Section 14.2 *Limits and Continuity*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. understand what is meant by limit of a function of two or more variables and

   a. learn how to compute these limits,

   b. get an idea about "How to check existence of these limits"

2. understand what is meant by a continuous function of two or more variables and

   a. learn how to check continuity of a function of two or more variables

Definition of $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$

Formal Definition

Assume the function f is defined at all points within a disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \mathrm{L}$$

if for any given number $\varepsilon > 0$, we can find number $\delta > 0$ such that $f(x,y)$ satisfies

$$|f(x,y) - L| < \varepsilon$$

whenever the distance between (x,y) and (x_0, y_0) satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Informal Definition

When we move any point (x,y) closer & closer to (x_0, y_0), without actually making it (x_0, y_0),

- Is there a real number L such that the value of $f(x,y)$ becomes closer and closer to L.
  - if yes then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$
  - if not then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist

Question: In how many ways can $(x,y) \to (x_0, y_0)$?

Answer: Infinite number of ways, e.g. along different lines or curves.

See class explanation

- Here we are concerned mainly with computations of limits
- We begin with learning "How to compute limit by approaching (x_0, y_0) along a given path"

Example 14.2.1 Find the limit of $f(x,y)=\dfrac{x^3 y}{2x^6+y^2}$ as $(x,y)\to(0,0)$ along

(a) the line $y=x$ **(b)** the curve $y=x^3$

Solution Done in class

Though we will be mostly consider $f(x,y)$, all of our definitions above and study below hold for functions of more variables

Our three main questions

1) When the limit does not exist?

2) Techniques for computation of limits?

(Assuming that limit exists)

3) Existence of limits question.

Below, we study these questions one by one.

Q.1: When the limit does not exit?

If $f(x,y)$ has different limits as $(x,y)\to(x_0,y_0)$ along two different paths then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exits

- We learn the method through different examples.

Show that the following limits do not exist

Example 14.2.2 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+3y^4}$

Use of paths along coordinate axes, lines etc

Example 14.2.3 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

Use of other paths e.g. lines, parabolas etc

Example 14.2.4 $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x^2+y^2}$

infinite limit

Example 14.2.5 $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^3+y^3+z^3}$

Use of different paths e.g. lines in 3-dim.

Example 14.2.6 Use polar coordinates to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Example 14.2.7 Use spherical coordinates to show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$$ does not exist.

All solutions done in class

Q.2: Techniques of computing limits $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

(Assuming limit exists)

Most general technique

- Choose any path (usually a simple one) through the point (x_0, y_0) and compute (like Example 14.2.1) along it.
- Below we demonstrate some alternate tricks for computing limits (these may lead to shorter calculations in some cases)

Alternate tricks

(I) Plug in values directly e.g. $\lim_{(x,y)\to(1,1)} \frac{xy}{x+y} = \frac{1}{2}$

(II) If "direct plugging" in leads to $\left(\frac{0}{0}\right)$ or other indeterminate form then

i. simplify & plug in values [See example 14.2.8]

ii. use substitutions

- polar coordinates [See example 14.2.9]
- spherical coordinates [See example 14.2.10]
- or some other appropriate substitution [see Qs.9-12 in the book]

Compute the following limits

Example 14.2.8 $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

Example 14.2.9 $\lim_{(x,y)\to(0,0)} y\ln\left(x^2 + y^2\right)$

Example 14.2.10 $\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2 + y^2 + z^2}\right]$

All solutions done in class

Q.3: Existence question of limits

To get an idea about "How to show existence of limits",

we look at the following example

Example 14.2.11 Does the limit $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

Solution:

- Computing along X-axis, Y-axis we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along the lines $y = kx$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$
- Computing along parabola $y = x^2$ we get $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

From above, we expect that limit exists and its value is zero but to ensure that the limit exists we should use formal definition.

Since showing existence "using definition" is not part of our syllabus so we don't proceed further. Our purpose was only to get an idea about "how to handle existence of limits questions".

Example 14.2.12 [In some cases, we can reduce problem to single variable case]

Find the limit $f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$, if it exists?

Solution: Done in class

Continuity

A function $f(x,y)$ is continuous at (x_0,y_0) if

i. $f(x_0,y_0)$ is defined

ii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

iii. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Means the graph has no hole or gap at the point (x_0,y_0)

Example 14.2.13 Let $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$ if $(x,y)\neq(0,0)$. Define $f(0,0)$ so that f is continuous at $(0,0)$.

Solution Done in class

Example 14.2.14 Find the region on which the function $f(x,y,z)=\dfrac{x+y+2}{x^2+z^2-4}$ is continuous.

Solution Done in class

End of Section 14.2

Do the whole exercise.