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Krull and Valuative Dimensions of the $A + XB[X]$ Rings

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0. INTRODUCTION

All the rings considered in this paper are integral domains, i.e. commutative rings with identity and non zero-divisors. Given a finite dimensional ring A, we say that A is a *Jaffard domain* if $\dim A = \dim_v A$ [2]. The previous property is not a local property and thus we say that A is a *locally Jaffard domain* if $A_{\mathfrak{p}}$ is a Jaffard domain, for each prime ideal $\mathfrak{p}$ of A. Noetherian domains and, in the locally finite dimensional case, Prüfer domains, stably strong S-domains and universally catenarian domains are examples of locally Jaffard domains. As a matter of fact, the locally Jaffard domains coincide with the *rings satisfying the inequality formula* [3], [4, Théorème 1.5], [16, Lemme 1.4]. Besides the locally Jaffard domains, further examples of Jaffard domains are given by the polynomial rings with the coefficients on a Jaffard domain and by some class of rings arising from the pullback diagrams of a special type (cf. [2], [3], [7], [9], [13] and [16]).

In [10] D. Costa, J. L. Mott and M. Zafrullah introduced the rings of the type $D^{(S)} := D + XD_S[X]$, where D is an integral domain and S is a multiplicatively closed subset of D. If $S := D \setminus \{0\}$ then $D^{(S)} = D + XK[X]$, where $K := \mathrm{qf}(D)$ (= quotient

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field of D). Some properties of the prime spectrum of $D^{(s)}$ were investigated in [10], even if the problem of an exact determination of $\dim D^{(s)}$ was not settled in the general case. In [13], the authors were interested in a more general situation concerning the rings $D^{(s,r)} := D + (X_1,\dots, X_r)D_S[X_1,\dots, X_r]$. After proving a formula for the Krull dimension of $D^{(s,r)}$ [13, Theorem 3.2], they showed that $D^{(s,r)}$ is a Jaffard domain with $\dim D^{(s,r)} = r + \dim D$ if and only if D is a Jaffard domain [13, Theorem 3.5].

In the present work, we investigate an even more general construction, considering the ring $R := A + XB[X] = \{f \in B[X] \mid f(0) \in A\}$, where $A \subset B$ is a ring extension, X is an indeterminant over B. The ring R is a particular case of the constructions B, I, D introduced by J.-P.Cahen in [9] (cf. also [12]).

The ring $\mathrm{Int}(B, A) := \{f \in B[X] \mid f(A) \subseteq A\}$ is a subring of $A + XB[X]$ and a deeper knowledge of the properties of the rings of the type $A + XB[X]$ may have some interesting consequences for the theory of the rings of integer-valued polynomials [1].

In Section 1, we study the structure of the prime spectrum of $R = A + XB[X]$, clarifying the relation among $\mathrm{Spec}(R)$ and the spectra of A and $B[X]$. Furthermore, we will provide upper and lower bounds to $\mathrm{ht}_R XB[X]$ by means of $\mathrm{tr.deg.}_A B$.

In the second section, we will take care of the theory of the dimension and of the transfer of the related properties in the constructions $A + XB[X]$. We will generalize some results previously established for the domains of the type $D + XK[X]$ and $D + XD_S[X]$, where K is a field containing D and S is a multiplicative subset of D. We will prove, among other facts, that $R = A + XB[X]$ is a Jaffard domain and $\dim R = 1 + \dim A$ if and only if A is a Jaffard domain and $\mathrm{deg.tr.}_A B = 0$.

Section 3 is devoted to the investigation of several examples showing the limits of some of the results previously established. We will show also that some of the results, holding for the constructions $D + XD_S[X]$ and for $D + XK[X]$, can not be extended in their classical form to the general construction $A + XB[X]$. We will take this opportunity to describe a new class of Jaffard domains, different from all the known classes.

1. THE PRIME SPECTRUM

We start by establishing some links between the prime ideals of $R = A + XB[X]$ and those of A and $B[X]$. The following lemma is a consequence of some general theorems concerning the pullback constructions [12].

LEMMA 1.1. *Let $A \subset B$ be an extension of rings, X is an indeterminant over B and $R = A + XB[X]$.*

(a) *The ideal $XB[X]$ is a prime ideal of R and $R/XB[X]$ is canonically isomorphic to A. From a topological point of view, the map ${}^a g : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$, corresponding to the canonical projection $g : R \to A$, is a closed embedding and it induces an order-isomorphism of $\mathrm{Spec}(A)$ onto $\mathfrak{X} := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid XB[X] \subset \mathfrak{p}\}$, $\mathfrak{p} \mapsto \mathfrak{p} + XB[X]$. In particular, $\mathfrak{X}$ is a subspace of $\mathrm{Spec}(R)$ stable under specialization.*

(b) *The set $S := \{X^n \mid n \geq 0\}$ is a multiplicatively closed subset of R and of $B[X]$ such that $S^{-1}R = S^{-1}B[X] = B[X, X^{-1}]$. Moreover, by contraction, we obtain an order-isomorphism $\{\mathfrak{Q} \in \mathrm{Spec}(B[X]) \mid X \notin \mathfrak{Q}\} \to \mathfrak{Y} := \{\mathfrak{P} \in \mathrm{Spec}(R) \mid X \notin \mathfrak{P}\}$, and thus $\mathfrak{Y}$ is a subspace of $\mathrm{Spec}(R)$ stable under generalization.*

(c) *The spectral space $\mathrm{Spec}(R)$ is canonically homeomorphic to the amalgamated sum of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B[X])$ with respect to $\mathrm{Spec}(B)$.*

PROOF. **(a)** The map $g : R \to A$, $X \mapsto 0$, is a surjective homomorphism with $\mathrm{Ker}(g) = XB[X]$. Therefore $R/XB[X]$ is isomorphic to A and $XB[X]$ is a prime ideal of R. It is clear that the continuous map ${}^a g : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is closed and injective.

(b) Since S is a multiplicatively closed subset of R then $S^{-1}R = A[X^{-1}] + S^{-1}(XB[X]) = S^{-1}(B[X]) = B[X, X^{-1}] = B[\mathbb{Z}]$. We deduce easily that $\mathfrak{Y}$, $\mathrm{Spec}(B[X, X^{-1}])$ and $\{\mathfrak{P} \in \mathrm{Spec}(B[X]) \mid X \notin \mathfrak{P}\}$ are bijectively equivalent.

(c) is a consequence of the general properties of the pullback constructions, cf. [12]. ∎

For the constructions $D + XK[X]$ and $D + XD_S[X]$, it is known that $\mathrm{ht}\, XK[X] = \mathrm{ht}\, XD_S[X] = 1$ ([10] and [13]; see also Step 1 of the proof of the following Lemma 1.3). Nevertheless, the previous result does not hold in the general case $R = A + XB[X]$. More precisely, we will see later (Example 1.5) that, for each $n \geq 1$, there exist $A \subset B$ such that $\mathrm{ht}_R\, XB[X] = n$. Next goal is an approximation of the height of $XB[X]$ inside R.

THEOREM 1.2. *Let* $R = A + XB[X]$, $N := A \setminus \{0\}$ *and* $k := \mathrm{qf}(A)$.

(a) $\mathrm{ht}_R\, XB[X] = \dim N^{-1}B[X] = \dim (B[X] \otimes_A k)$.

(b) $1 \leq \mathrm{ht}_R\, XB[X] \leq 1 + \mathrm{tr.deg.}_A B$.

