MATH 201 - Exam I

Solution for the 1st 6 Questions with Grading Policy

Q.1 $x = t^5 - 4t^3$; $y = t^2$

$\frac{dx}{dt} = 5t^4 - 12t^2$ $\qquad \frac{dy}{dt} = 2t \longrightarrow$ (1+1)

At point (0,4) ; $t = \pm 2 \longrightarrow$ (2)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{5t^4 - 12t^2}$$

$$\left.\frac{dy}{dx}\right|_{(0,4)} = \left.\frac{2t}{5t^4 - 12t^2}\right|_{t=2} = \frac{4}{80-48} = \frac{1}{8} \longrightarrow (2)$$

$$\frac{dy}{dx} = \Big|_{t=-2} = -\frac{4}{32} = -\frac{1}{8} \longrightarrow (2)$$

Equations of tangents at (0,4) are

$y - 4 = \frac{1}{8}(x-0)$ or $y = \frac{1}{8}x + 4 \longrightarrow$ (1)

and

$y - 4 = -\frac{1}{8}(x-0)$ $\qquad y = -\frac{1}{8}x + 4 \longrightarrow$ (2)

$x - 8y + 32 = 0$

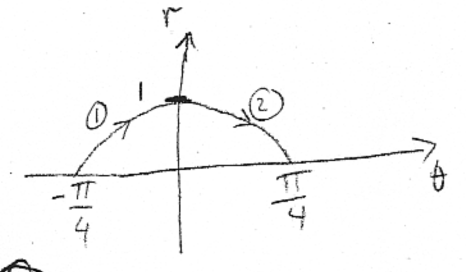
$-x - 8y + 32 = 0$

Q2. (14 pts) Consider the polar curve C: $r = f(\theta) = \cos 2\theta$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

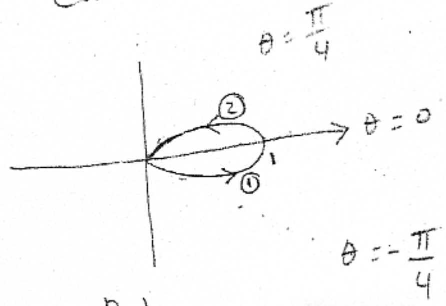
a) Sketch the curve.
b) Find the integral for the area enclosed by the curve (do not evaluate).
c) Setup an integral for the arc-length of the curve.

5 a)

θ	$r = \cos 2\theta$
$-\frac{\pi}{4} \to 0$	$0 \to 1$
$0 \to \frac{\pi}{4}$	$1 \to 0$



2 Cartesian



3

Polar

4 b) The area enclosed by the curve is given by

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\pi/4} r^2 d\theta \quad 2 \quad = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta \quad 2$$

5 c) An integral for the arc-length of the curve is

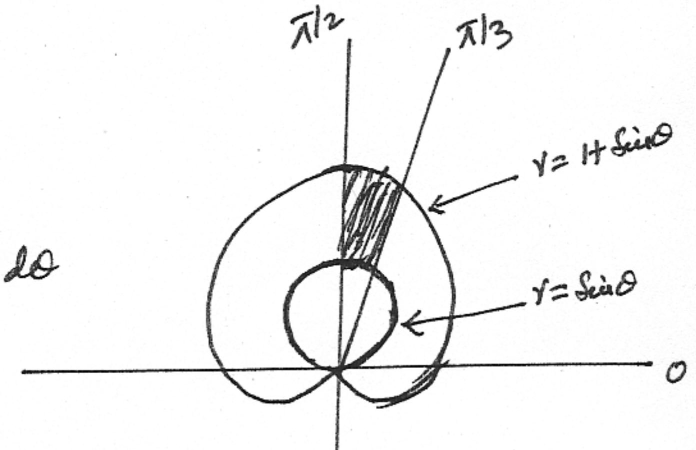
$$L = \int_{-\pi/4}^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad 2$$

$$= \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + (-2\sin 2\theta)^2}\, d\theta \quad 3$$

$$= \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\, d\theta$$

(Q3) (Pt 12) :- Find the area inside the curve $r = 1 + \sin\theta$ and outside the curve $r = \sin\theta$ when $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$.



Solution :-

$$\text{area} = \frac{1}{2}\int_{\pi/3}^{\pi/2} (1+\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/3}^{\pi/2} \sin^2\theta\, d\theta$$

(setup integral 4 points)

2 points for each graph (2+2).

