0. Introduction

All the rings and algebras considered in this paper will be commutative, with identity elements and ring-homomorphisms will be unital. If $A$ is a ring, then $\dim A$ will denote the (Krull) dimension of $A$, that is, the supremum of lengths of chains of prime ideals of $A$. An integral domain $D$ is said to have valuative dimension $n$ (in short, $\dim_v D = n$) if each valuation overring of $D$ has dimension at most $n$ and there exists a valuation overring of $D$ of dimension $n$. If no such integer $n$ exists, then $D$ is said to have infinite valuative dimension (see [G]). For reader’s convenience, recall that for any ring $A$, $\dim_v A = \sup\{\dim_v (A/P) \mid P \in \text{Spec}(A)\}$, and that a finite-dimensional domain $D$ is a Jaffard domain if $\dim D = \dim_v D$. As the class of Jaffard domains is not stable under localization, an integral domain $D$ is defined to be a locally Jaffard domain if $D_P$ is a Jaffard domain for each prime ideal $P$ of $D$ (see [ABDFK]). Analogous definitions are given in [C] for a finite-dimensional ring.

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2Partially supported by KFUPM.
In [8] Sharp proved that if $K$ and $L$ are two extension fields of a field $k$, then
\[
\dim(K \otimes_k L) = \min(t.d.(K : k), t.d.(L : k)).
\]
This result provided a natural starting point to explore dimensions of tensor products of somewhat general $k$-algebras and it was concretized by Wadsworth in [W], where the result of Sharp was extended to AF-domains (for "altitude formula"). Let $A$ be an AF-domain, that is, integral domain $A$ such that
\[
\text{ht } P + t.d.(A/P : k) = t.d.(A : k)
\]
for all prime ideals $P$ of $A$. He showed that if $A_1$ and $A_2$ are AF-domains, then
\[
\dim(A_1 \otimes_k A_2) = \dim(A_1) + t.d.(A_2 : k), \quad \dim(A_2) + t.d.(A_1 : k).
\]
He also stated a formula for $\dim(A \otimes_k R)$ which holds for an AF-domain $A$, with no restriction on $R$. At this point, it is worthwhile to recall that an $A$-algebra $B$ is said to be an AF-ring provided $ht_B = 0$ for all prime ideals $B$. We then developed quite general results for AF-rings, showing that the results do not extend trivially from integral domains to rings with zero-divisors.

Some phenomena do not hold in the field case since any $k$-algebra is a natural example of an AF-ring. We then extend, in a natural way, the definition of the "altitude formula", that is, integral domains $A$ of the form $A = R(A(X), \sigma)$, where $\sigma$ is an automorphism of a field $F$. The most remarkable outcome is that $A$ is an AF-ring. We begin by giving a simple generalization of a well-known result [ZS] for algebras over a field.

**Lemma 1.1.** Let $(A, \lambda_A)$ be an $R$-algebra and $P \in \text{Spec}(A)$. Then
\[
\text{ht } P + t(A/P : R) \leq t(A/P : R).
\]

**Proof.** Clearly, $t(A/P : R) = t.d.(A/P : R/\lambda_A^{-1}(P))$. If $P' \subseteq P$, then for the prime ideal $P/P'$ of the $R/\lambda_A^{-1}(P')$-algebra $A/P'$, by [ZS, p.10], we get
\[
\text{ht}(P/P') + t.d.(A/P : R/\lambda_A^{-1}(P)) \leq t.d.(A/P' : R/\lambda_A^{-1}(P')) = t.d.(A/P' : R/\lambda_A^{-1}(P')).
\]
The result then follows. □

The following elementary properties will be used frequently. These statements admit routine proofs.

