Section 15.1 *Double integrals*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. understand what is meant by double integral of $f(x,y)$
2. evaluate double integrals of $f(x,y)$ over rectangular regions
3. apply double integration to find volume under a non-negative function $f(x,y)$.

Motivation:

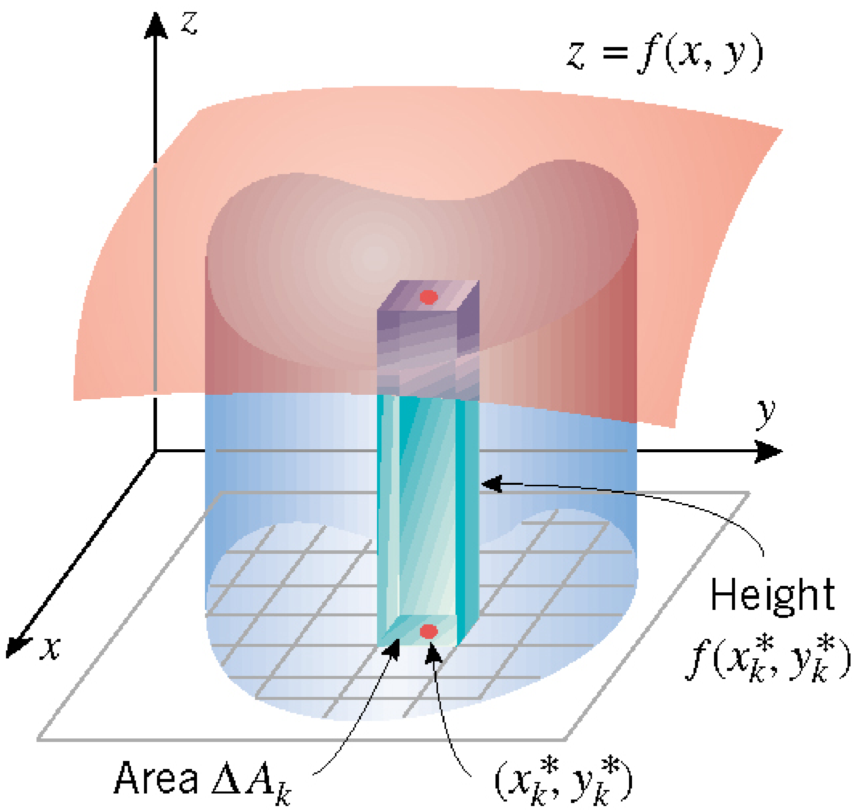
Volume under a surface $z = f(x, y) \geq 0$

- In Math101, we did

  area problem ----------------→ definite integration of $f(x)$

- Here we see that

  volume problem ----------------→ double integration over a region R



- Consider a non-negative function $z = f(x, y)$ over a region R
- Approximate the volume under $z = f(x, y)$, by adding the volumes of cubes {see figure for understanding}
- Volume of the kth cube = $f(x_k^*, y_k^*)\Delta A_k$
- Approximate volume = $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$
- Exact volume = $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$

See class explanation

Double integrals & basic properties

The double integral of a function $f(x,y)$ over a region R is defined as

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

If $f(x,y)\ge 0$ on R then the volume under $f(x,y)$ over the region R is given by

$$\iint_R f(x,y)\,dA$$

The integral $\iint_R 1\cdot dA = \iint_R dA$ gives area of the region R

Basic properties of double integrals

- $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$
- $\iint_R [f(x,y)\pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$
- $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Main Question:

Efficient methods of evaluating double integrals

Computing them by definition is too difficult and impractical

Evaluating double integrals over rectangular regions

more general regions in next section

If the region R is defined by $a \le x \le b$, $c \le y \le d$

then

$$\boxed{\iint\limits_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx}$$

i.e. R is rectangular region $[a,b]\times[c,d]$

Note:

- inner limits for inner differential
- outer limits for outer differential

These integrals are evaluated as iterated integrals

- $\boxed{\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy}$
- $\boxed{\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx}$

See examples 15.1.1, 15.1.2, 15.1.3 done in class

Do whole exercise

End of 15.1