In order to prove this theorem, we need the following lemma:

LEMMA 1.3. *In the same situation of* Theorem 1.2,

$$\mathrm{ht}_R\, XB[X] = 1 + \mathrm{Sup}\{\mathrm{ht}_{B[X]}\, \mathfrak{q}[X] \mid \mathfrak{q} \in \mathrm{Spec}(B) \text{ and } \mathfrak{q} \cap A = (0)\}.$$

PROOF. ***Step 1:*** If for each $\mathfrak{q} \in \mathrm{Spec(B)} \setminus \{(0)\}$ we have that $\mathfrak{q} \cap A \neq (0)$, then $\mathrm{ht}_R\, XB[X] = 1$.

As a matter of fact, in this situation, we have $N^{-1}B = \mathrm{qf}(B) =: L$. We deduce that $N^{-1}R = N^{-1}A + XN^{-1}B[X] = k + XL[X]$. Therefore $\mathrm{ht}_R\, XB[X] = \mathrm{ht}_{N^{-1}R}\, XN^{-1}B[X] = 1$ since $\dim N^{-1}R = 1$ [2, Proposition 2.15].

Step 2: There exists a non zero prime ideal $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \cap A = (0)$.

First at all, for each $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \cap A = (0)$ we have $\mathfrak{q}[X] \cap R \subset XB[X]$. As a matter of fact, $((\mathfrak{q}[X] + XB[X])/XB[X]) \cap A = (0)$, hence $(\mathfrak{q}[X] + XB[X]) \cap R = XB[X] \cap R = XB[X]$ thus $\mathfrak{q}[X] \cap R \subset XB[X]$.

If $\mathfrak{q} \in \mathrm{Spec}(B)$ is such that $\mathfrak{q} \cap A = (0)$, then we obtain $\mathrm{ht}_{B[X]}\, \mathfrak{q}[X] = \mathrm{ht}_R\, (\mathfrak{q}[X] \cap R)$ (Lemma 1.1). Therefore, $\mathrm{ht}_R\, XB[X] \geq 1 + \mathrm{ht}_{B[X]}\, \mathfrak{q}[X]$.

Let $(0) \subset \mathfrak{P}_1 \subset \ldots \subset \mathfrak{P}_n \subset XB[X]$ be a chain of prime ideals realizing the height of $XB[X]$ inside R. We claim that, for each $i \in \{1, \ldots, n\}$, $X \notin \mathfrak{P}_i$. If not, we would have $XR = XA + X^2B[X] \subset \mathfrak{P}_i \subset XB[X]$, hence $X^2B[X] \subset \mathfrak{P}_i \subset XB[X]$ and thus $\sqrt{(X^2B[X])} = \mathfrak{P}_i = XB[X]$ inside R: a contradiction. By Lemma 1.1, the previous chain lifts to a chain $(0) \subset \mathfrak{Q}_1 \subset \ldots \subset \mathfrak{Q}_n$ of the same length inside $B[X]$, hence

$\mathrm{ht}\, \mathfrak{P}_n = \mathrm{ht}\, \mathfrak{Q}_n = n$. By [15], $\mathrm{ht}_{B[X]}\, \mathfrak{Q}_n$ can be realized by a special chain $(0) \subset \subset \mathfrak{Q}'_1 \subset \ldots \subset \mathfrak{Q}'_{n-1} \subset \mathfrak{Q}_n$ of prime ideals of $B[X]$. Let $\mathfrak{q} := \mathfrak{Q}_n \cap B$, then either $\mathfrak{q}[X] = \mathfrak{Q}'_{n-1}$ or $\mathfrak{q}[X] = \mathfrak{Q}_n$. In any case, we have $(0) \subset \mathfrak{q} \cap A = \mathfrak{Q}_n \cap B \cap A = \mathfrak{Q}_n \cap R \cap A = \mathfrak{P}_n \cap A \subset XB[X] \cap A = (0)$.

If $\mathfrak{q}$ is not maximal among the prime ideals $\mathfrak{p}$ of B such that $\mathfrak{p} \cap A = (0)$, then let $\mathfrak{q}' \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \subsetneq \mathfrak{q}'$ and $\mathfrak{q}' \cap A = (0)$. We deduce that $\mathfrak{q}[X] \cap R \subset \mathfrak{q}'[X] \cap R \subset XB[X]$, $n - 1 = \mathrm{ht}_{B[X]}\, \mathfrak{q}[X] = \mathrm{ht}_R\, (\mathfrak{q}[X] \cap R)$ and $n \leq \mathrm{ht}_{B[X]}\, \mathfrak{q}'[X] = \mathrm{ht}_R\, (\mathfrak{q}'[X] \cap R) < \mathrm{ht}_R\, XB[X] = n + 1$. Therefore, $\mathrm{ht}_{B[X]}\, \mathfrak{q}'[X] = n$ and $\mathrm{ht}_R\, XB[X] = 2 + \mathrm{ht}_{B[X]}\, \mathfrak{q}[X] = 1 + \mathrm{ht}_{B[X]}\, \mathfrak{q}'[X]$.

Let $\mathfrak{q}$ be maximal among the prime ideals $\mathfrak{p}$ of B such that $\mathfrak{p} \cap A = (0)$. Necessarily we must have $\mathfrak{q}[X] = \mathfrak{Q}_n$: otherwise, $\mathfrak{q}[X] \subsetneq \mathfrak{Q}_n$ implies the existence of the following chain of prime ideals $\mathfrak{q}[X] \cap R \subset \mathfrak{Q}_n \cap R \subset XB[X]$. Therefore, we obtain the chain of prime ideals $(0) \subset (\mathfrak{Q}_n \cap R)/(\mathfrak{q}[X] \cap R) \subset (XB[X])/(\mathfrak{q}[X] \cap R)$ inside the integral domain $R/(\mathfrak{q}[X] \cap R)$ (isomorphic to $A + X(B/\mathfrak{q})[X]$). Because of the maximality of $\mathfrak{q}$, for each $\mathfrak{q}' \in \mathrm{Spec}(B/\mathfrak{q}) \setminus \{0\}$, we get that $\mathfrak{q}' \cap A \neq (0)$, hence we are in the situation of Step 1. We deduce that $\mathrm{ht}_{R'}\, X(B/\mathfrak{q})[X] = 1$, hence $R' := A + X(B/\mathfrak{q})[X]$. We reach a contradiction, since $(XB[X])/(\mathfrak{q}[X] \cap R)$ is isomorphic to $X(B/\mathfrak{q})[X]$ and $\mathrm{ht}((XB[X])/(\mathfrak{q}[X] \cap R)) \geq 2$. Therefore we have that $\mathfrak{q}[X] = \mathfrak{Q}_n$ and thus $\mathrm{ht}_R\, XB[X] = 1 + n = 1 + \mathrm{ht}_{B[X]}\, \mathfrak{Q}_n = 1 + \mathrm{ht}_{B[X]}\, \mathfrak{q}[X]$.

We proved that $\mathrm{ht}_R\, XB[X] \leq 1 + \mathrm{Sup}\{\mathrm{ht}_{B[X]}\, \mathfrak{q}[X] \mid \mathfrak{q} \in \mathrm{Spec}(B) \text{ et } \mathfrak{q} \cap A = (0)\}$ from which the conclusion follows easily. ∎

PROOF OF THEOREM 1.2. (a) We consider the ring $N^{-1}R = N^{-1}A + XN^{-1}B[X] = k + XN^{-1}B[X]$. It is obvious that, for each $\mathfrak{q} \in \mathrm{Spec}(N^{-1}B)$, we have that $\mathfrak{q} \cap k = (0)$. By Lemma 1.1 $\mathrm{ht}_R\, XB[X] = \mathrm{ht}_{N^{-1}R}\, XN^{-1}B[X]$ and, by Lemma 1.3, it follows that $\mathrm{ht}_{N^{-1}R}\, XN^{-1}B[X] = 1 + \mathrm{Sup}\{\mathrm{ht}_{N^{-1}B[X]}\, \mathfrak{q}[X] \mid \mathfrak{q} \in \mathrm{Spec}(N^{-1}B)\} = \dim N^{-1}B[X]$.

