Section 12.7 *Quadric surfaces*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know what are quadric surfaces
2. how to sketch quadric surfaces
3. how to identify different quadric surfaces

Key Idea

- It is not helpful to plot points to sketch a surface
- Mainly we use traces and intercepts to sketch surfaces

Here we only do quadric surfaces

What are quadric surfaces?

A quadric surface is the graph of a 2nd degree equation in 3 variables.

Its most general form is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Main technique for sketching

Determine XY-trace, YZ-trace, XZ-trace, other traces (if needed) and use this information to complete the sketch

The trace of a surface S in a plane is the intersection of S and the plane.

Most important traces

- XY-trace
- YZ-trace
- XZ-trace
- traces in planes parallel to coordinate planes

How to find traces?

- To find XY-trace put $z=0$ in the equation
  - put $z=k$, to get trace in a plane parallel to XY-plane at level k
- To find YZ-trace put $x=0$ in the equation
  - put $x=k$, to get trace in a plane parallel to YZ-plane at level k
- To find XZ-trace put $y=0$ in the equation
  - put $y=k$, to get trace in a plane parallel to XZ-plane at level k

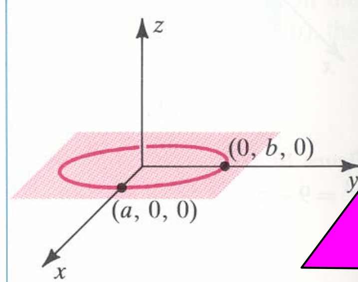
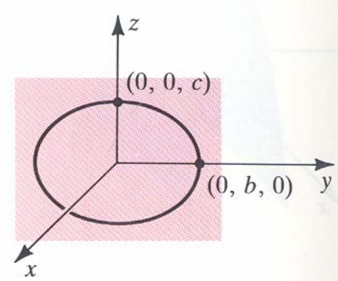
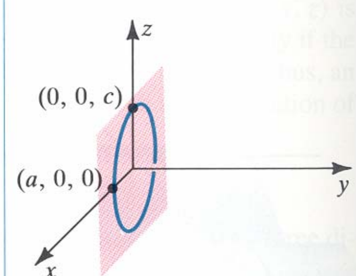
Example Find XY-trace, YZ-trace, XZ-trace of $z=x^2+y^2$.

What will be the traces in planes parallel to XY-plane.

Solution Done in class.

Sketching standard quadric surfaces

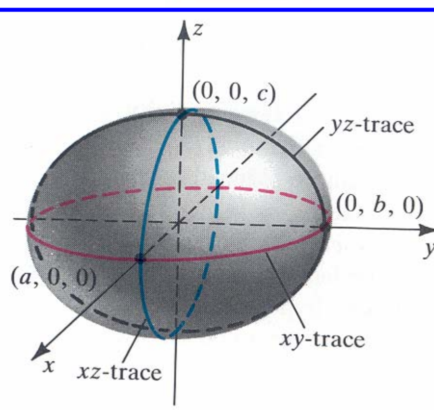
Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Ellipse	
yz-trace	$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipse	
xz-trace	$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$	Ellipse	

What are the traces in planes z=k for

- $|k|<c$
- k=c
- $|k|>c$

If you draw all traces on one coordinate axis, can you guess how the surface should look like?

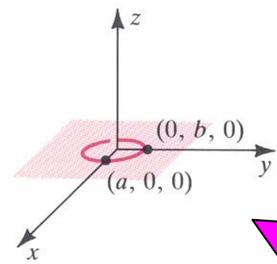
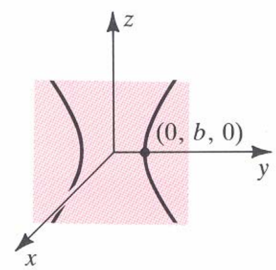
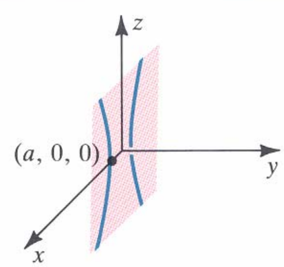


- See the Matlab illustration in class for more understanding of different traces & views
- download Matlab file eg_ellipsoid.m from WebCT and experiment with it.

