King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 101 EXAM I

Summer Term (083)

Time allowed: 120 Minutes

Name: KEY ID#: ________

Instructor: ________ Section: ________ Serial#: ________

- Show All Your WORK
- WRITE Clear Steps
- Calculator and Mobiles are not allowed

Q#	Marks	Maximum Marks
1		8
2		10
3		24
4		10
5		8
6		8
7		7
8		7
9		8
10		10
Total		100

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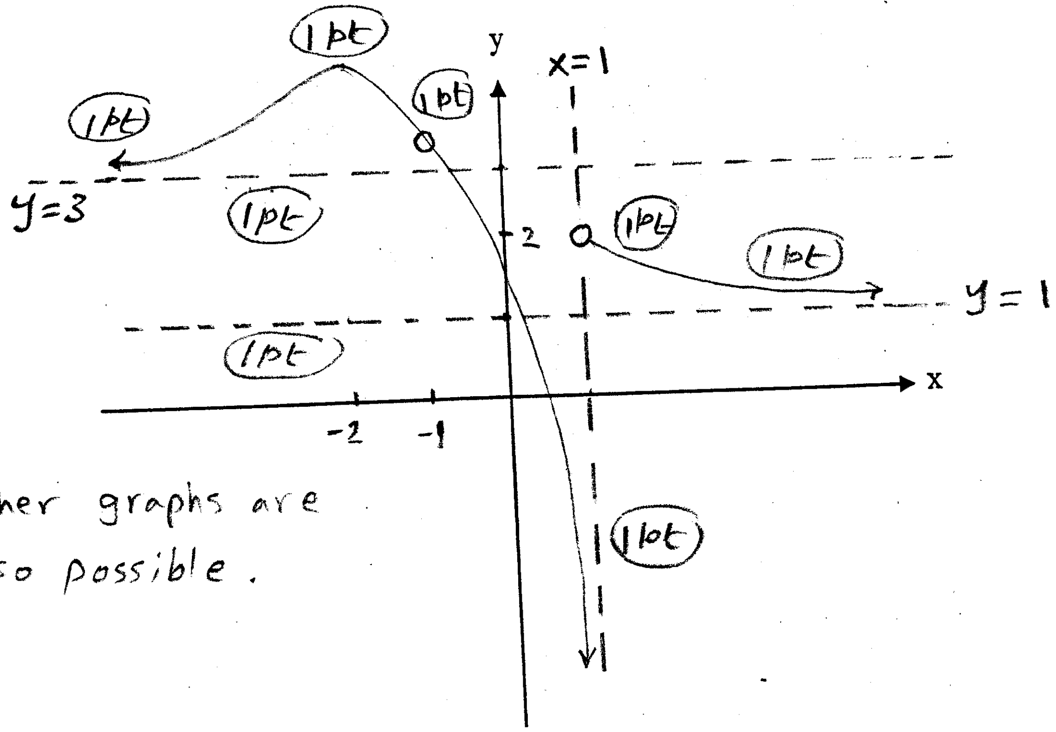
1. **(8 - points)** Sketch the graph of an example of a function f that satisfies the following conditions:

$$\lim_{x\to-\infty} f(x)=3; \quad \lim_{x\to\infty} f(x)=1; \quad \lim_{x\to1^-} f(x)=-\infty;$$

$$f'(-2)=0; \quad \lim_{x\to1^+} f(x)=2; \quad f \text{ has a removable discontinuity at } x=-1.$$



other graphs are also possible.

2. **(10 - points)** Use the Squeeze Theorem to show that

$$\lim_{x\to0^+}\left(\sqrt{x}\,e^{\sin\left(\frac{\pi}{\sqrt{x}}\right)}+1\right)=1$$

For any $x>0$ we have $e^{-1} \leq e^{\sin\left(\frac{\pi}{\sqrt{x}}\right)} \leq e$ (2 pts)

$\Rightarrow \sqrt{x}\,e^{-1} \leq \sqrt{x}\,e^{\sin\left(\frac{\pi}{\sqrt{x}}\right)} \leq \sqrt{x}\,e$ (2 pts)

$\Rightarrow \sqrt{x}\,e^{-1}+1 \leq \sqrt{x}\,e^{\sin\left(\frac{\pi}{\sqrt{x}}\right)}+1 \leq \sqrt{x}\,e+1$ (2 pts)

But $\lim_{x\to0^+}(\sqrt{x}\,e^{-1}+1) = \lim_{x\to0^+}(\sqrt{x}\,e+1) = 1$ (2 pts)

