Krull and valuative dimension of the Serre conjecture ring $R\langle n \rangle$

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Abstract. In this paper, we deal with the Serre conjecture ring $R\langle n \rangle$. The purpose is to give the Krull dimension and valuative dimension of the ring $R\langle n \rangle$. As a consequence, we characterize when it is Jaffard, and more precisely locally, residually or totally Jaffard.


Key words: Polynomials, Nagata ring, Serre conjecture ring, Jaffard rings, strong S-rings.

Introduction

Throughout this paper $R$ is a commutative ring with a unit element. We denote by $R[n]$ the ring of polynomials in $n$ indeterminates on $R$ (but rather by $R[X]$ the ring in one indeterminate). Letting $U$ be the multiplicative set of monic polynomials in $R[X]$, we denote by $R\langle X \rangle$ the localization $R\langle X \rangle = U^{-1}R[X]$ and we set $R\langle X_1, ..., X_n \rangle = R\langle X_1, ..., X_{n-1} \rangle\langle X_n \rangle$, where $X_1, ..., X_n$ are $n$ indeterminates. We note at once that the order of these indeterminates is in general pertinent in the definition of $R\langle X_1, ..., X_n \rangle$, since, for any two indeterminates $X$ and $Y$, $R\langle X \rangle\langle Y \rangle$ need not be equal $R\langle Y \rangle\langle X \rangle$ [9, Theorem 10]. Although this order is significant in general, it has no influence throughout this work, so we can denote $R\langle X_1, ..., X_n \rangle$ by $R\langle n \rangle$. We say that $R\langle n \rangle$ is the Serre conjecture ring in $n$ indeterminates on $R$. Letting $S$ be the multiplicative set in $R[n]$ formed by the polynomials whose coefficients generate $R$, we recall that the localization $R\langle n \rangle = S^{-1}R[n]$ is called the Nagata ring on $R$ with $n$ indeterminates on $R$. It is clear that $R\langle n \rangle$ is a localization of $R\langle n \rangle$ and that we always have $R[n] \subseteq R\langle n \rangle \subseteq R(n)$.

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We denote by $\dim R$ the Krull dimension of $R$ and by $\dim_v R$ its valuative dimension, i.e. the limit of the sequence $(\dim R[n] - n)$ (and we emphasize that $R$ need not be a domain with such a definition). In a first section we establish the Krull and valuative dimension of the rings $R\langle n \rangle$ and $R(n)$.

Recall that a finite dimensional ring $R$ is said to be Jaffard if $\dim R[n] = \dim R + n$, for all $n$ [1], or equivalently $\dim R = \dim_v R$, residually Jaffard if the quotient of $R$ by any prime $\mathfrak{p}$ is Jaffard, locally Jaffard if the localization of $R$ at any prime $\mathfrak{p}$ is Jaffard and lastly totally Jaffard if any quotient of any localization (equivalently any localization of any quotient) is Jaffard [8] (we may note that these last two definitions make sense if $R$ is only supposed to be locally finite dimensional). In a second section we investigate the transfer of the Jaffard (and more precisely of the locally, residually and totally Jaffard) properties from the Nagata ring $R(n)$ to the Serre conjecture ring $R\langle n \rangle$ and conversely.

Letting $R\langle \infty \rangle$ (resp. $R(\infty)$) be the union $R\langle \infty \rangle = \bigcup_n R\langle n \rangle$ (resp. $R(\infty) = \bigcup_n R(n)$), we say that $R\langle \infty \rangle$ (resp. $R(\infty)$) is the infinite Serre conjecture ring (resp. infinite Nagata ring) on $R$. In a third and last section we show the Krull dimension of these rings to be the valuative dimension of $R\langle n \rangle$ and $R(n)$.

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Terminology is standard as in [17]. We use “$\subseteq$” to denote proper containment. If $\mathfrak{P}$ is a prime ideal of $R\langle n \rangle$, $R[n]$, $R(n)$, $R(\infty)$ or $R(\infty)$ and $\mathfrak{p} = \mathfrak{P} \cap R$, we say that $\mathfrak{P}$ is above $\mathfrak{p}$. By convention, we let $R[0]$, $R(0)$ and $R(\infty)$ be the ring $R$. 
1 Krull and valuative dimensions

It is clear that every prime ideal upper to a maximal ideal in $R[X]$ contains a monic polynomial [17, theorem 28]. Hence every maximal ideal of $R[X]$ is either the extension $m(X)$ of a maximal ideal $m$ of $R$, or the localisation of a prime ideal $\mathfrak{p}$ of $R[X]$ which is an upper to a non maximal prime ideal $p$ of $R$. We thus get immediately the following, as already shown in [5, lemma1] and [18, Th.2.1].