Sketching standard quadric surfaces

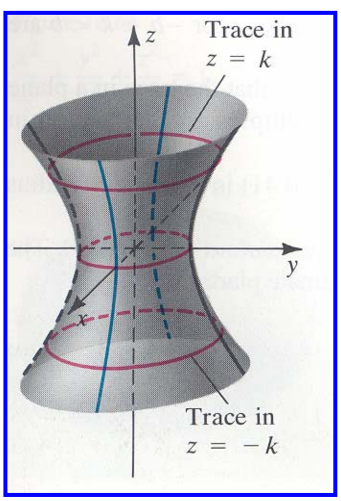
Hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Ellipse	
yz-trace	$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperbola	
xz-trace	$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$	Hyperbola	

If you draw all traces on one coordinate axis, can you guess how the surface should look like?

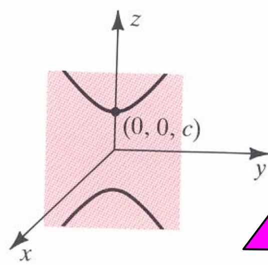
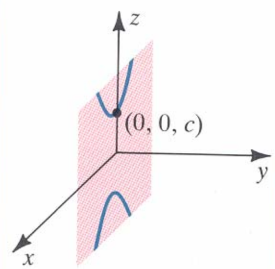
What are the traces in planes parallel to XY-plane? i.e. in the planes z=k



Sketching standard quadric surfaces

Hyperboloid of two sheets

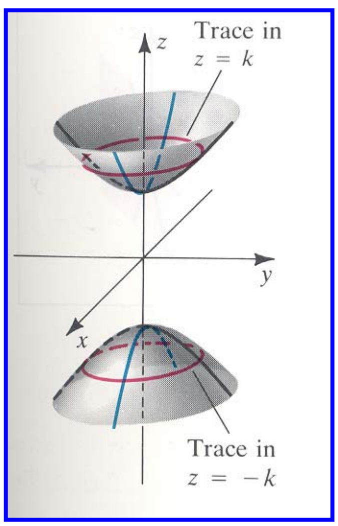
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	None	No graph
yz-trace	$-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Hyperbola	
xz-trace	$-\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$	Hyperbola	

If you draw all traces on one coordinate axis, can you guess how the surface should look like?

What are the traces in planes z=k for

- $|k|<c$
- k=c
- $|k|>c$

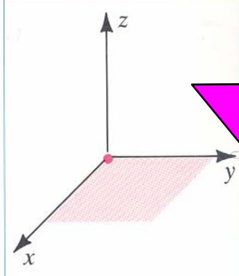
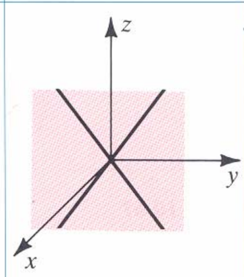
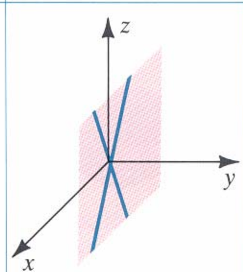


- See the Matlab illustration in class for more understanding of different traces & views
- download Matlab file eg_hyper2.m from WebCT and experiment with it.