(b) We have that $k \subset N^{-1}B[X] \subset L(X) = \mathrm{qf}(B[X])$. By [14, Theorem 20.9], $\dim N^{-1}B[X] \leq \mathrm{tr.deg.}_k\, L(X) = 1 + \mathrm{tr.deg.}_A B$, from the statement (a) the conclusion follows. ∎

We notice that Theorem 1.2 recovers, as a particular case, some of the known results concerning the domains of the type $D + XK[X]$ et $D + XD_S[X]$, where K is a field containing D and S is a multiplicatively closed subset of D. In the following Example 1.5 we will apply the full strength of Theorem 1.2 for computing the height of $XB[X]$ inside $A + XB[X]$, where B is not a field nor a localisation of A.

COROLLARY 1.4. *Let* $A \subset B$, $L := \mathrm{qf}(B)$ *and* S *be a multiplicatively closed subset of* A. *Set* $R := A + XB[X]$, $T := A + XL[X]$ *and* $A^{(S)} := A + XA_S[X]$.

(a1) *If* $\mathrm{qf}(A) \subset B$, *then* $\mathrm{ht}_R\, XB[X] = \dim B[X]$.

(a2) $\mathrm{ht}_T\, XL[X] = 1$.

(b1) *If* $A \subset B$ *is an algebraic extension of integral domains, then* $\mathrm{ht}_R\, XB[X] = 1$.

(b2) $\mathrm{ht}_{A^{(S)}}\, XA_S[X] = 1$.

PROOF. In order to prove **(a1)** it is sufficient to notice that $\mathrm{qf}(A) \subset B$ implies that $N^{-1}B = B$; **(b1)** follows by Theorem 1.2 (b). ∎

EXAMPLE 1.5. For each integer $n \geq 1$, there exist two integral domains $A \subset B$ such that $\mathrm{ht}_R\, XB[X] = n$, where $R := A + XB[X]$.

Let $A := \mathbb{Z}$, $B := \mathbb{Q}[X_1, ..., X_{n-1}]$. By Corollary 1.4 (a1), we obtain that $\mathrm{ht}_R\, XB[X] = \dim \mathbb{Q}[X_1, ..., X_{n-1}][X] = n$.

The following example shows that, for a domain of the type $R = A + XB[X]$, $\mathrm{ht}_R\, XB[X]$ can describe all the integer values between 1 and $1+ \mathrm{tr.deg.}_A B$. In particular, the boundaries established in Theorem 1.2 may not be improved.

EXAMPLE 1.6. Let $d \in \mathbb{N}$ and $t \in \{1, ..., 1+d\}$, then there exists an extension of integral domains $A \subset B$ such that $\mathrm{tr.deg.}_A B = d$ and $\mathrm{ht}_R\, XB[X] = t$, where $R := A + XB[X]$.

As a matter of fact, let k be a field and let $X, X_1, ..., X_{d+1}, Y_1, ..., Y_d$ be indeterminants over k. Set $A := k$ and $B := k(X_1, ..., X_{d-t+1})[Y_1, ..., Y_{t-1}]$. Then,

$\mathrm{tr.deg.}_A B = \mathrm{tr.deg.}_k\, k(X_1, ..., X_{d-t+1}, Y_1, ..., Y_{t-1}) = d$,

$\mathrm{ht}_R\, XB[X] = \dim B[X] = \dim k(X_1, ..., X_{d-t+1})[Y_1, ..., Y_{t-1}]\,[X] = t$ (Corollary 1.4 (a1)).

2. Krull and valuative dimension of $A+XB[X]$

In this section we establish two of the main results of the present work. If $A \subset B$ is a given extension of integral domains, then the first one gives an approximation of the Krull dimension of $A+XB[X]$. In the second result, we determine the valuative dimension of $R = A+XB[X]$ by means of $\dim_v A$ and $\mathrm{tr.deg.}_A B$. As a consequence, we will be able to study the transfer to R of the Jaffard and locally Jaffard properties.

THEOREM 2.1. *Let* $R := A+XB[X]$, $N := A \setminus \{0\}$ *and* $k := \mathrm{qf}(A)$.

(a) $\mathrm{Max}\{\dim N^{-1} B[X] + \dim A \;;\; \dim B[X]\} \leq \dim R \leq \dim A + \dim B[X]$;

(b) *If* $k \subset B$ *then* $\dim R = \dim A + \dim B[X]$.

PROOF. (a) We know that $\mathrm{ht}_R\, XB[X] + \dim R/XB[X] \leq \dim R$. Since $R/XB[X] \cong A$ (Lemma 1.1) and $\mathrm{ht}_R\, XB[X] = \dim N^{-1}B[X]$ (Theorem 1.2), then $\dim N^{-1}B[X] + \dim A \leq \dim R$. Furthermore, $S^{-1}R = B[X, X^{-1}]$ (Lemma 1.1) where $S := \{X^n \mid n \geq 0\}$. By [2, Proposition 1.14], $\dim B[X, X^{-1}] = \dim B[\mathbb{Z}] = \dim B[X]$. We deduce that $\dim B[X] \leq \dim R$, which implies the first inequality.

Let $\mathfrak{P}_0 = (0) \subset \mathfrak{P}_1 \subset ... \subset \mathfrak{P}_n$ be a chain of prime ideals of R which realizes the dimension of R. Let r be the maximum integer of $\{1, ..., n\}$ such that X does not belong to $\mathfrak{P}_r$, hence for each $m \leq r$, X does not belong to $\mathfrak{P}_m$. By using the order-isomorphisms $\{\mathfrak{Q} \in \mathrm{Spec}(B[X]) \mid X \notin \mathfrak{Q}\} \to \mathfrak{Y} = \{\mathfrak{P} \in \mathrm{Spec}(R) \mid X \notin \mathfrak{P}\}$ and $\mathrm{Spec}(A) \to \mathfrak{X} = \{\mathfrak{P} \in \mathrm{Spec}(R) \mid XB[X] \subset \mathfrak{P}\}$ (Lemma 1.10), we deduce that $n - r \leq \dim A$ and $r \leq \dim B[X]$. Therefore, $n = \dim R \leq \dim A + \dim B[X]$.

(b) If $k \subset B$, it is clear that, for each $\mathfrak{q} \in \mathrm{Spec}(B)$, $\mathfrak{q} \cap A = (0)$. By Lemma 1.3 we deduce that $\mathrm{ht}_R\, XB[X] = \dim B[X]$. The conclusion follows easily from Theorem 1.2 and from the point (a). ∎

For the constructions $D^{(S)} := D+XD_S[X]$, it has been proved in [13, Proposition 3.1] and in [10, Theorem 2.6, Corollary 2.9] that $\dim D^{(S)} \leq \mathrm{Min}\{\dim D[X] ; \dim D + \dim D_S[X]\}$. Example 3.1 shows that an inequality of the same type does not hold for the general constructions of the type $A+XB[X]$. More precisely, we will construct a domain $R = A+XB[X]$ such that $\dim R > \dim A[X] > \dim B[X]$, with $\mathrm{qf}(A) = \mathrm{qf}(B)$ (hence, $\dim R = \dim A + \dim B[X]$, Theorem 2.1. (b)). Furthermore, Example 3.1 shows that the double inequality of Theorem 2.1 (a) may be strict, with $\dim A[X] < \dim R$ and $\mathrm{tr.deg.}_A B = 0$.

