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MATH 301/Term 062/Hw#16(12.2)/

4. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if } \quad -1 \leq x < 0 \\ x, & \text{if } \quad 0 \leq x \leq 1. \end{cases}$$

The Fourier series of f on the interval $[-1, 1]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{1}x\right) + b_n \sin\left(\frac{n\pi}{1}x\right)$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x), \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{1}\int_{-1}^{1} f(x)dx = \int_0^1 x dx \\ &= \left[\frac{x^2}{2}\right]_0^1 \\ &= \frac{1}{2}. \end{aligned} \tag{2}$$

$$\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_0^1 x\cos(n\pi x)dx \\
&= \Big[x\frac{1}{n\pi}\sin(n\pi x)\Big]_0^1 - \int_0^1 \frac{1}{n\pi}\sin(n\pi x)dx \\
&= \sin(n\pi) - 0 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx \\
&= -\frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx \\
&= -\frac{1}{n\pi}\Big[-\frac{1}{n\pi}\cos(n\pi x)\Big]_0^1 \\
&= \frac{1}{n^2\pi^2}(\cos(n\pi) - 1) \\
&= \frac{(-1)^n - 1}{n^2\pi^2}. \qquad (3)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_0^1 x\sin(n\pi x)dx \\
&= \Big[-x\frac{1}{n\pi}\cos(n\pi x)\Big]_0^1 - \int_0^1 -\frac{1}{n\pi}\cos(n\pi x)dx \\
&= -\frac{1}{n\pi}\cos(n\pi) + 0 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi}\Big[\frac{1}{n\pi}\sin(n\pi x)\Big]_0^1 \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n^2\pi^2}(\sin(n\pi) - 0) \\
&= \frac{(-1)^{n+1}}{n\pi}. \qquad (4)
\end{aligned}$$

Taking into account (1), (2), (3) and (4), it follows that the Fourier series of f on the interval $[-1, 1]$ is given by

$$\frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi}\sin(n\pi x).$$

$\square$

6. Let f be the function defined by

$$f(x) = \begin{cases} \pi^2, & \text{if} \quad -\pi \leq x < 0 \\ \pi^2 - x^2, & \text{if} \quad 0 \leq x \leq \pi. \end{cases}$$

The Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\pi}x\right) + b_n \sin\left(\frac{n\pi}{\pi}x\right)$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2 dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)dx \\ &= \frac{1}{\pi}[\pi^2 x]_{-\pi}^{0} + \frac{1}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{1}{\pi}\pi^3 + \frac{1}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) \\ &= \frac{5\pi^2}{3}. \end{aligned} \tag{2}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\cos(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - x^2)\cos(nx)dx \\
&= \pi\int_{-\pi}^{0}\cos(nx)dx + \pi\int_{0}^{\pi}\cos(nx)dx - \frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\
&= \pi\int_{-\pi}^{\pi}\cos(nx)dx - \frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\
&= \Big[\frac{1}{n}\sin(nx)\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\
&= \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(-n\pi) - \frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\
&= -\frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx. \qquad (3)
\end{aligned}
$$

Integration by parts twice, we get

$$
\begin{aligned}
\int x^2\cos(nx)dx &= \frac{x^2}{n}\sin(nx) - \int\frac{2x}{n}\sin(nx)dx \\
&= \frac{x^2}{n}\sin(nx) - \frac{2}{n}\int x\sin(nx)dx \\
&= \frac{x^2}{n}\sin(nx) - \frac{2}{n}\Big(-\frac{x}{n}\cos(nx) - \int -\frac{1}{n}\cos(nx)dx\Big) \\
&= \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^2}\int\cos(nx)dx \\
&= \frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx). \qquad (4)
\end{aligned}
$$

We deduce from (4) that

$$\int_0^{\pi} x^2\cos(nx)dx = \frac{2\pi}{n^2}\cos(n\pi) = \frac{2\pi(-1)^n}{n^2}.$$

Using (3), we get

$$a_n = -\frac{1}{\pi}\frac{2\pi(-1)^n}{n^2} = \frac{2(-1)^{n+1}}{n^2}. \qquad (5)$$