Sketching standard quadric surfaces

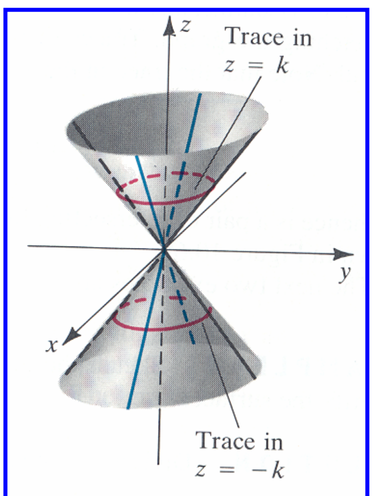
Elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	Equation of trace	Description of trace	Sketch of trace
xy-trace	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$	Origin	
yz-trace	$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	Two intersecting lines	
xz-trace	$\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$	Two intersecting lines	

What are the traces in planes parallel to XY-plane? i.e. in the planes z=k

If you draw all traces on one coordinate axis, can you guess how the surface should look like?

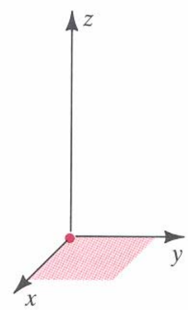
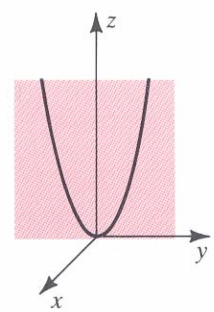
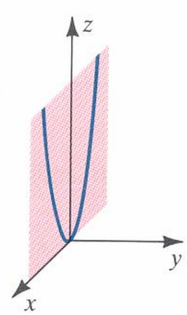


- See the Matlab illustration in class for more understanding of different traces & views
- download Matlab file eg_cone.m from WebCT and experiment with it.

Sketching standard quadric surfaces

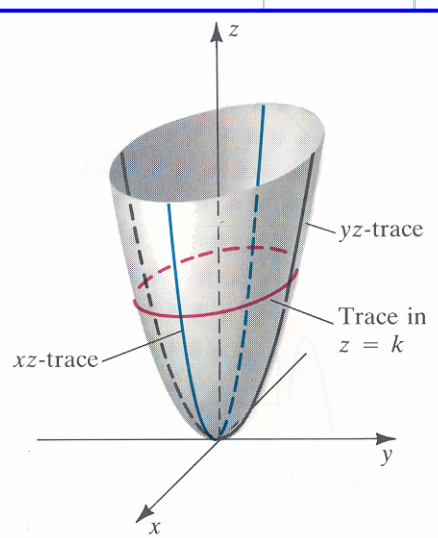
Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$$

Trace	Description of trace	Sketch of trace
xy-trace	Origin	
yz-trace	Parabola	
xz-trace	Parabola	

What are the traces in planes parallel to XY-plane? i.e. in the planes z=k

If you draw all traces on one coordinate axis, can you guess how the surface should look like?



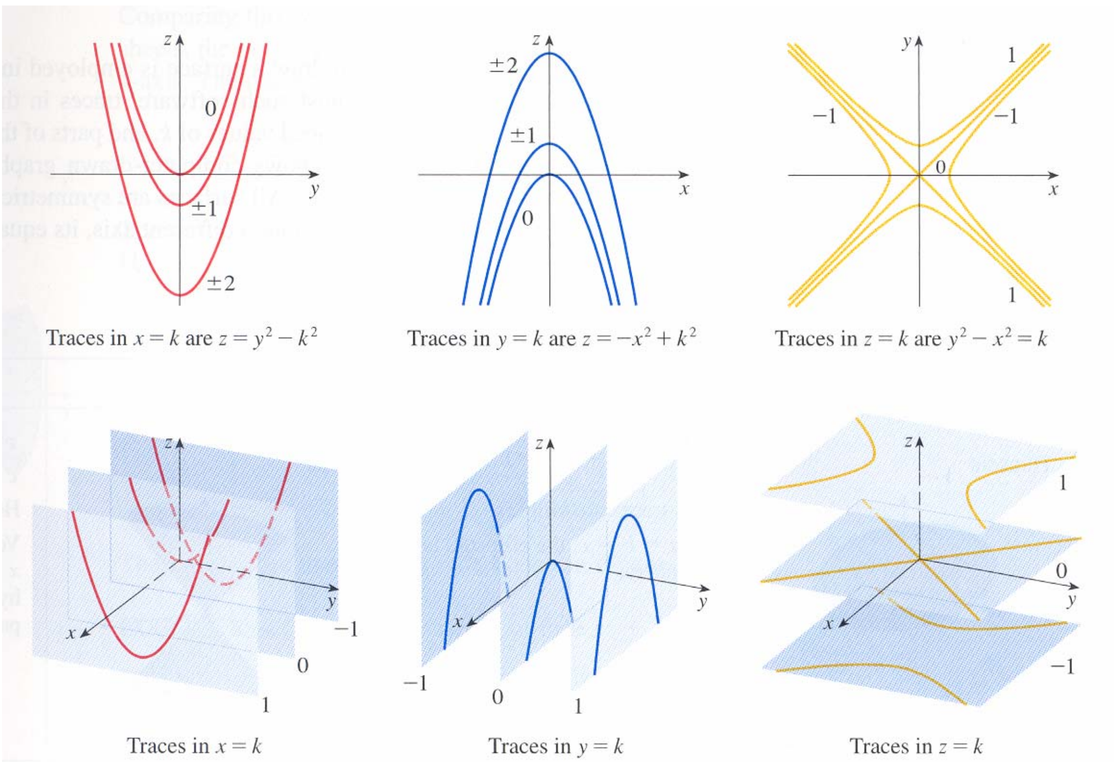
- See the Matlab illustration in class for more understanding of different traces & views
- download Matlab file eg_ paraboloid.m from WebCT and experiment with it.

