An Existence Result for the Generalized Vector Equilibrium Problem

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Abstract—In this paper, we prove an existence result for the generalized vector equilibrium problem by using Fan-Browder type fixed-point theorem due to [1]. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

Let X and Y be two topological vector spaces. Let K be a nonempty convex subset of X and $F : K \times K \to 2^Y$ be a multifunction, where 2^A denotes the set of all nonempty subsets of set A. Let $C : K \to 2^Y$ be multifunction such that for each $x \in K$, $C(x)$ is a closed convex cone with $\operatorname{int} C(x) \neq \emptyset$, where $\operatorname{int} C(x)$ denotes the interior of $C(x)$. Then, we consider the *generalized vector equilibrium problem* (in short, GVEP) is to find

$$\bar{x} \in K \text{ such that } F(\bar{x}, y) \nsubseteq -\operatorname{int} C(\bar{x}), \qquad \text{for all } y \in K. \tag{1}$$

This problem was considered by Ansari *et al.* [2] and Oettli and Schläger [3] which includes vector variational inequality and vector equilibrium problems as special cases (see, for example [4–15] and references therein).

The main object of this paper is to establish an existence result for GVEP by using Fan-Browder [16,17] type fixed-point theorem due to [1].

A multifunction $T : X \to 2^Y$ is said to be *upper semicontinuous* on X if, for each $x_0 \in X$ and any open set V in Y containing $F(x_0)$, there exists an open neighborhood U of x_0 in X such that $F(x) \subset V$ for all $x \in U$.

Let $T : X \to 2^Y$ be a multifunction. The *graph* of T, denoted by $\mathcal{G}(T)$, is

$$\mathcal{G}(T) = \{(x, z) \in X \times Y : x \in X, \ z \in T(x)\}.$$

The *inverse* T^{-1} of T is the multifunction from $\mathcal{R}(T)$, range of T, to X defined by

$$x \in T^{-1}(y) \text{ if and only if } y \in T(x).$$

We shall use the following Fan-Browder type fixed-point theorem due to [1] to prove the main result of this paper.

THEOREM 1.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^K$ be multifunction such that*

(a) *for each $x \in K$, $T(x)$ is nonempty convex subset of K;*
(b) *for each $y \in K$, $T^{-1}(y)$ contains relatively open subset O_y of K (O_y may be empty for some $y \in K$) such that $\bigcup_{x \in K} O_x = K$;*
(c) *K contains a nonempty subset D_0 which is contained in a compact convex subset D_1 of K such that the set $D = \bigcap_{x \in D_0} O_x^c$ is either empty or compact, where O_x^c denotes the complements of O_x in K.*

Then, there exists a point $x_0 \in K$ such that $x_0 \in T(x_0)$.

2. EXISTENCE RESULTS

Let X and Y be two topological vector spaces and K be a nonempty convex subset of X.

A multifunction $F : K \times K \to 2^Y$ is called C_x-*quasiconvex-like* if, for all $x, y_1, y_2 \in K$, and $\alpha \in [0, 1]$, we have either $F(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq F(x, y_1) - C(x)$ or $F(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq F(x, y_2) - C(x)$.

To show the class of C_x-quasiconvex-like multifunctions is nonempty, we give the following example.

EXAMPLE 2.1. *Let $K - [0, 1]$, $C(x) = [0, +\infty)$, for all $x \in K$. We define $F : K \times K \to 2^{\mathbb{R}}$ by*

$$F(x, y) = [x, y + 1], \qquad \text{for all } x, y \in K.$$

For all $x, y_1, y_2 \in K$ and $0 \le \alpha \le 1$, we note that

$$\text{if } y_1 \le y_2, \qquad \text{then } \alpha y_1 + (1 - \alpha)y_2 \le y_2$$

and

$$\text{if } y_1 > y_2, \quad \text{then } \alpha y_1 + (1 - \alpha)y_2 \le y_1.$$

Therefore, we have for each $t \in F(x, \alpha y_1 + (1 - \alpha)y_2)$,

$$t = \begin{cases} (y_2 + 1) - [(y_2 + 1) - t], & y_1 \le y_2, \\ (y_1 + 1) - [(y_1 + 1) - t], & y_1 > y_2. \end{cases}$$

Hence, we have either $F(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq F(x, y_1) - C(x)$ or $F(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq F(x, y_2) - C(x)$. Thus, F is C_x-quasiconvex-like.

Now we are ready to prove the following main result of this paper.

THEOREM 2.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space X and Y be a topological vector space. Let $F : K \times K \to 2^Y$ be a multifunction. Assume that*

(i) *$C : K \to 2^Y$ is a multifunction such that for each $x \in K$, $C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \ne \emptyset$;*

(ii) $W : K \to 2^Y$ is a multifunction defined as $W(x) = Y \setminus \{-\text{int } C(x)\}$, for each $x \in K$ such that the graph of W is closed in $K \times Y$;

(iii) for each $y \in K$, $F(\cdot, y)$ is upper semicontinuous with compact values on K;

(iv) there exists a multifunction $G : K \times K \to 2^Y$ such that

   (a) for each $x \in K$, $G(x, x) \not\subseteq -\text{int } C(x)$,

   (b) for each $x, y \in K$, $F(x, y) \subseteq -\text{int } C(x)$, implies $G(x, y) \subseteq -\text{int } C(x)$,

   (c) G is C_x-quasiconvex-like on K;

(v) there exists a subset D_0 contained in a nonempty compact convex subset D_1 of K such that for each $x \in K \setminus D_1$, there exists $y \in D_0$ with $F(x, y) \subseteq -\text{int } C(x)$.