COROLLARY 2.2. *Let D be an integral domain, K its field of fractions and S a multiplicatively closed subset of D.*

(a) $\dim D+XK[X] = 1 + \dim D$;

(b) $\mathrm{Max}\{1+ \dim D ; \dim D_S[X]\} \leq \dim D+XD_S[X] \leq \dim D + \dim D_S[X]$.

PROOF. (a) (respectively, **(b)**) follows from Theorem 2.1 (b) (respectively from Theorem 2.1 (a)). ∎

THEOREM 2.3. *Let $A \subset B$ be two integral domains and $R := A + XB[X]$.*

(a) $\dim_v R = \dim_v A + \mathrm{tr.deg.}_A B + 1$.

(b) *The following statements are equivalent :*

(i) *A is a Jaffard domain and $\mathrm{qf}(B)$ is an algebraic extension of $\mathrm{qf}(A)$;*

(ii) *R is a Jaffard domain and $\dim R = \dim A + 1$.*

PROOF. (a) We use induction on $\mathrm{d} := \mathrm{tr.deg.}_A B$.

Set $L := \mathrm{qf}(B)$ and $k := \mathrm{qf}(A)$.

Step 1 : $\mathrm{d} = 0$, i.e. L is an algebraic extension of k. We have that $A[X] \subset R \subset B[X]$ and $L(X) = \mathrm{qf}(B[X]) = \mathrm{qf}(R)$ is an algebraic extension of $k(X) = \mathrm{qf}(A[X])$. By [2, Definition-Theorem 0.1], $1 + \dim_v A = \dim_v A[X] = \mathrm{Sup}\{\dim V \mid V$ is a $L(X)$-valuation overring of $A[X]\} \geq \dim_v R = \mathrm{Sup}\{\dim V \mid V$ is a valuation overring of $R\}$. Let V be an L-valuation overring of A such that $\dim V = \dim_v A$ and set $T := V + XL[X]$. The integral domain T is clearly an overring of R. We deduce that $\dim_v R \geq \dim T = 1 + \dim V$ [2, Proposition 2.15], [10, Corollary 2.10], and thus $\dim_v R = 1 + \dim_v A$.

Step 2 : $\mathrm{d} = \mathrm{tr.deg.}_A B = 1$. Let $y \in B$ be a transcendental element over k (its existence follows from the fact that $L = \mathrm{qf}(B)$). We consider the integral domains $R[y] = A[y] + XB[X]$ and $R[y^{-1}] = A[y^{-1}] + XB[y^{-1}][X]$. In this situation, $\mathrm{tr.deg.}_{A[y]} B = 0$ and also $\mathrm{tr.deg.}_{A[y^{-1}]} B[y^{-1}] = 0$ since $\mathrm{tr.deg}_A A[y^{-1}] = 1$. By Step 1, $\dim_v R[y] = \dim_v A[y] + 1 = \dim_v A + 2$ and $\dim_v R[y^{-1}] = \dim_v A[y^{-1}] + 1 = \dim_v A + 2$. Moreover, every overring of R is a valuation overring of $R[y]$ or $R[y^{-1}]$. We deduce that $\dim_v R = \mathrm{Max}\{\dim_v R[y] ; \dim_v R[y^{-1}]\} = \dim_v A + 2$.

Step 3 : $\mathrm{d} = \mathrm{deg.tr.}_A B \geq 1$. We suppose that, for $A' \subset B'$ such that $\mathrm{r} = \mathrm{deg.tr.}_{A'} B' \leq \mathrm{d} - 1$ then $\dim_v R' = \dim_v A' + \mathrm{r} + 1$, where $R' := A' + XB'[X]$. Let $y \in B$ be a transcendental element over k, then $\mathrm{tr.deg.}_{A[y]} B = \mathrm{tr.deg.}_{A[y^{-1}]} B[y^{-1}] = \mathrm{d} - 1$. With the same notation of Step 2, the inductive hypothesis implies that $\dim_v R[y] = \dim_v A[y] + (\mathrm{d} - 1) + 1 = \dim_v A + \mathrm{d} + 1$ and $\dim_v R[y^{-1}] = \dim_v A[y^{-1}] + (\mathrm{d} - 1) + 1 = \dim_v A + \mathrm{d} + 1$. As in Step 2, we have $\dim_v R = \mathrm{Max}\{\dim_v R[y] ; \dim_v R[y^{-1}]\} = \dim_v A + \mathrm{d} + 1$. This completes the proof of **(a)**.

(b) (i) $\Longrightarrow$ (ii)

$$
\begin{aligned}
\dim_v R &= \dim_v A + \mathrm{tr.deg.}_A B + 1 \\
&= \dim A + 1, \quad \text{by hypothesis} \\
&= \dim A + \mathrm{ht}_R\, XB[X], \quad \text{by Corollary 1.4 (b)} \\
&\leq \dim R, \quad \text{by Theorem 2.1 (a)} \\
&\leq \dim_v R,
\end{aligned}
$$

thus R is a Jaffard domain and $\dim R = 1 + \dim A$.

(ii) $\Longrightarrow$ (i) $\dim_v R = \dim_v A + \mathrm{d} + 1 = \dim R = 1 + \dim A$, hence we deduce that $\dim_v A = \dim A$ and $\mathrm{d} = 0$. Therefore, A is a Jaffard domain and $\mathrm{qf}(B)$ is an algebraic extension of $\mathrm{qf}(A)$. ∎

Among the applications of Theorem 2.3, we recover some "classical" result concerning the integral domains $D + XD_S[X]$ and $D + XK[X]$ (cf. [10]) proved in [2, Proposition 2.15 and 2.16], [7, Proposition 2.11 and Corollary 2.12] and [13, Proposition 3.4 and Theorem 3.5 (a)]:

COROLLARY 2.4. *Let D be an integral domain, S a multiplicatively closed subset of D and $K := \mathrm{qf}(D)$.*

(a) $\dim_v D + XD_S[X] = \dim_v D + XK[X] = \dim_v D + 1$;

(b) *D is a Jaffard domain if and only if $D + XK[X]$ is a Jaffard domain* ;

(c) *D is a Jaffard domain if and only if $D + XD_S[X]$ is a Jaffard domain and* $\dim D + XD_S[X] = \dim D + 1$. ∎

COROLLARY 2.5. *Let A be an integral domain having k as its field of fractions and let L be a field extension of k. Set $R = A + XL[X]$.*

(a) $\dim_v R = \dim_v A + \mathrm{tr.deg.}_k L + 1$,

(b) *R is a Jaffard domain if and only if A is a Jaffard domain and L is an algebraic extension of k.* ∎

It was proved in [2, Proposition 2.16 (b)] that if $A \subset B$ and if $\mathrm{qf}(A) = \mathrm{qf}(B)$, when A is a Jaffard domain, then $R = A + XB[X]$ is also a Jaffard domain and $\dim R = \dim A + 1$. The following corollary establishes, among other facts, that the converse holds as well (cf. also [2, Remark 2.17]):

COROLLARY 2.6. *Let $A \subset B$ be an extension of integral domains with the same field of fractions. Set $R := A + XB[X]$.*

(a) $\dim_v R = \dim_v A + 1$,

(b) *A is a Jaffard domain if and only if R is a Jaffard domain and* $\dim R = \dim A + 1$. ∎

Lastly we have:

COROLLARY 2.7. *Let $A \subset B$ be an extension of integral domains such that $\mathrm{qf}(A) \subset B$. Set $R := A + XB[X]$. Then, R is a Jaffard domain if and only if A is a Jaffard domain and* $\dim B[X] = 1 + \mathrm{tr.deg.}_A B$.

