KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 345-(092)

Major Exam 1

Time: 90 Minutes

Name: Solution I.D. #

Show All Necessary Work

Question	Points
1	/20 3+4+13
2	/20 12+8
3	/10 2×5
4	/20 5+10+5
5	/16 8+8
6	/14 2×7
Total	/100

7C

1. **(a)** Define the order of an element in a group G.

The order of an element g in a group G is the smallest +ve integer n such that $g^n = e$. If no such integer exists, then g has infinite order.

(b) Find the order of the element 5 in the group $U(16)$.

$U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$

$|5| = 4$, since $5^4 = 1 \mod 16$.

(c) Let a be an element of order n in a group and let k be a positive integer. Prove that

$$<a^k> \; = \; <a^{\gcd(n,k)}> \text{ and } |a^k| = \frac{n}{\gcd(n,k)}.$$

See Your notes

2. (a) State and prove Lagrange's Theorem.

See your notes

(b) Let H be a subgroup of a group G. Prove that any two left cosets of H are either identical or disjoint.

See your notes

3. Consider the permutations:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 2 & 5 & 8 & 3 & 4 & 1 \end{bmatrix}, \text{ and } \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}.$$

(a) Write α and β as a product of disjoint cycles.

$$\alpha = (17458)(263)$$
$$\beta = (23847)(56)$$

(b) Write α, β and αβ as a product of transpositions.

$$\alpha = (18)(15)(14)(17)(23)(26)$$
$$\beta = (27)(24)(28)(23)(56)$$
$$\alpha\beta = (176853)(2)(4) \Rightarrow \alpha\beta = (13)(15)(18)(16)(17)$$

(c) Find the order of α.

$|\alpha| = 15$, Since $lcm(5,3) = 15$, from part (a).

(d) Decide whether α is even or odd permutation.

From (b), we can see clearly that α is even.

(e) Compute α^{107}.

Note that $\alpha^{15} = \epsilon$. So,

$$\alpha^{107} = \alpha^{105+2} = \alpha^{105}\alpha^{2} = \epsilon\alpha^{2} = \alpha^{2}$$
$$= (14875)(236)$$

4. (a) Show that the group $U(14)$ is cyclic and find all its generators.

$U(14) = \{1, 3, 5, 9, 11, 13\}$

$3^1 = 3$
$3^2 = 9$
$3^3 = 13 \pmod{14}$
$3^4 = 11$
$3^5 = 5$
$3^6 = 1$

$\Rightarrow \langle 3 \rangle = U(14) \Rightarrow U(14)$ is cyclic group generated by 3.

To find other generators: Recall that, since $|U(14)| = 6$, 3^k is a generator iff $(k,6) = 1$ iff $k = 1, 5$. i.e. $\underset{5}{3^5}$ and 3

$\therefore$ The generators are 3 and 5.

(b) Prove that a subgroup of a cyclic group is cyclic.

See your notes

(c) How many elements of order 10 in a cyclic group of order 40?

If G is cyclic group with $|G| = 40$,

the number of elements of order 10 is $\phi(10) = 4$.

Remember $\phi(10) = |U(10)| = 4$.

5. **(a)** Let G be the general linear group GL(2,R). Consider the subgroup:

$$H = SL(2,R) = \{y \in G : \det(Y) = 1\}.$$

If a and b are in G such that $aH = bH$, show that $\det(a) = \det(b)$.

Suppose that $aH = bH$. Then

$$a^{-1}b \in H$$

$$\Rightarrow \det(a^{-1}b) = 1$$

$$\Rightarrow \det(a^{-1})\det(b) = 1$$

$$\Rightarrow \frac{1}{\det(a)}\det(b) = 1$$

$$\Rightarrow \det(a) = \det(b).$$

(b) Prove that if each element in a group G is of order 2 then G must be abelian.

Suppose that each element in G is of order 2.

$\Rightarrow \quad x^2 = e \quad \forall x \in G$. Thus

$(ab)^2 = e \quad \forall a, b \in G.$

Also, $a^2b^2 = ee = e$

so, $(ab)^2 = a^2b^2$

$\Rightarrow abab = aabb$

$\Rightarrow a^{-1}(abab)b^{-1} = a^{-1}(aabb)b^{-1}$

$\Rightarrow \quad ba = ab \quad \forall a, b \in G$

$\Rightarrow \quad G$ is abelian.

6. **Write *True* or *False* for each of the following:**

(a) **Every abelian group is cyclic** .... (✗)

The Klein group K_4 is abelian but not cyclic

(b) **A group of order 20 must have exactly one element of order 2.** (✗)

(c) Z^*_{14} **is a group under multiplication modulo 14.** .... (✗)

(d) **The set** $H = \{x \in R^* : x \geq 1\}$ **is a subgroup of the group** $(R^*, .)$. .... (✗)

$3 \in H$ but it's inverse $\frac{1}{3} \notin H$

(e) $13 = -2 \bmod 15$. .... (✓)

(f) **The Klein group** K_4 **is abelian.** .... (✓)

(g) **A group of order 41 must be abelian.** .... (✓)