Sketching standard quadric surfaces

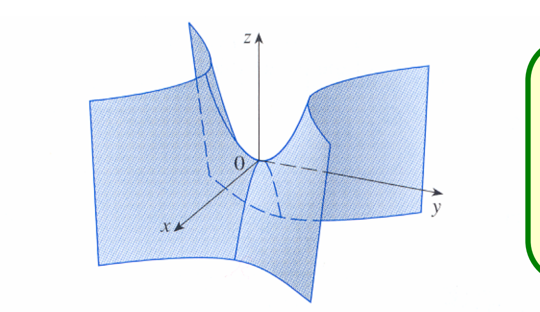
Hyperbolic paraboloid

As an **example** **we first sketch** $z = y^2 - x^2$

- **Traces**



- **Fitting the traces together**



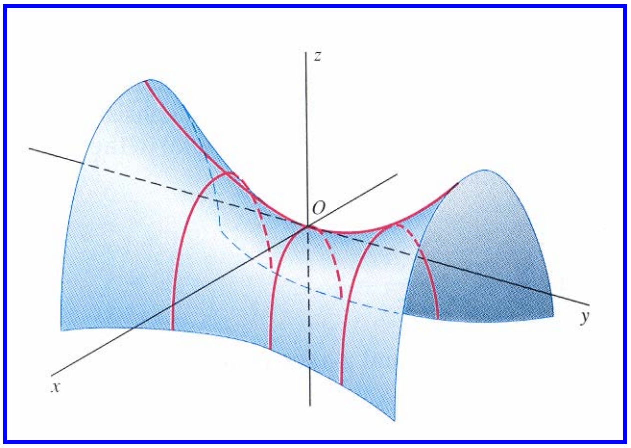
- See the Matlab illustration in class for more understanding of different traces & views
- download Matlab file eg_ hyp_paraboloid.m from WebCT and experiment with it.

Sketching standard quadric surfaces

Hyperbolic paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Similar to above example we have the following graph



See explanation given in class

where the traces

- traces in planes parallel to YZ-plane are parabolas opening upwards
- traces in planes parallel to XZ-plane are parabolas opening downwards
- traces in planes parallel to XY-plane are hyperbolas

Techniques for identifying and sketching quadric surfaces

For this, you must properly understand the graphs of standard quadric surfaces

- Bring the given equation in one of the standard forms
- Compare with standard equations to identify the surface
- Use the graph of standard form to complete the sketch

Complete the square (if needed)

