}$$

The above argument holds for a critical point, as $\lambda = 0$ on the critical set C_ϕ and the proof follows from Definition 1.3 because $\lambda^2 \mathbf{Scal^M}|_{\mathbf{H}}$ is a continuous function on C_ϕ. □

For later use we give the following results for totally geodesic horizontally conformal maps.

Definition 2.2. A map $\phi\colon M^m \to N^n$ is called *totally geodesic* if its second fundamental form $\nabla d\phi$ vanishes, where

$$\nabla d\phi(X, Y) = (\nabla_X d\phi)Y = \nabla_X^{\phi^{-1}TN}(d\phi \cdot Y) - d\phi(\nabla_X^M Y)$$

for $X, Y \in \mathcal{C}(TM)$.

Lemma 2.3. *A totally geodesic map has constant rank and constant energy density $e(\phi)$, where $e(\phi) = \frac{1}{2}\|d\phi\|^2$. In particular, a totally geodesic horizontally conformal map has constant dilation and so is a homothetic submersion.*

Proof. cf. [14, page-15] □

The Weitzenböck formula for totally geodesic horizontally conformal maps is given by the following.

Corollary 2.4. *Let ϕ: $M^m \to N^n$ be a non-constant totally geodesic horizontally conformal map between Riemannian manifolds. Then*

$$\mathbf{Scal^M}|_{\mathbf{H}} = \lambda^2 \mathbf{Scal^N} \tag{2.5}$$

where λ^2 is a non-zero constant.

Proof. Follows directly from Proposition 2.1 and Lemma 2.3. □

As a first application of the Weitzenböck formula developed in Proposition 2.1 we obtain:

Theorem 2.5. *Let $(M^m, \langle\cdot,\cdot\rangle^M)$ be an oriented compact Riemannian manifold without boundary. If $\mathbf{Scal^M}|_{\mathbf{H}} \geq 0$ and $(N^n, \langle\cdot,\cdot\rangle^N)$ is a Riemannian manifold with $\mathbf{Scal^N} \leq 0$, then every harmonic morphism ϕ: $M^m \to N^n$ is totally geodesic. Furthermore,*

1. *If $\mathbf{Scal^M}|_{\mathbf{H}} > 0$ at some point, then ϕ is constant.*
2. *If $\mathbf{Scal^N} < 0$ at some point, then ϕ is constant.*
3. *If ϕ is non-constant, then ϕ is a homothetic submersion and $\mathbf{Scal^M}|_{\mathbf{H}} \equiv 0$, $\mathbf{Scal^N} \equiv 0$.*

Proof. Integrating Equation (2.2) of Proposition 2.1 over M and making use of the fact that

$$\int_M (\Delta f) \upsilon^M = 0$$

for any smooth function f on an oriented compact manifold without boundary, we have

$$\int_M \|\nabla d\phi\|^2 \upsilon^M = \int_M \lambda^4 \mathbf{Scal^N} \upsilon^M - \int_M \lambda^2 \mathbf{Scal^M}|_{\mathbf{H}} \upsilon^M.$$

Since our hypothesis implies that the left hand side is non-negative and the right hand side is non-positive, this makes both sides zero. The left hand side of the above equation gives

$$\|\nabla d\phi\|^2 = 0,$$

making ϕ a totally geodesic map.

1. By an argument similar to the one above we have

$$\lambda^2 \mathbf{Scal^M}|_{\mathbf{H}} = 0.$$

   The proof follows from the hypothesis and Lemma 2.3.
2. Similar to part 1.
3. Immediate from Lemma 2.3.

$\square$

Remark 2.6. It might appear that the above Weitzenböck formula is applicable only for harmonic morphisms from a Riemannian manifold of non-negative Ricci curvature to a Riemannian manifold of non-positive scalar curvature. But under a weaker curvature restriction, namely $\mathbf{Scal^M}|_{\mathbf{H}} \geq \lambda^2 \mathbf{Scal^N}$, we can obtain applications for a harmonic morphism between Riemannian manifolds M and N, when $\mathbf{Scal^M}|_{\mathbf{H}}$ and $\mathbf{Scal^N}$ have the same sign.