For b_n, we have

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi^2 - x^2)\sin(nx)dx \\
&= \pi\int_{-\pi}^{0}\sin(nx)dx + \pi\int_{0}^{\pi}\sin(nx)dx - \frac{1}{\pi}\int_{0}^{\pi}x^2\sin(nx)dx \\
&= \pi\int_{-\pi}^{\pi}\sin(nx)dx - \frac{1}{\pi}\int_{0}^{\pi}x^2\sin(nx)dx \\
&= \Big[-\frac{1}{n}\cos(nx)\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi}x^2\sin(nx)dx \\
&= -\frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos(-n\pi) - \frac{1}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\
&= -\frac{1}{\pi}\int_{0}^{\pi}x^2\sin(nx)dx. \qquad (6)
\end{aligned}
$$

Integration by parts twice, we get

$$
\begin{aligned}
\int x^2\sin(nx)dx &= -\frac{x^2}{n}\cos(nx) - \int -\frac{2x}{n}\cos(nx)dx \\
&= -\frac{x^2}{n}\cos(nx) + \frac{2}{n}\int x\cos(nx)dx \\
&= -\frac{x^2}{n}\cos(nx) + \frac{2}{n}\Big(\frac{x}{n}\sin(nx) - \int\frac{1}{n}\sin(nx)dx\Big) \\
&= -\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) - \frac{2}{n^2}\int\sin(nx)dx \\
&= -\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx). \qquad (7)
\end{aligned}
$$

We deduce from (7) that

$$
\int_0^{\pi} x^2\sin(nx)dx = -\frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n^3}(\cos(n\pi) - 1) = \frac{\pi^2(-1)^{n+1}}{n} + \frac{2((-1)^n - 1)}{n^3}.
$$

Using (6), we get

$$b_n = -\frac{1}{\pi}\Big(\frac{\pi^2(-1)^{n+1}}{n} + \frac{2((-1)^n-1)}{n^3}\Big) = \frac{\pi(-1)^n}{n} + \frac{2}{\pi}\frac{(1-(-1)^n)}{n^3}. \tag{8}$$

Taking into account (1), (2), (5) and (8), it follows that the Fourier series of f on the interval $[-\pi,\pi]$ is given by

$$\frac{5\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2}\cos(nx) + \Big(\frac{\pi(-1)^n}{n} + \frac{2}{\pi}\frac{(1-(-1)^n)}{n^3}\Big)\sin(nx).$$

$\square$

16. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if } \quad -\pi \le x < 0 \\ e^x - 1, & \text{if } \quad 0 \le x \le \pi. \end{cases}$$

The Fourier series of f on the interval $[-\pi,\pi]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\big(\frac{n\pi}{\pi}x\big) + b_n \sin\big(\frac{n\pi}{\pi}x\big)$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{1}$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(e^x-1)dx \\ &= \frac{1}{\pi}[e^x - x]_0^{\pi} \\ &= \frac{e^\pi - \pi - 1}{\pi}. \end{aligned} \tag{2}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} (e^x - 1)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\cos(nx)dx - \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\cos(nx)dx - \frac{1}{\pi}\Big[\frac{1}{n}\sin(nx)\Big]_0^{\pi} \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\cos(nx)dx. \qquad (3)
\end{aligned}
$$

Integration by parts twice, we get

$$
\begin{aligned}
\int e^x\cos(nx)dx &= \frac{e^x}{n}\sin(nx) - \int \frac{1}{n}e^x\sin(nx)dx \\
&= \frac{e^x}{n}\sin(nx) - \frac{1}{n}\int e^x\sin(nx)dx \qquad (4) \\
&= \frac{e^x}{n}\sin(nx) - \frac{1}{n}\Big(- \frac{e^x}{n}\cos(nx) - \int -\frac{1}{n}e^x\cos(nx)dx\Big) \\
&= \frac{e^x}{n}\sin(nx) + \frac{e^x}{n^2}\cos(nx) - \frac{1}{n^2}\int e^x\cos(nx)dx. \qquad (5)
\end{aligned}
$$

