Department of Mathematical Sciences
KFUPM
Term 032

MATH 301-01, 04/ Exam#2/ Duration=2 Hours

Name: ID#:

1. $\mathcal{L}\left\{\cos^2(3t)\,\mathcal{U}_\pi(t)\right\} = e^{-\pi s}\,\mathcal{L}\left\{\cos^2(3(t+\pi))\right\}$

$= e^{-\pi s}\,\mathcal{L}\left\{\cos^2(3t+3\pi)\right\} = e^{-\pi s}\,\mathcal{L}\left\{\cos^2(3t)\right\}$

$= e^{-\pi s}\,\mathcal{L}\left\{\frac{1}{2}(1+\cos(6t))\right\} = \frac{1}{2}e^{-\pi s}\,\mathcal{L}\left\{1+\cos(6t)\right\}$

$= \frac{e^{-\pi s}}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right) = \frac{e^{-\pi s}(s^2+18)}{s(s^2+36)}$

$\mathcal{L}\left\{t^n e^{at}\right\} = \mathcal{L}\left\{t^n\right\}(s-a) = \frac{n!}{(s-a)^{n+1}}$

2. $s^2+s+1 = \left(s+\frac{1}{2}\right)^2 + \frac{3}{4} > 0.$

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} = \frac{(A+B)s^2+(A+C)s+A}{s(s^2+s+1)}$$

We obtain $\begin{cases} A=1 \\ A+B=0 \\ A+C=0 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=C=-1. \end{cases}$

So $$\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Note $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)$ if $F(s) = \mathcal{L}\left\{f(t)\right\}$.

So $$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} = e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}t}{2}\right) \quad \left(a=-\frac{1}{2},\ f(t)=\cos\left(\frac{\sqrt{3}}{2}t\right)\right)$$

and $$\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} = e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right) \quad \left(a=-\frac{1}{2},\ f(t)=\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Hence
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+s+1)}\right\} = 1 - e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

$$\frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

$$A = \frac{1}{-1+4} = \frac{1}{3}, \qquad B = \frac{1}{-4+1} = -\frac{1}{3}$$

Therefore $$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+4}$$

and $$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

Hence $$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\left(\cos(t) - \cos(2t)\right).$$

3. $$y'(t) + k^2 \int_0^t y(\tau)\, d\tau = \delta_a(t), \qquad y(0) = 1.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Then we have by applying the Laplace transform:

$$\mathcal{L}\{y'(t)\} + k^2 \mathcal{L}\left\{\int_0^t y(\tau)\, d\tau\right\} = \mathcal{L}\{\delta_a(t)\}$$

$$sY(s) - y(0) + k^2 \frac{Y(s)}{s} = e^{-as}$$

$$s^2 Y(s) - s + k^2 Y(s) = s e^{-as}$$

$$(s^2 + k^2) Y(s) = s + s e^{-as} \iff Y(s) = \frac{s}{s^2+k^2} + \frac{s e^{-as}}{s^2+k^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2} e^{-as}\right\}$$

$$= \cos(kt) + \cos(k(t-a))\, \mathcal{U}_a(t).$$

4. $$\int_{-\pi}^{\pi} \cos(3x)\sin(5x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(5x+3x) + \sin(5x-3x)\right)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin(8x) + \sin(2x)\right)dx$$

$$= \frac{1}{2}\left[\frac{-\cos(8x)}{8} + \frac{-\cos(2x)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(-\frac{1}{8}\left(\cos(8\pi) - \cos(-8\pi)\right) - \frac{1}{2}\left(\cos(2\pi) - \cos(-2\pi)\right)\right)$$

$$= \frac{1}{2}\left(0 - 0\right) = 0.$$

So $\cos(3x)$ and $\sin(5x)$ are orthogonal on $[-\pi, \pi]$.

The ~~se~~ square norm of $\sin(5x)$ is given by:

$$\int_{-\pi}^{\pi} \sin^2(5x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left(1 - \cos(10x)\right)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(10x)}{10}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{\sin(10\pi)}{10}\right) - \left(-\pi - \frac{\sin(-10\pi)}{10}\right)\right]$$

$$= \frac{2\pi}{2} = \pi$$

So the norm of $\sin(5x)$ is $\left(\int_{-\pi}^{\pi} \sin^2(5x)\,dx\right)^{1/2} = \sqrt{\pi}.$

5. $f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x < \pi . \end{cases}$

a. The half-range cosine series of $f(x)$ is given by:

$$\frac{a_0}{2} + \sum_{n \geqslant 1} a_n \cos\left(\frac{n\pi}{\pi} x\right) = \frac{a_0}{2} + \sum_{n \geqslant 1} a_n \cos(nx),$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi/2} dx = \frac{2}{\pi} \frac{\pi}{2} = 1.$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos\left(\frac{n\pi}{\pi} x\right) dx = \frac{2}{\pi} \int_0^{\pi/2} \cos(nx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{\sin(nx)}{n} \right]_0^{\pi/2} = \frac{2 \sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

So the half-range cosine series is given by:

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n \geqslant 1} \frac{\sin\left(\frac{n\pi}{2}\right)}{n} \cos(nx)$$

b. The half-range sine series of $f(x)$ is given by:

$$\sum_{n \geqslant 1} b_n \sin\left(\frac{n\pi}{\pi} x\right) = \sum_{n \geqslant 1} b_n \sin(nx),$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{n\pi}{\pi} x\right) dx = \frac{2}{\pi} \int_0^{\pi/2} \sin(nx)\, dx$

$$= \frac{2}{\pi} \left[\frac{-\cos(nx)}{n} \right]_0^{\pi/2} = \frac{2\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi}$$

So the half-range sine series is given by:

$$\frac{2}{\pi}\sum_{n\geq 1}\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\sin(nx).$$

c. The even extension of $f(x)$ to $(-\pi,\pi)$ is continuous except at $x=\pm\frac{\pi}{2}$ and so is $f'(x)$. We deduce then from the convergence theorem that the half-range cosine series of $f(x)$ converges to $\frac{f(x+)+f(x-)}{2}$ for any $x\in[0,\pi)$.

For $x=0$, we have $f(0+)=f(0^-)=1$. It follows then:

$$1=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos(0)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$

$$\Longleftrightarrow\quad\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}=\frac{\pi}{4}.$$

$$\sin\left(\frac{n\pi}{2}\right)=\begin{cases}\sin\left(\frac{2k\pi}{2}\right)=\sin(k\pi)=0 & \text{if } n=2k\\ \sin\left(\frac{(2k+1)\pi}{2}\right)=\sin\left(k\pi+\frac{\pi}{2}\right)=\cos(k\pi)=(-1)^k & \text{if } n=2k+1\end{cases}$$

So

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{\pi}{4}.$$

6. a) Eigenvalues and eigenfunctions of the BVP:

$$y'' + 2y' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

We first solve the equation : $r^2 + 2r + \lambda = 0$

Let $D = 4(1-\lambda)$.

- If $D = 0 \iff \lambda = 1$, then $r_1 = r_2 = -1$.

$$y(x) = (c_1 x + c_2) e^{-x}$$

$$\left.\begin{array}{l} y(0) = 0 \implies c_2 = 0 \\ y(1) = 0 \implies c_1 e^{-1} = 0 \implies c_1 = 0 \end{array}\right\} \implies y(x) \equiv 0.$$

- If $D > 0 \iff \lambda < 1$, then

$$r_1 = -1 - \sqrt{1-\lambda}, \quad r_2 = -1 + \sqrt{1-\lambda}$$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y(0) = 0 \implies c_1 + c_2 = 0 \implies c_2 = -c_1$$

$$y(1) = 0 \implies c_1 e^{r_1} + c_2 e^{r_2} = 0 \implies c_1 (e^{r_1} - e^{r_2}) = 0$$

$\implies c_1 = 0$ since $r_1 \neq r_2$.

So $c_1 = c_2 = 0$ and $y(x) \equiv 0$.

- If $D<0 \iff \lambda>1$

$r_1=-1-i\sqrt{\lambda-1}$, $\quad r_2=-1+i\sqrt{\lambda-1}$

$$y(x)=e^{-x}\left(c_1\cos(\sqrt{\lambda-1}\,x)+c_2\sin(\sqrt{\lambda-1}\,x)\right)$$

$y(0)=0 \Rightarrow c_1=0 \Rightarrow y(x)=c_2e^{-x}\sin(\sqrt{\lambda-1}\,x)$.

$y(1)=0 \Rightarrow c_2e^{-1}\sin(\sqrt{\lambda-1})=0 \Rightarrow c_2\sin(\sqrt{\lambda-1})=0$

If $c_2=0$, then $y(x)\equiv 0$.

If $c_2\neq 0$, then $\sin(\sqrt{\lambda-1})=0 \iff \sqrt{\lambda-1}=n\pi$, $n=1,2,\ldots$

$\iff \lambda=1+n^2\pi^2$, $n=1,2,3,\ldots$

So the eigenvalues of the BVP are $\lambda_n=1+n^2\pi^2$, $n=1,2,\ldots$

and the corresponding eigenfunctions are:

$$y_n(x)=e^{-x}\sin(\sqrt{\lambda_n-1}\,x)=e^{-x}\sin(n\pi x),\quad n=1,2,\ldots$$

- Self-adjoint form of $y''+2y'+\lambda y=0$:

$a(x)=1$, $\quad b(x)=2$, $\quad c(x)=0$, $\quad d(x)=1$

So $r(x)=\exp\left(\int\frac{b(x)}{a(x)}dx\right)=\exp\left(\int 2\,dx\right)=e^{2x}$

$q(x)=\frac{c(x)}{a(x)}r(x)=0$, $\quad p(x)=\frac{d(x)}{a(x)}r(x)=e^{2x}$.

We get the self-adjoint form: $\frac{d}{dx}\left[e^{2x}y'(x)\right]+\lambda e^{2x}y=0$.

. orthogonality relation : $r(x) = e^{2x} > 0$, $p(x) = e^{2x}$

So the set of eigenfunctions is orthogonal with respect to the weight $p(x) = e^{2x}$ on $[0,1]$. Therefore we obtain the orthogonality relation :

$$\int_0^1 e^{2x} \left(e^{-x} \sin(n\pi x)\right)\left(e^{-x} \sin(m\pi x)\right) dx = 0$$, for all n, m such that $n \neq m$

$$\Longleftrightarrow \int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx = 0$$ //.