For example, for a harmonic morphism whose dilation is bounded by the curvatures of the manifolds, we have

Theorem 2.7. *Let $\phi : (M^m, \langle\cdot,\cdot\rangle^M) \to (N^n, \langle\cdot,\cdot\rangle^N)$ be a submersive harmonic morphism between compact Riemannian manifolds with $\mathbf{Scal^M}|_{\mathbf{H}} \geq A$ and $0 < \mathbf{Scal^N} \leq B$ where $A, B > 0$. If the dilation λ^2 of ϕ is bounded by*

$$\lambda^2 \leq \frac{A}{B} \leq \frac{\mathbf{Scal^M}|_{\mathbf{H}}}{\mathbf{Scal^N}}$$

then ϕ is totally geodesic and either ϕ is constant or

1. *λ is constant with $\lambda^2 = \frac{A}{B}$ so that ϕ is a homothetic submersion and*
2. *The curvatures $\mathbf{Scal^M}|_{\mathbf{H}}$ and $\mathbf{Scal^N}$ are constants given by $\mathbf{Scal^M}|_{\mathbf{H}} = A > 0$ and $\mathbf{Scal^N} = B > 0$.*

Proof. From the hypothesis we have

$$\lambda^4 \mathbf{Scal^N} - \lambda^2 \mathbf{Scal^M}|_{\mathbf{H}} \leq -A\lambda^2 + B\lambda^4.$$

Therefore, the Weitzenböck formula for harmonic morphisms implies that

$$\frac{n}{2}\Delta\lambda^2 \leq -\|\nabla d\phi\|^2 + B\lambda^2(\lambda^2 - \frac{A}{B}). \tag{2.6}$$

Integration of Equation (2.6) over M forces each term on the right hand side to be zero. This proves that ϕ is totally geodesic and either $\lambda = 0$ or λ is the constant given by $\lambda^2 = \frac{A}{B}$. If $\lambda^2 = \frac{A}{B}$, Equation (2.2) implies that

$$\lambda^2 \mathbf{Scal^N} = \mathbf{Scal^M}|_{\mathbf{H}},$$

which shows that $\mathbf{Scal^M}|_{\mathbf{H}} \leq A$ and $\mathbf{Scal^N} \geq B$. Combining this with the hypothesis completes the proof. □

Corollary 2.4 gives us a non-existence result for totally geodesic horizontally conformal maps between Riemannian manifolds.

Corollary 2.8. *There exists no non-constant totally geodesic horizontally conformal map $\phi : (M^m, \langle \cdot, \cdot \rangle^M) \to (N^n, \langle \cdot, \cdot \rangle^N)$ in each of the following cases:*

1. $\mathbf{Ricci^M} > 0$ *and* $\mathbf{Scal^N} < 0$
2. $\mathbf{Ricci^M} < 0$ *and* $\mathbf{Scal^N} > 0$
3. $\mathbf{Ricci^M} = 0$ *and* $\mathbf{Scal^N} \neq 0$

4. $\mathbf{Ricci^M} \neq 0$ *and* $\mathbf{Scal^N} = 0$

Proof. Suppose on the contrary that there exists a non-constant totally geodesic horizontally conformal map in the cases under consideration. Since $\lambda^2 \neq 0$, in each case we get a contradiction from Corollary 2.4. □

3. Applications

In this section we present further applications of Proposition 2.1 and provide examples where it can be employed to study the classification of harmonic morphisms. For a harmonic morphism $\phi\colon M^m \to N^n$ we consider the cases $n \geq 3$ and $n = 2$ separately.

3.1. The case $n \geq 3$.

Theorem 3.1. *Let M^m be a compact Riemannian manifold with* $\mathbf{Ricci} \geq 0$. *Then there exists a metric* $\mathbf{h}$ *on the compact manifold N^n ($n \geq 3$) such that the only harmonic morphisms with respect to* $\mathbf{h}$ *are the constant maps.*

Proof. Every compact manifold of *dim* ≥ 3 carries a metric $\mathbf{h}$ of constant negative scalar curvature cf. p-389 of [1] or Corollary 5.4 of [24]. This fact combined with Proposition 2.1 gives the result. □

As an example there exists a metric on N^n ($n \geq 3$) such that there are no non-constant harmonic morphisms from an irreducible Riemannian symmetric space of compact type to N^n.