Lemma 1.1 For any ring $R$, $\dim R[X] = \dim R[X] - 1$.

We generalize the result of lemma 1.1 as follows:

Proposition 1.2 Let $R$ be a ring and $n$, $r$ two non negative integers, then $\dim R(n)[r] = \dim R(n)[r] = \dim R(n) - n$.

Proof. Since $R(n)$ is a localization of $R(n)$, we have $\dim R(n)[r] \leq \dim R(n)[r]$ (1)

We next prove that $\dim R(n) - n \leq \dim R(n)[r]$ (2)

Letting $m$ be a maximal ideal of $R$ such that $\dim R(n+r) = htm[n+r] + (n+r)$, then $\dim R(n)[r] \geq htm(n)[r] + r = htm[n][r] + r = htm[n + r] + r$, thus $\dim R(n)[r] \geq htm(n + r) - n$.

Lastly we prove, by induction on $n \geq 1$, that $\dim R(n)[r] \leq \dim R(n + r) - n$ (3)

Case $n = 1$. From the special chain theorem [6, theorem 1], we have, $\dim R(1)[r] = \sup \{htM[r] + r\}$ (4)

where $\mathfrak{M}$ runs among the maximal ideals of $R(1)$. As noticed above, two cases may occur.

a) $\mathfrak{M}$ is the extension $m(1)$ of a maximal ideal $m$ of $R$. In this first case, $h tm(1) = htm[1]$, and $h \mathfrak{M}[r] = htm[1][r] = htm[1 + r]$ (5)
b) \( \mathfrak{M} \) is the localisation of a prime ideal \( \mathfrak{P} \) of \( R[1] \), which is an upper to a non maximal prime ideal \( p \) of \( R \). Hence there is a maximal ideal \( m \) of \( R \) such that \( p \subset m \). Therefore, from [6, lemma 1], we get

\[
ht\mathfrak{M}[r] = ht\mathfrak{P} = htp[1][r] + 1 = htp[1 + r] + 1 \leq htm[1 + r]. \tag{6}
\]

In any case, (4), (5) and (6) lead to

\[
\dim R(1)[r] \leq Sup_{htf}\{htm[1 + r] + r\} \leq \dim R[1 + r] - 1.
\]

Case \( n \geq 2 \). From the case \( n = 1 \), we get

\[
\dim R\langle n \rangle[r] = \dim R\langle n - 1 \rangle\langle 1 \rangle[r] \leq \dim R\langle n - 1 \rangle[1 + r] - 1
\]

thus, by induction hypothesis,

\[
\dim R\langle n \rangle[r] \leq \dim R[(n - 1) + (1 + r)] - (n - 1) - 1 \leq \dim R[n + r] - n.
\]

This proves (3). The result follows, putting (1), (2) and (3) together. ⋆

In particular \( \dim R\langle n \rangle = \dim R(n) = \dim R[n] - n \). Thus we derive:

**Corollary 1.3** Let \( R \) be a ring and \( n \) a non negative integer, then

\[
\dim_v R = Sup_n \dim R\langle n \rangle = Sup_n \dim R(n).
\]

It results also clearly from proposition 1.2 that, if \( T = R(n) \) or \( T = R\langle n \rangle \), then \( \dim T[r] - r = \dim R[n + r] - n - r \), hence the limit of the sequence \( \dim T[r] - r \), is the same as the limit of the sequence \( \dim R[m] - m \). Thus we get the following:

**Corollary 1.4** Let \( R \) be a ring and \( n \) a non negative integer, then

\[
\dim_v R\langle n \rangle = \dim_v R(n) = \dim_v R.
\]

From proposition 1.2 and corollary 1.4, we note that \( R(n) \) and \( R\langle n \rangle \) have the same Krull dimension and the same valuative dimension.
2 Jaffard properties

It is clear that a finite dimensional ring $T$ is a Jaffard ring if and only if, for each non negative integer $k$, $\dim T[k] = \dim T + k$. From proposition 1.2 we thus get:

**Lemma 2.1** For any ring $T$, the following assertions are equivalent

(i) $T$ is a Jaffard ring,

(ii) for any non negative integer $k$, $\dim T(k) = \dim T$,

(iii) for any non negative integer $k$, $\dim T\langle k \rangle = \dim T$.