$\Rightarrow \lim_{x\to0^+}\left(\sqrt{x}\,e^{\sin\left(\frac{\pi}{\sqrt{x}}\right)}+1\right)=1$ by the squeezing Theorem. (2 pts)

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3. **(24 points: 6 points each)** Evaluate the limit, if it exists

(3a) $\lim_{x\to 1} \frac{x^3-1}{\sqrt{2x+2}-2}$ $= \lim_{x\to 1} \frac{(x^3-1)(\sqrt{2x+2}+2)}{(2x+2)-4}$ 2 pts

$= \lim_{x\to 1} \frac{(x^3-1)(\sqrt{2x+2}+2)}{2(x-1)}$

$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)(\sqrt{2x+2}+2)}{2(x-1)}$ 2 pts

$= \lim_{x\to 1} \left[\frac{1}{2}(x^2+x+1)(\sqrt{2x+2}+2)\right]$ 2 pts

$= \frac{1}{2}(3)(4) = 6.$

(3b) $\lim_{x\to 1^-} \frac{x^2-|x-1|-1}{|x-1|}$ $= \lim_{x\to 1^-} \frac{x^2+(x-1)-1}{-(x-1)}$ 2 pts

$= \lim_{x\to 1^-} \frac{x^2+x-2}{-(x-1)} = \lim_{x\to 1^-} \frac{(x-1)(x+2)}{-(x-1)}$ 2pts

$= \lim_{x\to 1^-} -(x+2) = -3$ 2pts

(3c) $\lim_{x\to \frac{1}{2}} (x - [|2x|])$, where $[|\ \ |]$ denotes the greatest integer function.

$\lim_{x\to \frac{1}{2}^-} (x - [\![2x]\!]) = \lim_{x\to \frac{1}{2}^-} (x-0) = \frac{1}{2} = L$ 2pts

$\lim_{x\to \frac{1}{2}^+} (x - [\![2x]\!]) = \lim_{x\to \frac{1}{2}^+} (x-1) = -\frac{1}{2} = R$ 2pts

$\Rightarrow L \neq R \Rightarrow$ the given limit DNE. 2pts

(3d) $\lim_{x\to\infty} \ln\left(\dfrac{e^{x+2}-8}{e^x+16}\right)$

Because ln is a continuous function $\Rightarrow$ (1 pt)

$$\lim_{x\to\infty} \ln\left(\frac{e^{x+2}-8}{e^x+16}\right) = \ln \lim_{x\to\infty}\left(\frac{e^{x+2}-8}{e^x+16}\right) \quad \text{(1 pt)}$$

$$= \ln \lim_{x\to\infty}\left(\frac{e^2-8e^{-x}}{1+6e^{-x}}\right) \quad \text{(2 pts)}$$

$$= \ln\left(\frac{e^2}{1}\right) = 2 \quad \text{(2 pts)}$$

4. **(10 - points)** The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $S(t) = \dfrac{3t-1}{t+2}$ where t is measured in seconds. Use limits to find the instantaneous velocity at $t = 3$.

$$v(3) = \lim_{h\to 0}\left[\frac{S(3+h)-S(3)}{h}\right] \quad \text{(2 pts)}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{9+3h-1}{3+h+2} - \frac{8}{5}\right) \quad \text{(3 pts)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{40+15h-(40+8h)}{5(h+5)} \quad \text{(2 pts)}$$

$$= \lim_{h\to 0} \frac{7h}{5h(h+5)} = \lim_{h\to 0} \frac{7}{5(5+h)} \quad \text{(2 pts)}$$

$$= \frac{7}{25} \text{ meters per second.} \quad \text{(1 pt)}$$

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5. **(8 - points)** Use the Intermediate Value Theorem to show that the graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = \cos x$ intersect on the interval $\left[0, \frac{\pi}{2}\right]$.

The functions $f(x) = \sqrt{x}$ and $g(x) = \cos x$ are both continuous on $[0, \frac{\pi}{2}]$
$\Rightarrow$ The function $h(x) = f(x) - g(x) = \sqrt{x} - \cos x$ is also continuous on $[0, \frac{\pi}{2}]$ (2pts)

But $h(0) = -1 < 0$ and $h(\frac{\pi}{2}) = \sqrt{\frac{\pi}{2}} > 0$ (2pts)

Thus by the Intermediate Value Theorem, there is a number $c \in (0, \frac{\pi}{2})$ Such that $h(c) = 0$ (2pts)

$\Rightarrow f(c) = g(c)$, i.e f and g intersect on $[0, \frac{\pi}{2}]$. (2pts)

6. Given that $f(x) = (x-1)^{\frac{2}{3}}$ and $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$.

(6a) **(3 - points)** Use limits to find, if any, the equation of the vertical tangent to the graph of f.

f is continuous at $x = 1$, while (1pt)

$\lim_{x \to 1} |f'(x)| = \lim_{x \to 1} \left|\frac{2}{3}(x-1)^{-1/3}\right| = \infty$ (1pt)

$\Rightarrow x = 1$ is a vertical tangent to the graph of f at $(1, 0)$ (1pt)

(6b) **(5 - points)** Find the equation of the normal line to the graph of f at $x = 9$.