Then, the solutions set $S = \{x \in K : F(x, y) \not\subseteq -\text{int } C(x)$, for all $y \in K\}$ is a nonempty and compact subset of D_1.

PROOF. We define $Q(y) = \{x \in K : F(x, y) \not\subseteq -\text{int } C(x)\}$, for all $y \in K$. Then the solution set $S = \bigcap_{y \in K} Q(y)$. We note that for each $y \in K$, $Q(y)$ is closed.

Indeed, let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in $Q(y)$ such that $\{x_\lambda\}$ converges to x. Then we have $F(x_\lambda, y) \not\subseteq -\text{int } C(x_\lambda)$ for each $y \in K$, that is, there exists $z^\lambda \in F(x^\lambda, y)$ such that $z^\lambda \notin -\text{int } C(x^\lambda)$, or $z^\lambda \in W(x^\lambda)$, for all $\lambda \in \Lambda$. Let $A = \{x^\lambda\} \cup \{x\}$. Then A is compact and $z_\lambda \in F(A, y)$ which is compact. Therefore, $\{z_\lambda\}$ has a convergent subnet with limit z. Without loss of generality, we may assume that $\{z_\lambda\}$ converges to z. Then, by the upper semicontinuity of $F(\cdot, y)$, we have $z \in F(x, y)$. Also since W has a closed graph in $K \times Y$, we have $z \in W(x)$. Consequently, $z \in F(x, y)$ and $z \notin -\text{int } C(x)$, i.e., $F(x, y) \not\subseteq -\text{int } C(x)$. Hence, $x \in Q(y)$ and so $Q(y)$ is closed as claimed.

Now we shall prove that the solution set S is nonempty. Assume to the contrary that $S = \emptyset$, if possible. Then, for each $x \in K$, the set

$$P(x) = \{y \in K : x \notin Q(y)\} = \{y \in K : F(x, y) \subseteq -\text{int } C(x)\} \neq \emptyset.$$

From Assumption (iv)(b), we have, for each $x \in K$,

$$H(x) = \{y \in K : G(x, y) \subseteq -\text{int } C(x)\} \supset \{y \in K : F(x, y) \subseteq -\text{ int } C(x)\} = P(x),$$

and hence, for each $x \in K$, $H(x)$ is nonempty. Also, for each $x \in K$, $H(x)$ is convex.

To see this, let $y_1, y_2 \in H(x)$, then for each $x \in K$, $G(x, y_1) \subseteq -\text{int } C(x)$ and $G(x, y_2) \subseteq -\text{int } C(x)$. Since G is C_x-quasiconvex-like, for all $\alpha \in [0, 1]$, we have either

$$G(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq G(x, y_1) - C(x) \subseteq -\text{int } C(x) - C(x) \subseteq -\text{int } C(x)$$

or

$$G(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq G(x, y_2) - C(x) \subseteq -\text{int } C(x) - C(x) \subseteq -\text{int } C(x).$$

In both cases, we get $G(x, \alpha y_1 + (1 - \alpha)y_2) \subseteq -\text{int } C(x)$. Hence, $\alpha y_1 + (1 - \alpha)y_2 \in H(x)$, and therefore, $H(x)$ is convex. Thus, $H : K \to 2^K$ defines a multifunction such that for each $x \in K$, $H(x)$ is nonempty and convex. Now for each $x \in K$, the set

$$
\begin{aligned}
H^{-1}(y) &= \{x \in K : y \in H(x)\} \\
&= \{x \in K : G(x, y) \subseteq -\text{int } C(x)\} \\
&\supset \{x \in K : F(x, y) \subseteq -\text{int } C(x)\} \\
&= \{x \in K : F(x, y) \not\subseteq -\text{int } C(x)\}^c \\
&= [Q(y)]^c \\
&= O_y,
\end{aligned}
$$

which is a relatively open set in K. From Assumption (v), for each $x \in K \setminus D_1$, there exists $y \in D_0$ such that $F(x, y) \subseteq -\text{int } C(x)$, that is, $x \notin Q(y)$. This implies that $D = \bigcap_{y \in D_0} O_y^c =$

$\bigcap_{y \in D_0} Q(y) \subset D_1$. Since $O_y = \{x \in K : F(x,y) \subseteq -\text{int}\, C(x)\} P^{-1}(y)$, we get $\bigcup_{y \in K} O_y = \bigcup_{y \in K} P^{-1}(y) = K$.

To see this, let $x \in K$. As $P(x) \neq \emptyset$, we can choose $y \in P(x)$. Hence, $x \in P^{-1}(y) = O_y$. By Theorem 1.1, there exists a point $\bar{x} \in H(\bar{x})$, that is , $G(\bar{x}, \bar{x}) \subset -\text{int}\, C(\bar{x})$, which is a contradiction of Assumption (iv)(a). Hence, the solution set S is nonempty. We conclude the proof by noting that $S = \bigcap_{y \in K} Q(y)$ being a closed subset of the compact set $D = \bigcap_{y \in D_0} Q(y)$ is compact and this completes the proof.

REMARK 2.1. When $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$, and F and G are single-valued maps from $K \times K$ to $\mathbb{R}$, Theorem 2.1 reduces to Lemma 2.1 in [18].

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