From the same proposition 1.2 we obtain also the following results for the transfer of the Jaffard (resp. locally Jaffard) property from the Nagata ring $R(n)$ to the Serre conjecture ring $R\langle n \rangle$.

**Proposition 2.2** Let $R$ be a finite dimensional ring and $n$ a non negative integer. Then the following assertions are equivalent:

(i) $R[n]$ is a Jaffard ring,

(ii) $R\langle n \rangle$ is a Jaffard ring,

(iii) $R\langle n \rangle$ is a Jaffard ring,

(iv) for any non negative integer $k$, $\dim R\langle n \rangle = \dim R\langle n + k \rangle$,

(v) for any non negative integer $k$, $\dim R\langle n \rangle = \dim R\langle n + k \rangle$.

**Proposition 2.3** Let $R$ be a finite dimensional ring and $n$ a non negative integer. Then the following assertions are equivalent:

(i) $R[n]$ is a locally Jaffard ring,

(ii) $R\langle n \rangle$ is a locally Jaffard ring,

(iii) $R\langle n \rangle$ is a locally Jaffard ring.

**Proof.** It is trivial that (i) implies (ii) and (ii) implies (iii). Conversely, if $R\langle n \rangle$ is a locally Jaffard ring, then $R_p\langle n \rangle$ is a Jaffard ring, for any prime ideal $p$ of $R$, and so is $R_p[n]$, from the previous proposition. Thus $R[n]$ is a locally Jaffard ring, by [3, lemma 1.11]. Therefore (iii) imples (i). \(\diamondsuit\)
Remarks 2.4 (i) From propositions 2.2 and 2.3, $R\langle n \rangle$ and $R(n)$ are Jaffard rings (resp. locally Jaffard rings) whenever $R$ is a Jaffard ring (resp. a locally Jaffard ring) or if $n \geq \dim_v R - 1$ [8, proposition 1].

(ii) If $R[n]$ is a totally Jaffard ring, then so are clearly $R(n)$ and $R\langle n \rangle$. The converse does not hold: [4, example 5.3] is a dimension 2, quasi-local and totally Jaffard domain such that $R[X]$ is not a strong S-ring. Thus $R[n]$ is not totally Jaffard for $n \geq 1$. According to the previous remark, $R(n)$ and $R\langle n \rangle$ are however dimension 2 locally Jaffard domains, for all $n$, thus even totally Jaffard domains from [8, corollaire 1] (and therefore strong S-rings).

We show next that $R\langle n \rangle$ is totally Jaffard for $n$ large if and only if it is a strong S-ring. First we set a lemma:

Lemma 2.5 Let $R$ be a finite dimensional ring such that $R\langle n \rangle$ is a strong S-ring for all $n$, then $R$ is totally Jaffard.

Proof. If $R\langle n \rangle$ is a strong S-ring for all $n$, so is $R(n)$ by localisation. For any prime $p$ of $R$, letting $\overline{R} = R/p$, $\overline{R}(n)$ is isomorphic to $R(n)/p(n)$, hence is also a strong S-ring. For any prime $q$ of $R$ containing $p$, letting $\overline{q} = q/p$ then $\overline{R}_{\overline{q}}(n)$ is isomorphic to the localisation of $\overline{R}(n)$ at the prime $\overline{q}(n)$, hence $\overline{R}_{\overline{q}}(n)$ is again a strong S-ring. From [16, theorem 2] it results that $\overline{R}_{\overline{q}} = R_{\overline{q}}/pR_{\overline{q}}$ is Jaffard. \hfill \Box

Since $R\langle n + m \rangle$ is clearly the same as $R\langle n \rangle R\langle m \rangle$ and totally Jaffard rings are strong S-rings, we derive immediately the following:

Proposition 2.6 Let $R$ be finite dimensional and $k$ be a non negative integer. The following assertions are equivalent:

(i) for $n \geq k$, $R\langle n \rangle$ is a strong S-ring

(ii) for $n \geq k$, $R\langle n \rangle$ is totally Jaffard.