4pts

$$= \frac{1}{2}\int_{\pi/3}^{\pi/2} (1+\cancel{\sin^2\theta} + 2\sin\theta - \cancel{\sin^2\theta})\, d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi/2} (1+2\sin\theta)\, d\theta = \frac{1}{2}\Big[\theta - 2\cos\theta\Big]_{\pi/3}^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 2\cos\pi/2 - \frac{\pi}{3} + 2\cos\frac{\pi}{3}\right] = \frac{1}{2}\left[\frac{\pi}{6} + 1\right]$$

Q.4 Find an equation of a sphere if one of its diameters has end points at $A(1,4,-2)$ and $B(-7,1,2)$. What is the intersection of this sphere with xz-plane?

Sol. The centre of sphere will be the midpoint of AB and so has coordinates $\left(-3, \frac{5}{2}, 0\right)$.

The diameter of the sphere is the distance between A and B and hence it is equal to $\sqrt{89}$.

So equation of the sphere is:

$$(x+3)^2 + \left(y-\frac{5}{2}\right)^2 + z^2 = \frac{89}{4} \qquad (1)$$

To find intersection of this sphere with the xz-plane, we put $y=0$ in (1) and get:

$$(x+3)^2 + \left(-\frac{5}{2}\right)^2 + z^2 = \frac{89}{4}, \quad y=0$$

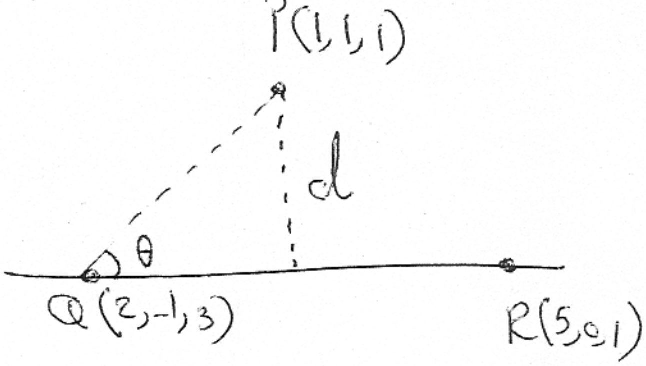
or

$$(x+3)^2 + z^2 = (4)^2$$

which represents a circle in the xz-plane with centre $(-3,0,0)$ and radius 4.

Centre = 2
Diameter = 2
Equation = 2
Intersection = 2
Interpretation = 1

Key Sol # 5



$|QP \times QR| = |QP||QR| \sin\theta$

also, $d = |QP| \sin\theta$

so $|QP \times QR| = |QR| \, d$

$\boxed{\dfrac{|QP \times QR|}{|QR|} = d}$ so 2 pts

$\overrightarrow{QR} = \langle 5-2, 0-(-1), 1-3 \rangle = \langle 3, 1, -2 \rangle$ 1 pts

$\overrightarrow{QP} = \langle 1-2, 1-(-1), 1-3 \rangle = \langle -1, 2, -2 \rangle$ 1 pts

$$\overrightarrow{QR} \times QP = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ -1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 2 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ -1 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} k$$

$$= 2i + 8j + 7k$$ 3 pts

so $|\overrightarrow{QR} \times \overrightarrow{QP}| = \sqrt{2^2 + 8^2 + 7^2} = \sqrt{4 + 64 + 49} = \sqrt{117}$ 1 pts

$|\overrightarrow{QR}| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$ 1 pts

so $d = \sqrt{\dfrac{117}{14}}$

Q.6 Find Cartesian equation of the curve whose polar equation is given as: $r = sec\theta - csc\theta$.

Sol: $r = \frac{1}{cos\theta} - \frac{1}{sin\theta} = \frac{sin\theta - cos\theta}{sin\theta cos\theta}$ 3

$rsin\theta cos\theta = sin\theta - cos\theta$ 2

$rsin\theta \, rcos\theta = rsin\theta - rcos\theta$ 2

$yx = y - x$. 1

$y - yx = x$

$y = \frac{x}{1-x}$

OR

$r = \frac{1}{\cos\theta} - \frac{1}{\sin\theta}$ 2

$= \frac{1}{\frac{x}{r}} - \frac{1}{\frac{y}{r}}$ 3

$= \frac{r}{x} - \frac{r}{y}$ 2

$= r\left(\frac{1}{x} - \frac{1}{y}\right)$ 1

$1 = \frac{1}{x} - \frac{1}{y} \Rightarrow xy = y - x$.

OR

$x = r\cos\theta = (\sec\theta - \csc\theta)\cos\theta = 1 - \cot\theta$ 2

$y = r\sin\theta = (\sec\theta - \csc\theta)\sin\theta = \tan\theta - 1$ 2

$\tan\theta = \frac{y}{x}$

$\cot\theta = \frac{x}{y}$

$x = 1 - \frac{x}{y}$ 3

$xy = y - x$. 1