Use the idea of translation of surfaces (when needed)

Translation of a surface

A change from (x, y, z) to $(x-h, y-k, z-l)$ means the surface has translated

- h units along X-axis
- k units along Y-axis
- l units along Z-axis

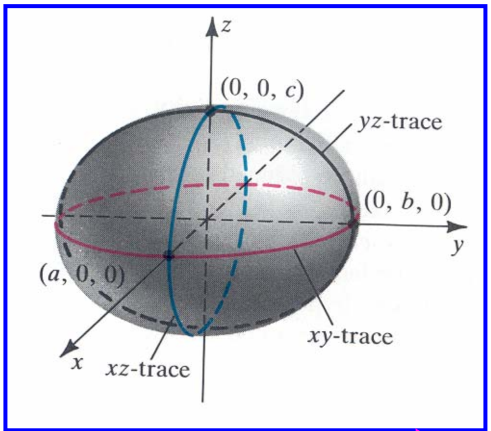
Note: h, k, l may be negative

Example:

$\frac{(x-1)^2}{a^2} + \frac{(y+2)^2}{b^2} + \frac{z^2}{c^2} = 1$ means an ellipsoid with centre at $(1, -2, 0)$.

Identifying quadric surfaces (Examples)

Recall the graph of Ellipsoid



Observations

- Equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- Form $f(x,y,z)=1$ with
  - all variables quadratic
  - all variables with positive coefficient

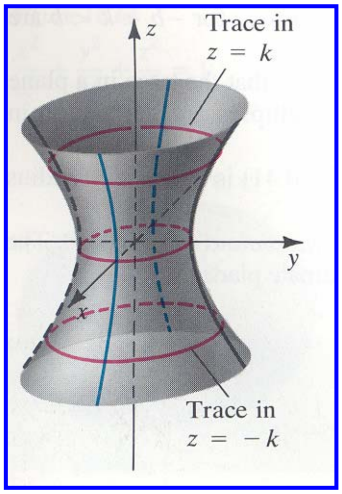
- Centre at $(0,0,0)$
- X-intercept: $\pm a$
- Y-intercept: $\pm b$
- X-intercept: $\pm c$

Example 12.7.1 Describe the surface $4x^2+4y^2+z^2+8y-4z=-4$

Solution Done in class

Identifying quadric surfaces (Examples)

Recall the graph of Hyperboloid of one sheet



Observations

- Equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
- Form $f(x,y,z)=1$ with
  - all variables quadratic
  - one variable with negative coefficient

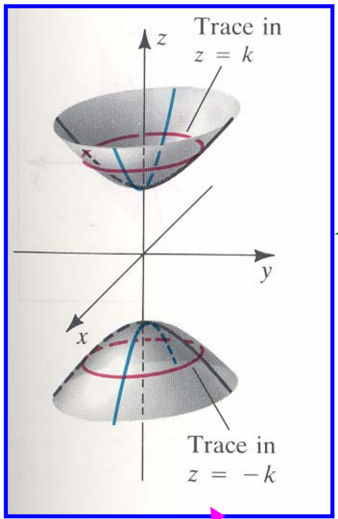
- Axis of symmetry corresponds to variable with negative coefficient
- Opens along axis of symmetry

Example 12.7.2 Sketch the graph of $16x^2-9y^2+36z^2=144$.

Solution Done in class

Identifying quadric surfaces (Examples)

Recall the graph of Hyperboloid of two sheets



Observations

- Equation $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Form $f(x, y, z) = 1$ with
  - all variables quadratic
  - two variable with negative coefficient

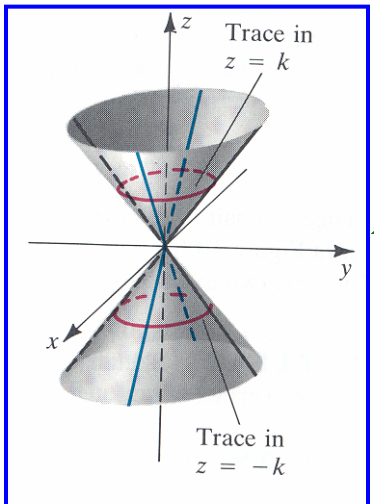
- Axis of symmetry corresponds to variable with positive coefficient
- Opens along axis of symmetry

Example 12.7.3 Identify the surface given by $x^2 - 4y^2 - 2z^2 = 4$.