We close this section with some questions and an example:

Question 2.7 Are $R\langle n \rangle$ and $R(n)$ residually Jaffard rings, when $R[n]$ is?

We note that conversely, $R\langle n \rangle$ and $R(n)$ may be residually (even totally) Jaffard rings, whereas $R[n]$ is not: indeed, if $R$ is a domain such that $\dim R = 1$ and $\dim_v R = 2$. From remark 2.4 (i), $R\langle n \rangle$ and $R(n)$ are thus dimension 2 locally Jaffard domains, for $n \geq 1$, hence totally Jaffard domains [8, corollaire 1]. But $R$ is not Jaffard, thus $R[n]$ is not residually Jaffard, for any $n$. 

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Question 2.8 Is $R\langle n \rangle$ a residually Jaffard (resp. a totally Jaffard ring, resp. a strong S-ring), if and only if $R(n)$ is?

Question 2.9 Is $R\langle n \rangle$ a totally Jaffard ring (or equivalently a strong S-ring) for $n \geq \dim_v R$ or at least for $n$ large?

Lastly the following example presents a totally Jaffard domain $R$ such that $R[X]$, $R\langle X \rangle$ and $R(X)$ are not residually Jaffard domains.

Example 2.10 As in [8, example 8], we let $k$ be a field, $u, v, w$ indeterminates and $S$ the multiplicative subset complement of the union $m_1 \cup n_1$ of the prime ideals $m_1 = (u - 1)$ and $n_1 = (u, v, w)$ of $k[u, v, w]$. The localisation $B = S^{-1}k[u, v, w]$ is a three dimensional semi-local domain, with two maximal ideals $m = S^{-1}m_1$ and $n = S^{-1}n_1$ such that $ht_m = 1$ and $ht_n = 3$. Finally, let $I = m \cap n$ and $R = k + I$. Then $R$ is a 3 dimensional quasi-local totally Jaffard domain such that $R[X]$ is not a residually Jaffard domain. More precisely, it has been established in [8, example 8] that there exists a prime ideal $P$ in $R\langle X \rangle$ such that $P \subset I\langle X \rangle$ are consecutive in $R\langle X \rangle$, whereas $P[Y] \subset I[X,Y]$ are not in $R[X,Y]$. This prime $P$ lifts as a prime $P'$ of $R(X)$ (resp. $R\langle X \rangle$). Clearly $\dim R(X)/P' = htI[X]/P = 1$ (resp. $\dim R\langle X \rangle/P' = 1$). On the other hand $\dim (R(X)/P')Y \geq htI[X,Y]/P[Y] + 1 \geq 3$ (resp. $\dim (R\langle X \rangle/P')Y \geq 3$). Therefore $R(X)/P'$ (resp. $R\langle X \rangle/P'$) is not a Jaffard ring.

3 Infinitely many indeterminates

We first give the Krull dimension of $R(\infty)$ and $R(\infty)$, as already done by D.E. Dobbs at al. in [9, corollary 2.5] for the infinite Nagata ring in the particular case of a domain. We also give the height of the extended primes:

Proposition 3.1 For any ring $R$,

(i) $\dim_v R = \dim R(\infty) = \dim R(\infty)$,

(ii) for any prime $p$ of $R$, $ht_v p = htp(\infty) = htp(\infty)$.

Proof. Since $R(\infty)$ (resp. $R(\infty)$) is the union of the rings $R\langle n \rangle$ (resp. $R(n)$), by [9, lemma 2.1] we have the inequality $\dim R(\infty) \leq Sup_n \{\dim R(n)\}$ (resp. $\dim R(\infty) \leq Sup_n \{\dim R(n)\}$). Moreover, any chain of primes in $R\langle n \rangle$ (resp. in $R(n)$) lifts in $R(\infty)$ (resp. in $R(\infty)$), hence the reverse inequality proving
Let $R$, $p$ a prime of $R$, $\text{ht}_{p} p = \dim_{p} Rp = \dim R_{p}(\infty)$. But $\dim R_{p}(\infty) = \text{ht}(\infty)$, since $R_{p}(\infty)$ is the localization of $R(\infty)$ with respect to the prime $p(\infty)$. Thus $\text{ht}_{p} p = \text{ht}(\infty)$. On the other hand, $\text{ht}(\infty) = \text{ht}(\infty)$, since $p(\infty)$ is a localization of $p(\infty)$. This proves (ii). "\]