3.2. $n = 2$. Recall that the notion of a harmonic morphisms to a Riemann surface N^2 does not depend on any specific Hermitian metric on N^2, as the composition of harmonic morphisms is a harmonic morphism and a weakly conformal map between surfaces is a harmonic morphism.

Theorem 3.2. *Let N^2 be a compact Riemann surface. If N^2 has genus $g \geq 2$, then there exists no non-constant harmonic morphism from a compact Riemannian manifold M^m of non-negative Ricci curvature to N^2.*

Proof. Every compact Riemann surface of genus $g \geq 2$ has a Hermitian metric of constant negative curvature. The theorem then follows from Theorem 2.5. □

A rich collection of examples, to apply our results, is given by symmetric spaces and Lie groups.

Corollary 3.3.

1. *There does not exist a non-constant harmonic morphism from an irreducible Riemannian symmetric space of compact type to a compact Riemann surface N^2 of genus $g \geq 1$.*
2. *In particular, there exists no non-constant harmonic morphism from a compact connected Lie group, endowed with a bi-invariant metric, to a compact Riemann surface of genus $g \geq 1$.*

Proof.

1. If $g \geq 2$ we get the result from the above. Suppose $g = 1$. Then N^2 carries a Hermitian metric of zero curvature and the proof follows from an argument similar to Theorem 3.2.
2. The proof follows from the fact that every compact connected Lie group is a Riemannian symmetric space of positive Ricci curvature with respect to each bi-invariant structure.

□

It is known that a complete Riemannian locally symmetric space is Riemannian covered by a Riemannian globally symmetric space. In view of Corollary 3.3, we comment that if a complete Riemannian locally symmetric space M is covered by a globally symmetric space of compact type, there exists no non-constant harmonic morphism from M to a compact Riemann surface of genus $g \geq 1$.

Note that many examples of compact Einstein manifolds with positive scalar curvature are given in [8]. A non-existence result for harmonic morphisms from such manifolds is given by the following.

Corollary 3.4. *Let M^m be a compact Einstein manifold with positive* scalar *curvature. There exist no non-constant harmonic morphisms from M^m to a compact Riemann surface of genus $g \geq 1$.*

4. Bochner technique for morphisms of p-harmonic functions

In Section 2 we obtained a method of generalizing the Weitzenböck formula for harmonic maps (also called 2-harmonic maps) to harmonic morphisms (or morphisms of 2-harmonic functions). In this section we show that this method works well to obtain a Weitzenböck formula for morphisms of p-harmonic functions.

For a smooth map $\phi\colon M^m \to N^n$ the p-energy of ϕ on a compact domain $\Omega \subset M$ is defined as

$$E_p(\phi) = \frac{1}{p}\int_\Omega \|d\phi\|^p v^M$$

where, at a point $x \in \Omega$,

$$\|d\phi\|^p = (\sum_{s=1}^{m} \langle d\phi \cdot e_s, d\phi \cdot e_s\rangle)^{\frac{p}{2}}$$

for an orthonormal basis $(e_s)_{s=1}^m$ of T_xM.

Definition 4.1. A map $\phi\colon M^m \to N^n$ is called *p-harmonic* if it is a critical point of the p-energy functional.

The first variational formula of the p-energy, cf. [3], implies that a smooth map $\phi\colon M^m \to N^n$ is p-harmonic if and only if its tension field $\tau_p(\phi) = 0$ where

$$\tau_p(\phi) = \sum_{s=1}^{m} \{\nabla_{e_s}(\|d\phi\|^{p-2} d\phi)\}(e_s)$$

and ∇ is the connection on the bundle $T^*M \otimes\ \phi^{-1}TN$.

Definition 4.2. A map $\phi\colon M^m \to N^n$ is a *morphism of p-harmonic functions* if for a p-harmonic function f on an open set V of N, the function $f \circ \phi$ is a p-harmonic function on $\phi^{-1}(V)$, with $\phi^{-1}(V)$ non-empty.