PROOF. By Theorem 1.2 (b), Lemma 1.3 and Corollary 1.4 (a1), we deduce that $\mathrm{ht}_R XB[X] = 1 + \mathrm{Sup}\{\mathrm{ht}\, \mathfrak{q}[X] \mid \mathfrak{q} \in \mathrm{Spec}(B) \text{ and } \mathfrak{q} \cap A = (0)\} = \dim B[X] \leq 1 + \mathrm{tr.deg.}_A B$. Theorem 2.3 (a) and Theorem 2.1 (b) lead to the conclusion, since $\dim_v R - \dim R = \dim_v A - \dim A + (1 + d - \dim B[X])$. ∎

In Section 3, we will give several examples showing the limits of the previous results.

THEOREM 2.8. *Let $A \subset B$ be an extension of integral domains and let $R := A + XB[X]$. We suppose that A is a locally Jaffard domain. The following statements are equivalent:*

(i) *$B[X]$ is locally Jaffard and* $\mathrm{ht}_R XB[X] = 1 + \mathrm{tr.deg.}_A B$;

(ii) *R is locally Jaffard.*

In order to prove this theorem we need the following lemma (cf. also [2, Corollary 1.16]):

LEMMA 2.9. *Let B be an integral domain, then $B[X]$ is locally Jaffard if and only if $B[X, X^{-1}]$ is locally Jaffard.*

PROOF. We suppose that $B[X, X^{-1}]$ is a locally Jaffard domain. Since $B[X, X^{-1}]$ is integral over $B[X + X^{-1}]$, for each $\mathfrak{q} \in \mathrm{Spec}(B[X + X^{-1}])$, if $T := (B[X + X^{-1}] \setminus \mathfrak{q})$, then $T^{-1}B[X, X^{-1}]$ is integral over $T^{-1}B[X + X^{-1}] = B[X + X^{-1}]_{\mathfrak{q}}$. Now $B[X, X^{-1}]$ is locally Jaffard, then $B[X + X^{-1}]_{\mathfrak{q}}$ is the same [2, Proposition 1.1 and Proposition 1.5 (a)]. Since $B[X + X^{-1}]$ is isomorphic to $B[X]$, then $B[X]$ is also locally Jaffard and the lemma is proved. ∎

PROOF OF THEOREM 2.8.

(i) ⟹ (ii). Let A and $B[X]$ be locally Jaffard domains and $\mathrm{ht}_R XB[X] = 1 + \mathrm{tr.deg.}_A B$, we want to prove that, for each $\mathfrak{P} \in \mathrm{Spec}(R)$, $R_{\mathfrak{P}}$ is a Jaffard domain. For such a prime $\mathfrak{P}$ two cases are possible :

Case 1 : $X \in \mathfrak{P}$. There exists $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{P} = \mathfrak{p} + XB[X]$ (Lemma 1.1 (a)). It is easy to verify that $R_{\mathfrak{P}} = A_{\mathfrak{p}} + XT^{-1}B[X]$, where $T^{-1} := R \setminus (\mathfrak{p} + XB[X])$. We have that $A_{\mathfrak{p}} + XB_{\mathfrak{p}}[X] \subset R_{\mathfrak{P}} \subset A_{\mathfrak{p}} + XL[X]_{(X)}$, with $B_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1}B$ and $L = \mathrm{qf}(B)$, and we notice that all these rings have the same field of fractions. By the previous remarks, we deduce that :

- $\dim_v(A_{\mathfrak{p}} + XB_{\mathfrak{p}}[X]) = \dim_v A_{\mathfrak{p}} + \mathrm{tr.deg.}_A B + 1$ (Theorem 2.3 (a));
- $A_{\mathfrak{p}} + XL[X]_{(X)}$ is the pullback of the inclusion $A_{\mathfrak{p}} \to L$ with respect to the canonical projection $L[X]_{(X)} \to L$.

By [2, Theorem 2.6], $\dim_v(A_{\mathfrak{p}} + XL[X]_{(X)}) = \dim_v A_{\mathfrak{p}} + \dim_v L[X]_{(X)} + \mathrm{tr.deg.}_A L = \dim_v A_{\mathfrak{p}} + 1 + \mathrm{tr.deg.}_A B$. On the other hand, $\dim R_{\mathfrak{P}} = \mathrm{ht}_R \mathfrak{P} = \mathrm{ht}_R(\mathfrak{p} + XB[X]) \geq \mathrm{ht}_A \mathfrak{p} + \mathrm{ht}_R XB[X] = \dim A_{\mathfrak{p}} + \mathrm{tr.deg.}_A B + 1 = \dim_v A_{\mathfrak{p}} + \mathrm{tr.deg.}_A B + 1 = \dim_v R_{\mathfrak{P}}$, thus $R_{\mathfrak{P}}$ is a Jaffard domain.

Case 2 : $X \notin \mathfrak{P}$. Set $S := \{X^n \mid n \geq 0\}$. Then $S \cap \mathfrak{P} = \emptyset$ and $R_{\mathfrak{P}} = (S^{-1}R)_{S^{-1}\mathfrak{P}}$ [14, Corollary 5.3]. On the other hand, $S^{-1}R = S^{-1}(B[X]) = B[X, X^{-1}]$ (Lemma 1.1 (b)), and $B[X]$ is locally Jaffard. It follows that $R_{\mathfrak{P}}$ is a Jaffard domain.

(ii) $\Longrightarrow$ (i). If R is locally Jaffard and $S := \{X^n \mid n \geq 0\}$, then $S^{-1}R = S^{-1}(B[X]) = B[X, X^{-1}]$ is the same. Lemma 2.9 allows to conclude that $B[X]$ is a locally Jaffard domain. Since $R_{XB[X]}$ is a Jaffard domain, then $\mathrm{ht}_R XB[X] = \dim R_{XB[X]} = \dim_v R_{XB[X]}$. By replacing $\mathfrak{p}$ with (0) in (i) $\Longrightarrow$ (ii) of Case 1, we obtain that $\dim_v R_{XB[X]} = \dim_v \mathrm{qf}(A) + \mathrm{tr.deg.}_A B + 1 = \mathrm{tr.deg.}_A B + 1$ and the proof of the theorem is complete. ∎ .

We notice that the hypothesis that A is a Jaffard domain (instead of a locally Jaffard domain) in Theorem 2.8 is not sufficient for the conclusion. As a matter of fact, it is sufficient to consider a Jaffard non locally Jaffard domain A [2, Example 3.2], and $B := \mathrm{qf}(A)$. In this situation $B[X]$ is locally Jaffard and $\mathrm{ht}_R XB[X] = 1 = 1 + \mathrm{tr.deg.}_A B$ but R is not locally Jaffard [2, Corollary 2.12 (a)]. However, the hypothesis that A is locally Jaffard is not necessary in order that R is also locally Jaffard (cf. Example 3.1 (d)).

COROLLARY 2.10. *Let D be a domain and $K := \mathrm{qf}(D)$. The following are equivalent:*

(i) *D is locally Jaffard;*

(ii) *$D + XD_S[X]$ is locally Jaffard, for each multiplicatively closed subset S of D ;*

(iii) *$D + XK[X]$ is locally Jaffard.*

PROOF. (i) $\Longrightarrow$ (ii) is a consequence of Theorem 2. 8, (i) $\Longrightarrow$ (ii) (cf. also [9, Proposition 1 (i)] ; (ii) $\Longrightarrow$ (iii) holds trivially and (iii) $\Longrightarrow$ (i) is a particular case of [2, Corollary 2.12 (a)] . ∎

Theorem 2.8 will give us the possibility to construct new examples of locally Jaffard domains (cf. Examples 3.1 and 3.5).

