Section 15.5 *Triple integrals*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. evaluate triple integrals over rectangular regions
2. evaluate triple integrals over general regions like
   - **a.** simple XY-regions
   - **b.** simple YZ-regions
   - **c.** simple XZ-regions
3. apply triple integrals to find volume of 3-dimensional regions

Triple integrals

- We have used double integrals to integrate $f(x, y)$ over a 2-d region
- To integrate $f(x, y, z)$ over a 3-d region G we require triple integration
  - o Notation: $\boxed{\iiint_G f(x, y, z)\,dV}$

Evaluating triple integrals over rectangular boxes

Evaluated as iterated integrals

Let G be the rectangular box $a \le x \le b$, $c \le y \le d$, $k \le z \le l$ then

$$\boxed{\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx}$$

- Or any other ordering with proper adjustment of limits of integration.
- See class explanation of this.

Example 15.5.1 Evaluate $I = \iiint_G 8xyz\,dV$ over the region R given by $2 \le x \le 3$, $0 \le y \le 1$, $1 \le z \le 2$.

Solution: Done in class.

Evaluating triple integrals over simple XY-region

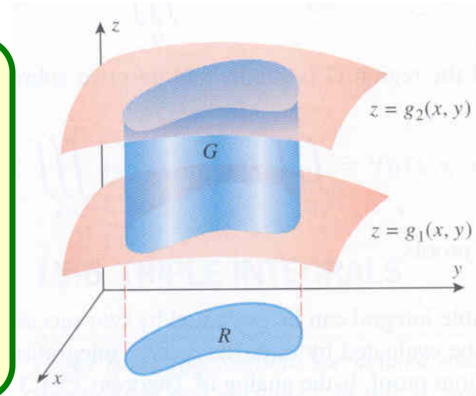
See class explanation

Simple XY-region

A region G given by

- $G = \{(x,y,z) : (x,y) \in R \text{ and } g_1(x,y) \le z \le g_2(x,y)\}$

where R is projection of G on XY-plane

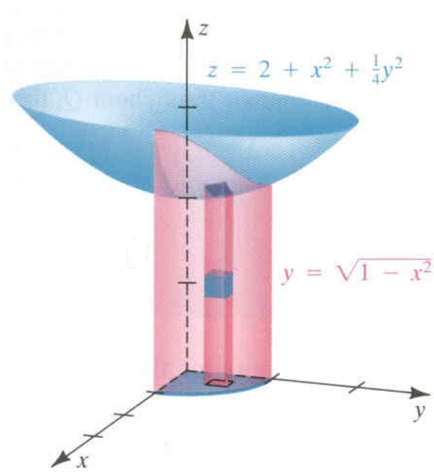


Evaluation of triple integral over such regions as

- $\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right] dA$

Example 15.5.2 Express $I = \iiint\limits_G f(x,y,z)\,dV$ as an iterated integral if G is the region in the first octant bounded by the coordinate planes, the paraboloid $z = 2 + x^2 + \dfrac{y^2}{4}$ and the cylinder $x^2 + y^2 = 1$.

Figure:



Solution: Done in class.

Evaluating triple integrals over simple YZ-region

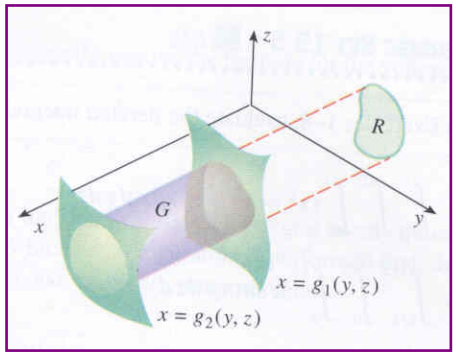
See class explanation

Simple YZ-region

A region G given by

- $G = \{(x,y,z) : (y,z) \in R \text{ and } g_1(y,z) \le x \le g_2(y,z)\}$

where R is projection of G on YZ-plane



Evaluation of triple integral over such regions as

- $$\iiint\limits_G f(x,y,z)\,dV = \iint\limits_R \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right] dA$$

Example 15.5.3 Let G be the region in the first octant cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate $I = \iiint\limits_G z\,dV$.

- Try to project the region G on XY, YZ, XZ planes
- and see which gives you easier calculations and

  better visualization of G & R.
- **Easiest:** thinking as integral over YZ-region

Solution: Done in class

Evaluating triple integrals over simple XZ-region

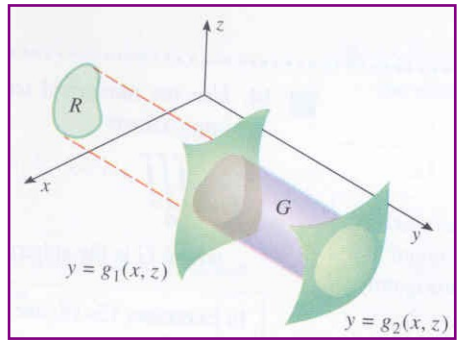
See class explanation

Simple XZ-region

A region G given by

- $G=\{(x,y,z):(x,z)\in R \text{ and } g_1(x,z)\le y\le g_2(x,z)\}$

where R is projection of G on XZ-plane



Evaluation of triple integral over such regions as

- $\iiint_G f(x,y,z)dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)dy\right]dA$

Exercise

Evaluate $I=\iiint_G \sqrt{x^2+z^2}\,dV$ where G be the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$.

- Try to project the region G on XY, YZ, XZ planes
- and see which gives you easier calculations and better visualization of G & R.
- **Easiest:** thinking as integral over XZ-region

Volume as triple integral

The volume of a 3-dimensional region G is given by

$$\boxed{V = \iiint_G dV}$$

Example 15.5.4 Set up an integral for finding the volume of the region G bounded by $x^2 + y^2 = 4$, $x + z = 9$, $y - z = 6$.

- **Easiest:** thinking as integral over XY-region (Why?)

Solution: Done in class

Do Qs: 1-24 & 31, 32

End of 15.5