$x = 9 \Rightarrow f(9) = 8^{2/3} = 4$ (1pt)

The slope of the tangent line at $(9,4)$

$= f'(9) = \frac{2}{3}(8)^{-1/3} = \frac{1}{3}$ (2pts)

$\Rightarrow$ The slope of the normal line at $(9,4)$

$= -3$ (1pt)

$\Rightarrow$ The required eqn is $y - 4 = -3(x - 9)$ (1pt)

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7. **(7 - points)** Determine the intervals on which the function $f(x) = \dfrac{\ln(x) + \tan^{-1}(3x)}{x^2 - 4}$ is continuous.

$\ln x$ is continuous on $(0, \infty)$ (1pt)

$\tan^{-1}(3x)$ is continuous on $(-\infty, \infty)$ (1pt)

$\Rightarrow$ $\ln x + \tan^{-1}(3x)$ is continuous on $(0,\infty)$ (2pts)

But the zeros of the denominator x^2-4 are -2 and 2 (1pt)

$\Rightarrow$ The given function is continuous on $(0,2)$ and on $(2,\infty)$. (2pts)

8. **(7 - points)** Use limits to determine whether or not the following function is continuous at $x = 2$

$$f(x) = \begin{cases} \dfrac{10}{3x-1}, & \text{if } x < 2 \\ \sqrt{3x-2}, & \text{if } x \geq 2 \end{cases}$$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \dfrac{10}{3x-1} = \dfrac{10}{5} = 2$, (2pts)

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \sqrt{3x-2} = \sqrt{4} = 2$, (2pts)

and $f(2) = \sqrt{4} = 2$ (1pt)

$\Rightarrow$ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$ (1pt)

$\Rightarrow$ f is continuous at $x = 2$. (1pt)

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9. **(8 - points)** Given that $\lim_{x\to 2}\left(3x - \frac{2}{5}\right) = \frac{28}{5}$ and $\epsilon = 0.009$. Find the largest possible value of δ that satisfies the conditions given in the $\epsilon - \delta$ definition of a limit.

Need to find $\delta > 0$ such that

$\left|\left(3x - \frac{2}{5}\right) - \frac{28}{5}\right| < 0.009$ whenever $0 < |x-2| < \delta$ (2pts)

But $\left|\left(3x - \frac{2}{5}\right) - \frac{28}{5}\right| = |3x-6| = 3|x-2|$ (2pts)

So, we want $3|x-2| < 0.009$ whenever $0 < |x-2| < \delta$

or $|x-2| < 0.003$, whenever $0 < |x-2| < \delta$ (2pts)

Thus the largest possible value of δ is

$\delta = 0.003$ (2pt)

10. **(10 - points)** Use limits to find all vertical and horizontal asymptotes of the graph of

$$f(x) = \frac{6x}{\sqrt{2x^2 - 8}}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{6x}{|x|\sqrt{2-\frac{8}{x^2}}} = \lim_{x\to-\infty} \frac{6x}{-x\sqrt{2-\frac{8}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{-6}{\sqrt{2-\frac{8}{x^2}}} = -\frac{6}{\sqrt{2}},$$ (2pts)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x}{x\sqrt{2-\frac{8}{x^2}}} = \lim_{x\to\infty} \frac{6}{\sqrt{2-\frac{8}{x^2}}} = \frac{6}{\sqrt{2}}$$ (2pts)

$\Rightarrow$ Horizontal asymptotes: $y = -\frac{6}{\sqrt{2}}$ and $y = \frac{6}{\sqrt{2}}$. (2pts)

Now $f(x) = \frac{6x}{\sqrt{2}\sqrt{x^2-4}} = \frac{6x}{\sqrt{2}\sqrt{(x-2)(x+2)}} \Longrightarrow$

$\lim_{x\to-2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = \infty$ (2pts)

$\Longrightarrow$ Vertical Asymptotes: $x = -2$ and $x = 2$ (2pts)