**Lemma 1.2.** Let $(A, \lambda_A)$ be an $R$-algebra.

1. If $P$ is a prime ideal of $A$ and $p = \lambda_A^{-1}(P)$, then
\[
\text{ht } P = d.(P/pA) \quad \text{and} \quad t(A/P : R) = t((A/pA)/pA : R).
\]

2. If $P$ is a prime ideal of $A$ and $p = \lambda_A^{-1}(P)$, then for each $n \geq 1$
\[
\text{ht } P[X_1, \ldots, X_n] = \text{ht}(P/pA)[X_1, \ldots, X_n].
\]
(3) If $A$ is a locally Jaffard ring, then $A/pA$ is a locally Jaffard ring for each prime ideal $p$ of $R$ such that $pA \neq A$.

Let $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ be $R$-algebras. For $i = 1, 2$, we denote by $\mu_i : A_i \to A_i \otimes_R A_2$ the canonical $A_i$-algebra homomorphism. The $R$-algebra $A_i \otimes_R A_2$, when not specifically indicated, has $\lambda_{A_i \otimes_R A_2} = \mu_1 \circ \lambda_1 = \mu_2 \circ \lambda_2$ as its associated ring homomorphism. If $P_i \in \text{Spec}(A_i)$, $i = 1, 2$, denotes the inclusion of $P_i$ into $A_i$, $t_{P_i}$ denotes the transcendence degree of the local ring $A_{P_i}$ over $R$ and $k(P_i)$ denotes the residue field of $A_{P_i}$. At last, we set $\Gamma (A_1, A_2) = \{(P_i, P_2) \mid P_i \in \text{Spec}(A_1), P_2 \in \text{Spec}(A_2) \text{ and } \lambda_i^{-1}(P_i) = \lambda_j^{-1}(P_2)\}$.

A tensor product of $R$-algebras may be zero. We are interested in $R$-algebras $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ such that $A_1 \otimes_R A_2 \neq 0$, and say that such algebras are tensorially compatible. The next result provides some elementary and useful characterizations of tensorially compatible $R$-algebras. For a more general result, we refer the reader to [GD, Corollary 3.2.7.1].

**Proposition 1.3.** Let $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ be $R$-algebras. The following conditions are equivalent:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
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<tbody>
<tr>
<td>(1)</td>
<td>$(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ are tensorially compatible.</td>
</tr>
<tr>
<td>(2)</td>
<td>$\lambda_1(\text{Spec}(A_1)) \cap \lambda_2(\text{Spec}(A_2)) = \emptyset$.</td>
</tr>
<tr>
<td>(3)</td>
<td>There exists a prime ideal $P_1$ of $A_1$ such that $\lambda_1^{-1}(P_1)A_2 \neq A_2$.</td>
</tr>
<tr>
<td>(4)</td>
<td>There exists a prime ideal $P_2$ of $A_2$ such that $\lambda_2^{-1}(P_2)A_1 \neq A_1$.</td>
</tr>
<tr>
<td>(5)</td>
<td>There exists a prime ideal $p$ of $R$ such that $pA_1 \neq A_1$ and $pA_2 \neq A_2$.</td>
</tr>
<tr>
<td>(6)</td>
<td>$\ker \lambda_1 + \ker \lambda_2 \neq R$.</td>
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**Proof.** (1) $\implies$ (2). If (1) holds, then there exists a prime ideal $Q$ of $A_1 \otimes_R A_2$; therefore $\mu_1^{-1}(Q) \in \text{Spec}(A_1)$ and $\mu_2^{-1}(Q) \in \text{Spec}(A_2)$ are such that $\lambda_1^{-1}(\mu_1^{-1}(Q)) = \lambda_2^{-1}(\mu_2^{-1}(Q))$, and hence, $\lambda_1(\text{Spec}(A_1)) \cap \lambda_2(\text{Spec}(A_2)) \neq \emptyset$. (2) $\implies$ (3). Let $P_1$ be a prime ideal of $A_1$ and $P_2$ a prime ideal of $A_2$ such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)$; then $\lambda_1^{-1}(P_1)A_2 \subseteq P_2$ and so $\lambda_1^{-1}(P_1)A_2 \neq A_2$. The implications (3) $\implies$ (4), (4) $\implies$ (5) and (5) $\implies$ (6) are apparent. Finally, assume (6). Since $\ker \lambda_1 + \ker \lambda_2 \neq R$, there exists a prime ideal $p$ of $R$ such that $\ker \lambda_1 + \ker \lambda_2 \subseteq p$; this ensures that there exist a prime ideal $P_2$ of $A_1$ and a prime ideal $P_2$ of $A_2$ such that $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = p$. Then