We deduce from (5) that

$$
\Big(1 + \frac{1}{n^2}\big)\int e^x\cos(nx)dx = \frac{e^x}{n}\sin(nx) + \frac{e^x}{n^2}\cos(nx)
$$

which leads to

$$
\int e^x\cos(nx)dx = \frac{ne^x}{1+n^2}\sin(nx) + \frac{e^x}{1+n^2}\cos(nx). \qquad (6)
$$

Using (3), we get

$$
a_n = \frac{1}{\pi}\frac{e^{\pi}(-1)^n - 1}{1+n^2}. \qquad (7)
$$

For b_n, we have

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} (e^x-1)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\sin(nx)dx - \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\sin(nx)dx + \frac{1}{\pi}\Big[\frac{1}{n}\cos(nx)\Big]_0^{\pi} \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^x\sin(nx)dx + \frac{1}{\pi}\frac{(-1)^n-1}{n}. \qquad (8)
\end{aligned}
$$

Using (4) and (6), we obtain

$$
\begin{aligned}
\int e^x\sin(nx)dx &= e^x\sin(nx) - n\int e^x\cos(nx)dx \\
&= e^x\sin(nx) - \frac{n^2e^x}{1+n^2}\sin(nx) - \frac{ne^x}{1+n^2}\cos(nx). \qquad (9)
\end{aligned}
$$

We deduce from (9) that

$$\int_0^{\pi} e^x\sin(nx)dx = \frac{n}{1+n^2} - \frac{ne^{\pi}}{1+n^2}\cos(n\pi) = \frac{n(1-e^{\pi}(-1)^n)}{1+n^2}.$$

Using (8), we get

$$b_n = \frac{1}{\pi}\frac{n(1-e^{\pi}(-1)^n)}{1+n^2} + \frac{1}{\pi}\frac{(-1)^n-1}{n}. \qquad (10)$$

Taking into account (1), (2), (7) and (10), it follows that the Fourier series of f on the interval $[-\pi,\pi]$ is given by

$$\frac{e^{\pi}-\pi-1}{2\pi} + \sum_{n=1}^{\infty}\frac{1}{\pi}\frac{e^{\pi}(-1)^n-1}{1+n^2}\cos(nx) + \Big(\frac{1}{\pi}\frac{n(1-e^{\pi}(-1)^n)}{1+n^2} + \frac{1}{\pi}\frac{(-1)^n-1}{n}\Big)\sin(nx).$$

□

20. Let f be the function defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad -\pi \le x < 0 \\ \sin(x), & \text{if} \quad 0 \le x \le \pi. \end{cases}$$

One can verify (Pb. 9) that the Fourier series of f on the interval $[-\pi, \pi]$ is given by

$$\frac{1}{\pi} + \frac{1}{2}\sin(x) + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos(nx). \tag{1}$$

The function f is continuous on the interval $[-\pi, \pi]$ and differentiable except at $x = 0$ with

$$f'(x) = \begin{cases} 0, & \text{if} \quad -\pi \le x < 0 \\ \cos(x), & \text{if} \quad 0 < x \le \pi. \end{cases}$$

In particular f is continuous at $\frac{\pi}{2}$. Therefore we get from (1)

$$f\Big(\frac{\pi}{2}\Big) = \frac{1}{\pi} + \frac{1}{2}\sin\Big(\frac{\pi}{2}\Big) + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos\Big(\frac{n\pi}{2}\Big). \tag{2}$$

Since we have for each integer, $1 + (-1)^{(2k+1)} = 0$ and $\cos\left(\frac{2k\pi}{2}\right) = \cos(k\pi) = (-1)^k$, we deduce from (2) that

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{1+(-1)^{2k}}{1-(2k)^2}(-1)^k.$$

which can be written as

$$2\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2-1} = \pi\Big(\frac{1}{\pi} + \frac{1}{2} - 1\Big) = 1 - \frac{\pi}{2}$$

or

$$\begin{aligned} \frac{\pi}{4} &= \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2-1} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1).(2k+1)} \\ &= \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - .... \end{aligned}$$

$\square$