Math 514 (091)

Advanced Mathematical Methods

Updated on May 29, 2010

HW # 1: Complex Variables — Due: Monday, Mar 15.

(1) Define the principal value of $\arg z$ to lie in $(0, 2\pi]$. Find $\mathrm{Ln}\,1$, $\mathrm{Ln}(-2)$, and $\mathrm{Ln}(-i)$.

(2) Use the branch cut structure of the square root function and logarithmic function to find a branch cut structure for $\ln\left(4+\sqrt{z^2-9}\right)$.

(3) Exercise 6.4.3.

HW # 2: Complex Variables — Due: Saturday, Mar 27.

(1) Consider the integral $\int_C \frac{z^p dz}{\sinh z - ia}$, $a \neq 0$ and real, over the contour which is the boundary of the rectangular region $-R \le \mathrm{Re}\, z \le R$ and $0 \le \mathrm{Im}\, z \le 2\pi$. Use the integrals with $p = 1, 2$ to evaluate $\int_{-\infty}^{\infty} \frac{x dx}{\sinh x - ia}$.

(2) Exercise 6.3.5 (hint. $\mathrm{Re}\, w - c\, \mathrm{Im}\, w = 0$ on circles through ± 1 with centers at $z = ic$. $\arg w$ can be found by considering selected points on each circle such as $(1-\sqrt{2})i$ and i.)

(3) Exercise 6.3.6.a

HW # 3: Fourier Transform Due: Monday, Apr 5.

(1) Show that $\mathscr{F}\left\{te^{-a|t|}\right\} = -\dfrac{4ai\omega}{(\omega^2+a^2)^2}$, $a>0$.

(2) Show that the Fourier transform of

$$f(t) = \begin{cases} \cos(at), & |t|<1 \\ 0 & |t|>1 \end{cases}$$

is

$$F(\omega) = \frac{\sin(\omega-a)}{\omega-a} + \frac{\sin(\omega+a)}{\omega+a}.$$

(3) Use the definition of Fourier transform and $\mathscr{F}\left\{H(t)\right\} = \pi\,\delta(\omega) - \dfrac{i}{\omega}$ to show that

$$\int_0^\infty e^{-i\omega t}dt = \pi\delta(\omega) - \frac{i}{\omega}.$$

(4) Given that $\mathscr{F}\left\{\dfrac{1}{1+t^2}\right\} = \pi e^{|\omega|}$, find $\mathscr{F}\left\{\dfrac{\cos(at)}{1+t^2}\right\}$, a is real.

(5) Use contour integration to find $\mathscr{F}^{-1}\left\{\dfrac{\omega}{\omega^2+1}\right\}$.

(6) Use the definition of the convolution to show that $e^t H(t) * e^{-2t}H(t) = (e^{-t} - e^{-2t})H(t)$.

Homework # 4 (Fourier transform II) Due: Saturday May 1.

(1) Find the inverse of $F(\omega) = \dfrac{e^{i\omega}}{\omega^2+1}$ using the residue theorem.

(2) Find the particular solution for $y'' - 4y' + 4y = e^{-t}H(t)$.

(3) Solve

$$u_t = 4u_{xx}, \qquad -\infty < x < \infty,\ t>0,$$

$$u(x,0) = e^{-|x|}, \qquad -\infty < x < \infty.$$

HW # 5: Laplace and Mellin Transforms Due: Saturday, May 1.

(1) Find $Y(s)$ for

$$y'' + 4y' + 4y = H(t-1), \qquad t > 0,$$
$$y(0) = 0, \qquad y(3) = 2.$$

(2) Solve the integral equation $f(t) = 1 + \int_0^t f(x)\sin(t-x)dx$.

(3) Show that $\mathscr{M}\left\{\int_0^\infty f(xu)\, g(u)\, du\right\} = F(p)G(1-p)$, where $F(p) = \mathscr{M}\{f(x)\}$ and

$G(p) = \mathscr{M}\{g(x)\}$.

(4) Use the definition of Mellin Transform to solve the integral equation

$$\int_0^\infty f(u)\, g\left(\frac{x}{u}\right) du = h(x), \qquad x > 0,$$

where $f(x)$ is unknown and $g(x)$ and $h(x)$ are given functions.

HW # 6: Hankel Transform Due: Saturday, May 9.

1. Let $f(r)$ be defined for $r > 0$ and such that $r f(r)$ and $r f'(r)$ vanish as $r \to 0$ and $r \to \infty$.

   Show that $\mathscr{H}_n\left\{\left(\Delta - \frac{n^2}{r^2}\right) f(r)\right\} = -k^2 F_n(k)$, where $\Delta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$.

2. Solve:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 < r < \infty, \qquad 0 < z < \infty,$$
$$u(r,0) = H(1-r), \qquad 0 < r < \infty.$$

HW # 7: Wiener-Hopf Technique and Asymptotic Expansions Due: Saturday, Jun 5.

1. DuT: problem 1, p. 570.
2. DuT: problem 1, p. 583. Use steps 5 and 9 without showing them.
3. Exercise 10.2.1
4. Exercise 10.3.1