We notice that, if $X_1, \ldots, X_n$ are indeterminants over $R = A + XB[X]$, then $R[X_1, \ldots, X_n] = A[X_1, \ldots, X_n] + XB[X_1, \ldots, X_n][X]$ is a ring of the same type (i.e. $R[X_1, \ldots, X_n] = A' + XB'[X]$ with $A' := A[X_1, \ldots, X_n]$ and $B' := B[X_1, \ldots, X_n]$). From the previous remark, we are led to studying the ring $R[X_1, \ldots, X_n]$ by means of the techniques introduced above.

As a consequence of Theorem 2.1 (a) and 2.3 (a), and of the fact that $\dim R[X_1, \ldots, X_n] \leq \dim_v R[X_1, \ldots, X_n]$, we deduce :

(*) $\mathrm{Max}\{\dim A[X_1,\ldots, X_n] + \mathrm{ht}_{R[X_1,\ldots,X_n]} XB[X_1,\ldots, X_n][X]\ ;\ \dim B[X_1, \ldots, X_n][X]\} \leq$

$$\leq \dim R[X_1, \ldots, X_n] \leq$$

$$\leq \mathrm{Min}\{\dim_v A[X_1,\ldots, X_n] + \mathrm{tr.deg.}_A B + 1\ ;\ \dim A[X_1,\ldots, X_n] + \dim B[X_1,\ldots, X_n][X]\}.$$

Therefore, for n big enough, it is possible to evaluate the dimension of $R[X_1, \ldots, X_n]$:

COROLLARY 2.11. *Let $A \subset B$ be an extension of integral domains, $R := A + XB[X]$, n an integer and let $X_1, \ldots, X_n$ be indeterminants over R .*

(a) *If* $n \geq \mathrm{Max}\{\mathrm{tr.deg.}_A B\ ;\ \dim_v A - 1\}$, *then* $\dim R[X_1, \ldots, X_n] = \dim A[X_1, \ldots, X_n] + \mathrm{tr.deg.}_A B + 1$ *and $R[X_1, \ldots, X_n]$ is a Jaffard domain.*

(b) *If* $n < \mathrm{tr.deg.}_A B$ *and if B is a field, then* $\dim R[X_1, \ldots, X_n] = \dim A[X_1, \ldots, X_n] + n + 1$ *and $R[X_1, \ldots, X_n]$ is not a Jaffard domain.*

PROOF. (a). By Theorem 1.2 (b), we deduce that $\mathrm{ht}_{R[X_1, \ldots, X_n]}XB[X_1, \ldots, X_n][X] \leq \text{tr.deg.}_A B + 1$. The integral domains R and $B[X]$ have in common the ideal $XB[X]$, then by [9, Lemma 3] we obtain:

$\mathrm{ht}_{R[X_1,\ldots,X_n]} XB[X_1, \ldots, X_n][X] \geq \mathrm{ht}_{B[X_1,\ldots,X_n][X]}XB[X_1, \ldots, X_n][X] + \mathrm{Min}\{n, \text{tr.deg.}_A B\}$.

Therefore, $\mathrm{ht}_{R[X_1,\ldots,X_n]} XB[X_1, \ldots, X_n][X] \geq \text{tr.deg.}_A B + 1$, thus we deduce the equality.

On the other hand, by (*) we get $\dim A[X_1, \ldots, X_n] + \text{tr.deg.}_A B + 1 \leq$ $\leq \dim R[X_1, \ldots, X_n] \leq \dim_v R[X_1, \ldots, X_n] = \dim_v A[X_1, \ldots, X_n] + \text{tr.deg.}_A B + 1 =$ $\dim A[X_1, \ldots, X_n] + \text{tr.deg.}_A B + 1$, (where $\dim_v A[X_1, \ldots, X_n] = \dim A[X_1, \ldots, X_n]$ since $n \geq \dim_v A - 1$, [2, Definition-Theorem 0.1]) and thus $\dim R[X_1, \ldots, X_n] = \dim_v R[X_1, \ldots, X_n]$.

(b) Let $n < \text{tr.deg.}_A B$, by the same reason as in (a) we get $\mathrm{ht}_{R[X_1, \ldots, X_n]} XB[X_1, \ldots, X_n][X] \geq n + 1$. We deduce that $1 + n + \dim A[X_1, \ldots, X_n] \leq \dim R[X_1, \ldots, X_n] \leq \dim A[X_1, \ldots, X_n] + \dim B[X_1, \ldots, X_n][X]$. Therefore, if B is a field, then $\dim B[X_1, \ldots, X_n][X] = n + 1$ and $\dim R[X_1, \ldots, X_n] = \dim A[X_1, \ldots, X_n] + n + 1 < \dim_v A[X_1, \ldots, X_n] + \text{tr.deg.}_A B + 1 = \dim_v R[X_1, \ldots, X_n]$. As a consequence, we obtain that $R[X_1, \ldots, X_n]$ is not a Jaffard domain. ∎

REMARK 2.12. It is known by [13, Theorem 3.5 (a)] that D is a Jaffard domain if and only if $D + XD_S[X]$ is a Jaffard domain and $\dim(D + XD_S[X]) = 1 + \dim D$. This result can not be directely generalized to the general constructions $A + XB[X]$ with $\text{tr.deg.}_A B > 0$ (cf. Examples 3.4 and 3.5).

3. EXAMPLES AND COUNTER-EXAMPLES

In this section we construct several examples showing the limits of the results proved in Sections 1 and 2. We give also a few counter-examples showing that some results

concerning the domains of the type $D + XD_S[X]$ can not be extended to the general constructions $A + XB[X]$.

EXAMPLE 3.1. Let K be a field and let X, X_1, X_2, X_3, X_4 be indeterminants over K. Set,

$A := K[X_1]_{(X_1)} + X_4K(X_1, X_2, X_3)[X_4]_{(X_4)}$,

$B := K(X_1)[X_2]_{(X_2)} + X_3K(X_1, X_2)[X_3]_{(X_3)} + X_4K(X_1, X_2, X_3)[X_4]_{(X_4)}$,

$R := A + XB[X]$.

Then:

(a) $\mathrm{Max}\{\dim A + \mathrm{ht}_R XB[X], \dim B[X]\} < \dim R < \dim A + \dim B[X]$.

(b) $\dim A[X] < \dim R$, (notice that for the construction $D + XD_S[X]$, it happens that $\dim D + XD_S[X] \leq \dim D[X]$, [13, Proposition 3.1]).

(c) R shows that [13, Theorem 3.5 (b) (j) $\Longrightarrow$ (jj)], concerning the domains $D + XD_S[X]$, can not be extended to the constructions $A + XB[X]$.

(d) R is a locally Jaffard domain, even if A is not a locally Jaffard domain (cf. Theorem 2.8 and [2, Proposition 2.16]).