$$ (A_1 \otimes_R A_2)/(\text{Im}(j_1 \otimes id_{A_2}) + \text{Im}(id_{A_1} \otimes j_2)) \cong (A_1/P_1) \otimes_R (A_2/P_2) \cong (A_1/P_1) \otimes_R (A_2/P_2) \neq 0 $$

and so $A_1 \otimes_R A_2 \neq 0$. \(\Box\)

By induction, we obtain the following:
Proof. Let \((P_1, P_2) \in \Gamma(\mathcal{A}_1, \mathcal{A}_2)\). Then by Proposition 1.5 there exists a prime ideal \(Q\) of \(\mathcal{A}_1 \otimes_R \mathcal{A}_2\) such that \(\langle (\mathcal{A}_1 \otimes_R \mathcal{A}_2) / Q \rangle = \langle \mathcal{A}_1 / P_1 \rangle + \langle \mathcal{A}_2 / P_2 \rangle\). So \(\sup \{t(\mathcal{A}_1 / P_1) : R \} = \sup \{t(\mathcal{A}_2 / P_2) : R \}\) if and only if there exist \(P_1 \in \text{Spec}(\mathcal{A}_1)\) and \(P_2 \in \text{Spec}(\mathcal{A}_2)\) such that \(t(\mathcal{A}_1 / P_1) + t(\mathcal{A}_2 / P_2) = t(\mathcal{A}_1 \otimes_R \mathcal{A}_2)\). The prime ideals \(P_1 = \mu^{A_1}(Q)\) and \(P_2 = \mu^{A_2}(Q)\) are such that \(\lambda^{-1}(P_1) = \lambda^{A_1}(P_2) = p\); let \(T = A_1 \otimes_R A_2\); then, using [W, Corollary 2.4], we obtain:

\[
\sup \{t(A_1 / P_1 + A_2 / P_2) : R \} = \\
\sup \{t(A_1 / P_1) + t(A_2 / P_2) : R \} = \sup \{t(A_1 \otimes_R A_2) : R \}.
\]

as desired. □

Remark 1. Let \((A_1, \lambda_1)\) and \((A_2, \lambda_2)\) be tensorially compatible \(R\)-algebras. Clearly, \(t(\mathcal{A}_1 \otimes_R \mathcal{A}_2) : R = t(\mathcal{A}_1 / P_1 + \mathcal{A}_2 / P_2) : R\) if and only if there exist \(P_1 \in \text{Spec}(\mathcal{A}_1)\) and \(P_2 \in \text{Spec}(\mathcal{A}_2)\) such that \(\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2)\) and \(t(\mathcal{A}_1 / P_1) + t(\mathcal{A}_2 / P_2) = t(\mathcal{A}_1 \otimes_R \mathcal{A}_2)\). The second condition holds, for instance, if \(A_1\) and \(A_2\) are integral domains or if \(\text{Spec}(R)\) is reduced to only one prime ideal. In general, the equality fails as it is shown in the next example. Moreover, when \(R\) is a field, we have \(\dim(\mathcal{A}_1 \otimes_R \mathcal{A}_2) \geq \dim A_1 + \dim A_2\) [W, Corollary 2.5]. This is not, in general, true in the zero-dimensional case. The next example deals with these matters.