We may note, as D.E. Dobbs et al. for the infinite Nagata ring, in the special case of a domain [9, corollary 2.5], that it results easily from this proposition that $R(\infty)$ and $R(\infty)$ are Jaffard rings (if their dimension are finite). We will show that they are in fact stably strong S-rings. First, we set the following:

**Lemma 3.2** Let $\mathfrak{p} \subset \mathfrak{q}$ be consecutive primes of finite height in $R[\infty]$; then $\mathfrak{p}[1] \subset \mathfrak{q}[1]$ are consecutive in $R[\infty][1]$.

**Proof.** We note first that there is an integer $k$ such that $\mathfrak{p}$ is the extension of a prime ideal of $R[k]$. Indeed, letting $\mathfrak{p}_{n}$ be the intersection $\mathfrak{p}_{n} = \mathfrak{p} \cap R[n]$, if the extension $\mathfrak{p}_{n}[1]$ of $\mathfrak{p}_{n}$ to $R[n+1] = R[n][1]$ is such that $\mathfrak{p}_{n}[1] \subset \mathfrak{p}_{n+1}$, then $\text{ht}\mathfrak{p}_{n+1} > \text{ht}\mathfrak{p}_{n}$, since any chain of $R[n]$ lifts in $R[n+1]$ (taking the extension of each prime of the chain). If the set of integers such that $\mathfrak{p}_{n}[1] \subset \mathfrak{p}_{n+1}$ were infinite, so would be $\text{ht}\mathfrak{p}$, contrary to the hypothesis. Therefore, there is an integer $k$ such that $\mathfrak{p}_{k}[n] = \mathfrak{p}_{k+1}$, for all $n$, thus $\mathfrak{p} = \bigcup_{n} \mathfrak{p}_{k+n} = \bigcup_{n} \mathfrak{p}_{k}[n] = \mathfrak{p}_{k}[\infty]$. For the same reason, there is an integer $k$ such that both $\mathfrak{p}$ and $\mathfrak{q}$ are extensions of primes of $R[k]$ to $R[\infty]$. Replacing $R$ by $R[k]$, since $R[\infty]$ and $R[\infty][k]$ are clearly isomorphic, we may thus consider that $\mathfrak{p} = p[\infty]$ and $\mathfrak{q} = q[\infty]$, where $\mathfrak{p}$ and $\mathfrak{q}$ are respectively above the primes $p$ and $q$ of $R$. The infinite polynomial ring $R[\infty]$ is the set theoretic union of the rings $R[n]$ and $R[\infty][1]$ the set theoretic union of the rings $R[n][1]$. Since $R[n][1]$ is isomorphic to $R[n+1]$, $R[\infty][1]$ is thus isomorphic to $R[\infty]$. Similarly $\mathfrak{p} = p[\infty]$ and $\mathfrak{q} = q[\infty]$ are respectively the union of the primes $p[n]$ and $q[n]$, whereas $\mathfrak{p}[1]$ and $\mathfrak{q}[1]$ are respectively the union of the primes $p[n][1]$ and $q[n][1]$, thus $\mathfrak{p}[1]$ and $\mathfrak{q}[1]$ correspond to the primes $\mathfrak{p}$ and $\mathfrak{q}$ under the isomorphism of $R[\infty][1]$ with $R[\infty]$.

Since $R(\infty)[m]$ (resp. $R(\infty)[m]$) is a localization of $R[\infty][m]$, which is isomorphic to $R[\infty]$, consecutive primes of $R(\infty)[m]$ (resp. $R(\infty)[m]$) correspond to consecutive primes of $R[\infty]$. Thus we get:

**Theorem 3.3** If $R$ is a ring such that $\dim_{p} R$ is finite, then $R(\infty)$ and $R(\infty)$ are stably strong S-rings.

**Corollary 3.4** If $R$ is a ring such that $\dim_{p} R$ is finite, then $R(\infty)$ and $R(\infty)$ are totally Jaffard rings.
References