As a matter of fact, set $L := K(X_1, X_2, X_3)$, $k := K(X_1, X_2)$, $\mathfrak{M} := X_4L[X_4]_{(X_4)}$, $\mathfrak{N} := X_3k[X_3]_{(X_3)}$, $V := L + \mathfrak{M}$, $V_1 := k + \mathfrak{N}$, $D := K(X_1)[X_2]_{(X_2)} + \mathfrak{N}$, $B := D + \mathfrak{M}$ and $A := K[X_1]_{(X_1)} + \mathfrak{M}$. By some well known result concerning the $D + \mathfrak{M}$ domains, by [2, Corollary 2.8] and [12, Proposition 2.1 (5) and Theorem 2.4 (1)] we obtain that V, V_1, D and B are valuation domains of dimensions 1, 1, 2 and 3 respectively. Moreover:

- $\dim B[X] = \dim B + 1 = 4$,
- $\dim A = \dim K[X_1]_{(X_1)} + \dim V = 2$ [2, Corollary 2.8],
- $\dim_v A = \dim_v K[X_1]_{(X_1)} + \dim V + \text{tr.deg.}_{K(X_1)} L = 4$ [2, Proposition 2.14 (a)],
- $\dim A[X] = \dim V + \dim K[X_1]_{(X_1)}[X] + \mathrm{Min}\{1, \text{tr.deg.}_{K(X_1)} L\} = 4$ [2, Corollary 2.8],
- $\mathrm{Spec}(B) = \{(0); \mathfrak{M}; \mathfrak{P}_1 := \mathfrak{N} + \mathfrak{M}; \mathfrak{P}_2 := X_2K(X_1)[X_2]_{(X_2)} + \mathfrak{P}_1\}$,
- $\mathrm{Spec}(A) = \{(0); \mathfrak{M}; \mathfrak{Q} = X_1K[X_1]_{(X_1)} + \mathfrak{M}\}$,
- $\mathfrak{M} \cap A = \mathfrak{P}_1 \cap A = \mathfrak{P}_2 \cap A = \mathfrak{M}$ [12, Theorem 1.4].

(a) and **(b)**. We notice that $qf(A) = qf(B) = qf(V)$, since they A, B and V have the ideal $\mathfrak{M}$ in common. Inside $\mathrm{Spec}(R)$ we have the following chain of prime ideals :

© $(0) \subset \mathfrak{M}[X] \cap R \subset \mathfrak{P}_1[X] \cap R \subset \mathfrak{P}_2[X] \cap R \subset \mathfrak{M} + XB[X] \subset \mathfrak{Q} + XB[X]$.

Therefore $\dim R \geq 5$. By Theorem 2.3 (a), we deduce that $\dim_v R = \dim_v A + \mathrm{tr.deg.}_A B + 1 = 5$, thus $\dim R = 5 = \dim_v R$. As a consequence, we have :

$$(\dim A[X] = 4) < (\dim R = 5) < (\dim A + \dim B[X] = 6).$$

On the other hand (Theorem 1.2 (b)), we have

$$(\dim A + \mathrm{ht}_R\, XB[X] = 2 + 1 = 3) < (\dim B[X] = 4) < (\dim R = 5).$$

(c) Since $\dim R = \dim_v R = 5$, R is then a Jaffard domain. Furthermore, $\dim A[X] = 4$ and $\dim_v A[X] = \dim_v A + 1 = 5$, then $A[X]$ is not a Jaffard domain. This example shows that [13, Theorem 3.5 (b) (j) $\Longrightarrow$ (jj)] can not be extended to the constructions $A + XB[X]$.

(d) The domain A is not a locally Jaffard domain [2, Proposition 1.5 (b)], since it is not a Jaffard domain. In order to show that R is a locally Jaffard domain, it is sufficient to see what happens for the prime ideals of the type $\mathfrak{p} + XB[X]$ with $\mathfrak{p} \in \mathrm{Spec}(A)$. As a matter of fact, if $\mathfrak{P} \in \mathrm{Spec}(R)$ and if $X \notin \mathfrak{P}$, by setting $S := \{ X^n \mid n \geq 0 \}$, we get that $R_{\mathfrak{P}} = (S^{-1}R)_{S^{-1}\mathfrak{P}} = B[X, X^{-1}]_{S^{-1}\mathfrak{P}}$ which is a Jaffard domain (because B is a valuation domain). By the proof of Theorem 2.8 (Case1), we deduce that $\dim_v R_{(\mathfrak{p}+XB[X])} = \dim_v A_{\mathfrak{p}} + \mathrm{tr.deg.}_A B + 1$. We claim that $R_{(\mathfrak{p}+XB[X])}$ is a Jaffard domain, for each $\mathfrak{p} \in \mathrm{Spec}(A)$.

- $\mathfrak{P} = XB[X]$, i.e. $\mathfrak{p} = (0)$. In this case, $\dim R_{\mathfrak{P}} = \dim R_{XB[X]} = \mathrm{ht}_R\, XB[X] = 1 = \dim_v A_{(0)} + \mathrm{tr.deg.}_A B + 1 = \dim_v R_{\mathfrak{P}}$.
- $\mathfrak{P} = \mathfrak{M} + XB[X]$. In the present situation, the chain © shows that $\mathrm{ht}\, \mathfrak{P} = 4$ (since $\dim R = 5$), hence $\dim R_{\mathfrak{P}} = 4 = \dim_v K(X_1) + \dim V + \mathrm{tr.deg.}_{K(X_1)} L + 1 = \dim_v A_{\mathfrak{M}} + 1 = \dim_v R_{\mathfrak{P}}$.
- $\mathfrak{P} = \mathfrak{Q} + XB[X]$. In this case, $\dim R_{\mathfrak{P}} = \mathrm{ht}\, \mathfrak{P} = 5$ (see the chain ©) and $\dim_v R_{\mathfrak{P}} = \dim_v A_{\mathfrak{Q}} + 1 = \dim_v K[X_1]_{(X_1)} + \dim V + \mathrm{tr.deg.}_{K(X_1)} L + 1 = 5$, since $A_{\mathfrak{Q}} = A$ (cf. also [2, Theorem 2.6 (a)])

In all the cases, $R_{\mathfrak{P}}$ is a Jaffard domain. ∎

Theorem 2.3 shows the way to construct new classes of Jaffard domains.

EXAMPLE 3.2. Let,

- $A_1 := \mathbb{Z}$, $B_1 := \mathbb{Z}^\nabla$ and $R_1 := \mathbb{Z} + X\mathbb{Z}^\nabla[X]$,

where $\mathbb{Z}^\nabla$ is the integral closure of $\mathbb{Z}$ inside an algebraic extension of $\mathbb{Q}$.

Since A_1 is a Jaffard domain, then by Theorem 2.3 (b) we deduce that R_1 is a Jaffard domain and $\dim R_1 = 1 + \dim A_1 = 2 < \dim A_1 + \dim B_1[X] = 1 + 2 = 3$ (cf. also Theorem 2.1 (a)). It is possible to show that R_1 is a Noetherian domain if and only if $\mathbb{Z}^\nabla$ is the integral closure of $\mathbb{Z}$ inside a finite extension of $\mathbb{Q}$.

We consider

- $A_2 := \mathbb{R}$, $B_2 := \mathbb{C}[Y]$ and $R_2 := \mathbb{R} + X\mathbb{C}[Y][X]$,

Since $qf(A_2) \subset B_2$ then, by Theorem 2.1 (b), we deduce that $\dim R_2 = \dim A_2 + \dim B_2[X] = 2$ and, by Theorem 2.3 (a), that $\dim_v R_2 = \dim_v A_2 + \mathrm{tr.deg.}_{A_2} B_2 + 1 = 2$. Therefore R_2 is a Jaffard domain. Moreover $A_2[X] = \mathbb{R}[X]$ is obviously a Jaffard domain, but $\dim R_2 \neq \dim A_2[X]$. (Notice that, for the domain R_2, the inequalities of Theorem 2.1 (a) are both equalities.) ∎

The following two examples show the limits of some of the results established in Section 2.

EXAMPLE 3.3. Let :

- $A := \mathbb{Z}$, $B := \mathbb{Q}(Y)$ and $R := \mathbb{Z} + X\mathbb{Q}(Y)[X]$.

Then :

(a) The bounds of the inequalities established in Theorem 2.1 (a) can be effectively reached.

(b) R shows that [13, Theorem 3.5 (a), (i) $\Longrightarrow$ (ii)] and [13, Theorem 3.5 (b), (jj) $\Longrightarrow$ (j)] can not be extended to the case $A + B[X]$.