Solution Done in class

Identifying quadric surfaces (Examples)

Recall the graph of elliptic cone



Observations

- Equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$
- Form $f(x,y,z)=0$ with
  - all variables quadratic
  - sign of coefficient of one variable different from other two.

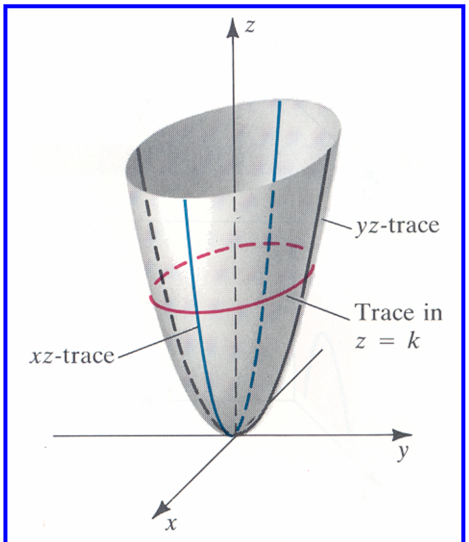
- Axis of symmetry corresponds to variable with different sign
- Opens along axis of symmetry

Example 12.7.4 Sketch the surface $x=\sqrt{y^2+z^2}$.

Solution Done in class

Identifying quadric surfaces (Examples)

Recall the graph of elliptic paraboloid



Observations

- Equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$ with
  - one linear variable
  - two quadratic variables with same sign

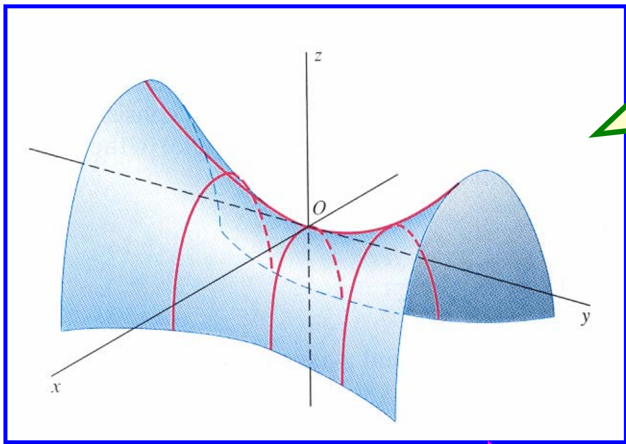
- Axis of symmetry corresponds to variable which is linear

Example 12.7.5 Describe the surface $x^2+2z^2-6x-y+10=0$.

Solution Done in class

Identifying quadric surfaces (Examples)

Recall the graph of hyperbolic paraboloid



Observations

- Equation $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ with
  - one linear variable
  - two quadratic variables with opposite signs

- Such surfaces are like a horse saddle
- To get an idea about orientation
  - imagine the horse saddle and the axis along which the horse would stand
  - e.g. in above figure the horse would stand along Y-axis

Example 12.7.6 Sketch the surface $z = \frac{x^2}{9} - \frac{y^2}{4}$.

Solution Done in class

Final remarks

The identification tricks learnt here are good short cuts but the best approach is to sketch the surfaces by using traces (as we did in the earlier part of these notes.)

Exercise: For the following equations

- Find the (appropriate) traces
- Sketch these traces
- Use these traces to sketch the surface

(a) $z = -x^2 - y^2$

(b) $-\dfrac{x^2}{4} - \dfrac{y^2}{16} + z^2 = 0$

Do Qs: 1-36

End of 12.7