Section 15.3 *Double integrals in polar coordinates*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. evaluate double integrals in polar coordinates
2. convert double integrals from rectangular coordinates to polar coordinates.

Double integrals in polar coordinates

- Integrand & the region of integration given in polar coordinates

Advantage

- Some double integrals are easier to evaluate polar coordinates (as we will see below)

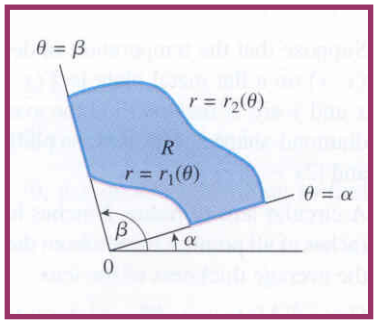
Evaluating double integrals in polar coordinates

Simple polar region

A region defined by

- $\{(r,\theta):\alpha \le \theta \le \beta \text{ and } 0 \le r_1(\theta) \le r \le r_2(\theta)\}$

  where $\beta - \alpha \le 2\pi$



Formula for evaluating double integral in polar coordinates

Given $f(r,\theta)$ over a simple region R given by

$$\{(r,\theta):\alpha \le \theta \le \beta \text{ and } 0 \le r_1(\theta) \le r \le r_2(\theta)\}.$$

Then $$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\,dr\,d\theta$$

See in class
- figure
- use of line to put limits

Note: Area element *dA*

- in rectangular coordinates: $dA = dxdy$
- in polar coordinates: $dA = rdrd\theta$

See class explanation

Example 15.3.1 Evaluate $I = \iint_R \cos\theta dA$ over the region R bounded by

$$r = 1 + \sin\theta,\ r = 3\sin\theta,\ \theta = \frac{\pi}{6} \text{ and } \theta = \frac{\pi}{2}.$$

Solution: Done in class.

Application

- If $f(r,\theta) \geq 0$ then $\iint_R f(r,\theta)\,dA$ gives volume under surface $z = f(r,\theta)$ over the region R.
- $\iint_R dA$ gives area of the region R.

Example 15.3.2 Set up an integral to determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution: Done in class.

Converting double integrals into polar coordinates

Some double integrals are easier to evaluate in polar coordinates.

Especially,

- If the region R consists of a circle or a part of the circle.
  i.e contains an equation of the form $x^2 + y^2 = a^2$
- If the integrand contains expressions of the form $x^2 + y^2 = a^2$.

How to convert?

Use

- $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $x^2 + y^2 = r^2$

Example 15.3.3 Evaluate $I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \left(x^2+y^2\right)^{3/2} dydx$ by converting to polar coordinates.

Solution: Done in class.

Important steps for conversion
- identify the region R
- use it to find limits of integration in polar coordinates

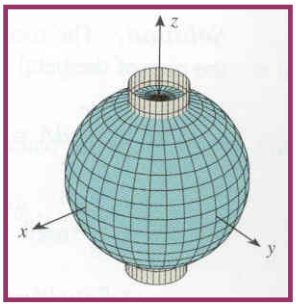
Exercise Find $I = \iint_R e^{x^2+y^2} dA$ over the circle $x^2 + y^2 = 1$

Answer: $I = \int_0^{2\pi}\int_0^r re^{r^2} drd\theta = \pi(e-1)$

Example 15.3.4 Find the volume of solid that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 4$

Solution:

- Figure:



- In rectangular coordinates the volume is given by $V = \iint_R \sqrt{9 - x^2 - y^2}\, dA$ where R is the region inside the circle $x^2 + y^2 = 4$.

Not a nice integration

- Do we get easier integration in polar coordinates?
- See class notes for more.

Do Qs: 1-35

End of Section 15.3