Example 2. There exist two tensorially compatible \(R\)-algebras \((A_1, \lambda_1)\) and \((A_2, \lambda_2)\) with

\[
t(\mathcal{A}_1 \otimes_R \mathcal{A}_2) : R = t(\mathcal{A}_1) + t(\mathcal{A}_2) = \dim(\mathcal{A}_1 \otimes_R \mathcal{A}_2) : R = \dim A_1 + \dim A_2.
\]

Let \(R = R \times R\), \(A_1 = R\) and \(A_2 = R \times R [X]\). Let \(\lambda_1 : R \rightarrow A_1\) be the ring homomorphism defined by \(\lambda_1(x, y) = x\) and let \(\lambda_2 : R \rightarrow A_2\) be the ring homomorphism defined by \(\lambda_2(x, y) = (x, y)\). Then

\[
t(\mathcal{A}_1 \otimes_R \mathcal{A}_2) = t(R \times R) = 0 = \dim(\mathcal{A}_1 \otimes_R \mathcal{A}_2).
\]

and

\[
t(\mathcal{A}_1 \otimes_R \mathcal{A}_2) = t(R \times R) = 0 = \dim(\mathcal{A}_1 \otimes_R \mathcal{A}_2).
\]

Moreover, by Corollary 1.7,

\[
t(A_1 \otimes_R A_2) = \sup \{t(A_1 / P_1 + A_2 / P_2) : R \} = \sup \{t(A_1 / P_1) + t(A_2 / P_2) : R \} = \sup \{t(A_1 / P_1) + t(A_2 / P_2) : R \} = 0.
\]

Further

\[
\dim(\mathcal{A}_1 \otimes_R A_2) : R = \dim(\mathcal{A}_1 \otimes_R A_2) : R = 0 < \dim A_1 + \dim A_2 = 1. \tag{\ref{definition}}
\]

2. Tensor products of AF-rings

Definition 2.8. An \(R\)-algebra \((A, \lambda)\) is an AF-ring if for every \(P \in \text{Spec}(A)\) and \(\mathfrak{p} \in \mathfrak{p}

Remark 2. The AF-ring concept does not depend on the structure of algebra over \(R\) defined by the associated ring homomorphism.

Indeed, let \(A\) be a ring and let \(\lambda\) and \(\lambda'\) be two ring homomorphisms defining two different structures of algebra over \(R\) on \(A\). Let \(P \in \text{Spec}(A)\). Let \(\pi : A \rightarrow A/P\) be the natural ring homomorphism. Then \(p = \ker(\pi \circ \lambda) = \lambda^{-1}(P)\) and \(q = \ker(\pi \circ \lambda') = \lambda'^{-1}(P)\). We can view \(R/p\) and \(R/q\) as subfields of \(A/P\). Let \(k = R/p \cap R/q\). We have:

\[
t(A/P \otimes_R \mathcal{A}) = \sup \{t(A/P : \mathcal{A}) : R \} = \sup \{t(A/P : \mathcal{A}) : R/p \} - \sup \{t(A/P : \mathcal{A}) : R/q \} - \sup \{t(A/P : \mathcal{A}) : R/k \}.
\]

On the other hand

\[
t(A/P \otimes_R \mathcal{A}) = \sup \{t(A/P : \mathcal{A}) : R \} = \sup \{t(A/P : \mathcal{A}) : R/p \} - \sup \{t(A/P : \mathcal{A}) : R/q \} - \sup \{t(A/P : \mathcal{A}) : R/k \}.
\]

Therefore \(t(A/P \otimes_R \mathcal{A}) = t(A/P \otimes_R \mathcal{A})\) in case if and only if \((A, \lambda, \lambda')\) is an AF-ring.

Let \(\mathcal{R}\) be the class of \(R\)-algebras that are AF-rings. Since \(R [X_1, \ldots, X_n]\) satisfies the first chain condition for prime ideals [G, Corollary 31.17], any finitely generated \(R\)-algebra or any integral extension of such an algebra is an AF-ring. Moreover the class \(\mathcal{R}\) is stable under localization and direct product.

The next result presents some properties of the class \(\mathcal{R}\), and our proof of Proposition 2.3 uses the following lemma.
Lemma 2.9. Let \( (A, \lambda_A) \) be an R-algebra. Then \( A \) is an AF-ring if and only if \( A/pA \) is an AF-ring over the field \( R/p \), for each prime ideal \( p \) of \( R \) such that \( pA \neq A \).