For $p = 2$ morphisms of p-harmonic functions are the harmonic morphisms defined in the usual sense. Despite some work being done in this direction, a characterization of morphisms of p-harmonic functions for $p > 2$, analogous to Theorem 1.4, has not yet been obtained. The only result in this direction is the following.

Theorem 4.3. [31] *Let ϕ be a p-harmonic and horizontally conformal map from M^m to N^n. Then ϕ is a morphism of p-harmonic functions.*

We obtain the following Weitzenböck formula for horizontally conformal p-harmonic maps, from a Weitzenböck formula for p-harmonic maps [31].

Proposition 4.4. *Let ϕ be a p-harmonic and horizontally conformal map between Riemannian manifolds $(M^m, \langle\cdot,\cdot\rangle^M)$ and $(N^n, \langle\cdot,\cdot\rangle^N)$ with dilation λ. Then*

$$\begin{aligned}\frac{n^{\frac{p}{2}}}{2p}\Delta\lambda^p =& -\mathbf{div}(w^\sharp)\\ &-(p-2)n^{\frac{p-4}{2}}\lambda^{p-4}\sum_{t=1}^m\Big(\sum_{s=1}^n\langle\nabla_{e_t}d\phi\cdot e_s, d\phi\cdot e_s\rangle\Big)^2 \qquad (4.1)\\ &+n^{\frac{p-2}{2}}\lambda^{p-2}\{-\|\nabla d\phi\|^2+\lambda^4\mathbf{Scal^N}-\lambda^2\mathbf{Scal^M}|_{\mathbf{H}}\}\end{aligned}$$

where $w^\sharp$ is a vector field on M defined by

$$\langle w^\sharp, X\rangle^M = w(X) = \|d\phi\|^{p-2}\sum_{t=1}^m\langle(\nabla_{e_t}d\phi)e_t, d\phi\cdot X\rangle$$

for any vector field X on M.

Proof. Let $(e_s)_{s=1}^m$ be an orthonormal basis of T_xM. The Weitzenböck formula for p-harmonic maps can be written, cf. [31], as

$$
\begin{aligned}
\frac{1}{p}\Delta\|d\phi\|^p = &- \mathbf{div}(w^\sharp) \\
&- (p-2)\|d\phi\|^{p-4}\sum_{t=1}^{m}(\sum_{s=1}^{m}\langle\nabla_{e_t}d\phi\cdot e_s, d\phi\cdot e_s\rangle)^2 \\
&+ \|d\phi\|^{p-2}\{-\|\nabla d\phi\|^2 + \sum_{s,t=1}^{m}\langle\mathbf{R^N}(d\phi\cdot e_s, d\phi\cdot e_t)d\phi\cdot e_s, d\phi\cdot e_t\rangle^N \\
&- \sum_{s=1}^{m}\langle d\phi\cdot\mathbf{Ricci^M}e_s, d\phi\cdot e_s\rangle^M\}
\end{aligned}
$$

where $w^\sharp$ is a vector field on M defined as in the statement.

By applying the horizontal conformality condition to the above Weitzenböck formula and doing calculations similar to those of Proposition 2.1, we obtain the Weitzenböck formula for horizontally conformal p-harmonic maps. $\square$

This result can be used to determine conditions under which a horizontally conformal p-harmonic map reduces to a harmonic morphism.

Theorem 4.5. *Let $(M^m, \langle\cdot,\cdot\rangle^M)$ and $(N^n, \langle\cdot,\cdot\rangle^N)$ be Riemannian manifolds with $\mathbf{Scal^M}|_{\mathbf{H}} \geq 0$ and $\mathbf{Scal^N} \leq 0$. Suppose that M is compact. Then every horizontally conformal p-harmonic map $\phi\colon M^m \to N^n$ is a harmonic morphism.*

Proof. If we integrate Equation (4.1) over M, then an analysis similar to the proof of Theorem 2.5 tells us that ϕ is totally geodesic. Hence for any $X \in \mathcal{C}(TM)$, $X(\|d\phi\|^p) = 0$ cf. Lemma 2.3. Therefore, $\|d\phi\|^p$ is constant and ϕ is a harmonic morphism. $\square$

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