As a matter of fact, let $N = A \setminus \{0\}$:

(a) $\dim R = \dim A + \dim B[X] = 2$ (Theorem 2.1 (b)) and $\dim N^{-1}B[X] + \dim A = 1 + 1 = 2$. Therefore, $\dim N^{-1}B[X] = 1$, since $\dim N^{-1}B[X] + \dim A \leq \dim R$.

Henceforth, $\mathrm{Max}\{\dim N^{-1}B[X] + \dim A\ ,\ \dim B[X]\} = \dim N^{-1}B[X] + \dim A = \dim R = \dim A + \dim B[X] = 2$, thus the bounds of the inequalities established in Theorem 2.1 (a) can be effectively reached.

(b) By (a), $\dim R = 2$ and Theorem 2.3 (a) shows that $\dim_v R = \dim_v \mathbb{Z} + \text{tr.deg.}_{\mathbb{Q}}\ \mathbb{Q}(Y) + 1 = 3$, thus R is not a Jaffard domain. However, A is a Jaffard domain [13, Theorem 3.5 (a), (i) $\Longrightarrow$ (ii)]. Moreover, $A[X]$ is a Jaffard domain and $\dim R = \dim A[X]$, in contrast with [13, Theorem 3.5 (b), (jj) $\Longrightarrow$ (j)] for the domains of the type $D + XD_S[X]$. ∎

EXAMPLE 3.4. Let :

- $A := \mathbb{Z}$, $B := \mathbb{Q}[Y]$ and $R := \mathbb{Z} + X\mathbb{Q}[Y][X]$.

Alors,

(a) R shows that [13, Theorem 3.5 (a), (i) $\Longrightarrow$ (ii)], [13, Theorem 3.5 (b), (j) $\Longrightarrow$ (jj)] and [2, Proposition 2.15 (a)] can not be extended to the case $A + B[X]$.

(b) $\dim R > \dim A[X]$ with $\text{tr.deg.}_A B > 0$.

As a matter of fact,

(a) $\mathrm{qf}(A) \subset B$, then by Theorem 2.1 (b) we deduce that $\dim R = \dim A + \dim B[X] = 1 + 2 = 3$, and hence Theorem 2.3 (a) shows that $\dim_v R = \dim_v A + \text{tr.deg.}_A B + 1 = 3$. We deduce that R is a Jaffard domain. Since $A = \mathbb{Z}$ is a Jaffard domain, but $\dim R \neq \dim A + 1$, the domain R shows that [13, Theorem 3.5 (a), (i) $\Longrightarrow$ (ii)] can not be extended from the case $D + XD_S[X]$ to the general situation $A + B[X]$.

The rings R and $A[X]$ are Jaffard domains, but $\dim R \neq \dim A[X]$, hence [13, Theorem 3.5 (b), (j) $\Longrightarrow$ (jj)] can not be extended to the general construction $A + B[X]$.

If A and R are the Jaffard domains introduced above, since $\text{tr.deg.}_A B \neq 0$, then it is clear that [2,Proposition 2.15] does not hold for the general construction $A + B[X]$.

(b) We know that $3 = \dim R > \dim A[X] = 2$, like in Example 3.1 (b), but in this case $\text{tr.deg.}_A B \neq 0$. ∎

EXAMPLE 3.5. Let K be a field and let X, Y, Z, W be indeterminants over K . Set :

$A := K[Y]_{(Y)}$, $B := K[Y]_{(Y)} + ZK(Y)[Z]_{(Z)} + WK(Y, Z)[W]_{(W)}$ and $R := A + XB[X]$.

Then,

(a) $\mathrm{ht}_R\, XB[X] = 1 + \text{tr.deg.}_A B > 1$.

(b) $\dim R = \dim A + \mathrm{ht}_R\, XB[X]$.

(c) R is a locally Jaffard domain, different from all the examples already known.

As a matter of fact, set :

$V := K(X, Y)[W]_{(W)}$, $\mathfrak{M} := WK(X, Y)[W]_{(W)}$, $V' := K[Y]_{(Y)} + ZK(Y)[Z]_{(Z)}$,

then,

(a) $A \subset B = V' + \mathfrak{M}$ are both valuation domains, with $\dim A = 1$ and $\dim B = 3$ (cf. [12, Proposition 2.1 (5), Theorem 2.4 (1)]), and $d := \text{tr.deg.}_A B = 2$. Moreover :

$\mathrm{Spec}(B) = \{(0)\ ;\ \mathfrak{M}\ ;\ \mathfrak{P} := ZK(Y)[Z]_{(Z)} + \mathfrak{M}\ ;\ \mathfrak{Q} := YK[Y]_{(Y)} + \mathfrak{P}\}$,

$\mathrm{Spec}(A) = \{(0)\ ;\ YK[Y]_{(Y)}\}$,

$\mathfrak{M} \cap A = \mathfrak{P} \cap A = (0)$, et $\mathfrak{Q} \cap A = YK[Y]_{(Y)}$,

thus $\mathrm{ht}_R\, XB[X] = 3 = 1 + \text{tr.deg.}_A B$ (Theorem 1.2 (b) and Lemma 1.3) .

(b) We notice that $\dim R \geq \dim A + \mathrm{ht}_R\, XB[X] = 1 + 3 = 4$ (Theorem 1.2 (a)), and that $\dim_v R = \dim_v A + \text{deg.tr.}_A B + 1 = 4$ (Theorem 2.3 (a)). Therefore R is a Jaffard domain with Krull dimension $4 = \dim A + \mathrm{ht}_R\, XB[X] < \dim A + \dim B[X] = 5$ (thus the hypothesis $\mathrm{qf}(A) \subset B$ in Corollary 1.4 (a1) is essential).

(c) We notice that A and B are locally Jaffard domains, since they are both valuation domains. It is clear that $B[X]$ is a locally Jaffard domain [9, Proposition 1 (i)] and by Theorem 2.8 it follows that R is a locally Jaffard domain, since $\mathrm{ht}_R\, XB[X] = 1 + \text{tr.deg.}_A B$ (cf. (a)) . ∎

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13
Divisorial Ideals and Class Groups of Mori Domains

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The properties of divisorial ideals and class groups of noetherian integrally closed domains, more generally of Krull domains, have been extensively studied in the past and are now well known [F]. More recently, the notion of class group has been introduced for any domain in [Bv] and [BvZ] and its general properties have been studied by several authors (see for example [A], [AA], [AAZ], [ARy], [G], [GRt], [NA], [Ry]). However the structure of the class group is known for only a few special families of domains.

A first class of domains for which this investigation has been carried out is that of Mori domains, namely those domains with the ascending chain condition on divisorial ideals [Rl], [BG1], [BG2], [BGR2]. A motivation to consider this kind of domain is that noetherian and Krull domains are Mori. Thus, the results obtained hold in particular for noetherian domains and moreover, since a Krull domain is a Mori completely integrally closed domain, this study puts in evidence which properties of Krull domains depend uniquely on the ascending chain condition on divisorial ideals and which ones depend also on the condition of being completely integrally closed.

In this paper we will survey recent and older results on this subject.

Let R be an integral domain and K its quotient field.

An ideal I of R is *divisorial* if $I = I_v := R:(R:I) = \cap\{xR : x \in K, xR \supset I\}$ and a divisorial ideal I is v-*finite* if there is a finitely generated ideal J such that $I = J_v$. We denote by D(R) the set of all divisorial ideals of R and by $D_f(R)$ the set of all v-finite ideals of R. The sets D(R) and $D_f(R)$ are semigroups, with unit R, with respect to the operation $I*J = (IJ)_v$. The group P(R) of principal ideals of R is a subgroup of $D_f(R)$ and