Proof. Let \( B \) be a prime ideal of \( A \) and let \( p = \lambda_A^{-1}(B) \). According to Lemma 1.2, \( \text{ht} P = \text{ht}(P/pA) + t(A : P : R) = t((A/pA)_p : R/p) \).

Assume that \( P \) is an AF-ring and let \( p \) be a prime ideal of \( R \) such that \( pA \neq A \). Let \( P \) be a prime ideal of \( A \) containing \( pA \); then

\[
\text{ht}(P/pA) + t \cdot (A/pA)_p : R/p) = \text{ht} P + t(A : P : R) = t(A : P : R) = t((A/pA)_p : R/p).
\]

Conversely, let \( P \) be a prime ideal of \( A \) and let \( p = \lambda_A^{-1}(P) \). Then \( pA \neq A \); so by hypothesis \( A/pA \) is an AF-ring over \( R/p \), hence

\[
\text{ht} P + t(A : P : R) = \text{ht}(P/pA) + t \cdot (A/pA)_p : R/p) = \text{ht}(P/pA) + t \cdot ((A/pA)_p : R/p) = t((A/pA)_p : R/p).
\]

Proposition 2.10. The class \( \mathcal{R} \) satisfies the following properties:

1. Let \( (A_1, \lambda_1), \ldots, (A_n, \lambda_n) \) be tensorially compatible R-algebras. If \( A_1, \ldots, A_n \) are AF-rings, then \( A_1 \otimes_R \cdots \otimes_R A_n \) is an AF-ring.

2. Let \( A \) be an AF-ring. Then the polynomial ring \( A[X] \) is an AF-ring and for each prime ideal \( P \) of \( A \), \( \text{ht} P = \text{ht} P[X] \).

3. An AF-ring \( A \) is a locally Jaffard ring.

Proof. (1) By induction, it suffices to consider the case \( n = 2 \). Let \( (A_1, \lambda_1) \) and \( (A_2, \lambda_2) \) be tensorially compatible AF-rings. Let \( p \) be a prime ideal of \( R \) such that \( p(A_1 \otimes_R A_2) \neq A_1 \otimes_R A_2 \). By Lemma 2.2, \( A/pA \) and \( A_2/pA_2 \) are AF-rings over the field \( R/p \); hence by [W, Proposition 3.1], \( (A/pA) \otimes_R (A_2/pA_2) \) is an AF-ring over \( R/p \), so that \( (A/pA) \otimes_R (A_2/pA_2) \) is an AF-ring over \( R/p \). The proof is complete via Lemma 2.2.

(2) Since \( A[X] \cong A \otimes_R R[X] \), the result follows from (1). Let \( P \) be any prime ideal of \( A \); so

\[
\text{ht} P \leq \text{ht} P[A[X]] = t(A[X]_{pA[X]} : R) - t(A[X]/P[AX] : R) \\
\leq t(A_P : R) - t((A/P)[X] : R) = \text{ht} P.
\]
Further, for any \( n \geq 1 \), let \( I \{ X_1, \ldots, X_n \} = \text{ht}(I/pB)[X_1, \ldots, X_n] \). Hence
\[
\delta(P/pA, I/pB) = \text{ht}(P, I) = \text{ht}(\{ X_1, \ldots, X_n \}) \supset \text{ht}(I/pB). 
\]

It follows that \( \delta(P, I) = \text{ht}(P, I) \), as asserted. Consequently, using the definitions of \( \delta, \Delta \) and \( D \) and the stated condition on \( \delta(P, I) \), yields
\[
\dim(A \otimes R B) = \sup \{ \delta(P, I) \mid (P, I) \in \Gamma(A, B) \} = \sup \{ \Delta(t, P, I) \mid (P, I) \in \Gamma(A, B) \} = \sup \{ D(t, P, B/pB) \mid P \in \text{Spec}(A), \Delta(P) \neq \text{ht}(P/pB) \} = \sup \{ \text{ht}(I) \mid (I, P) \in \Gamma(A, B) \} \}
\]

as we wished to show. \( \square \)

It is worthwhile to note that \( \dim(A \otimes R B) \) depends on the \( R \)-module structure of \( A \) and \( B \). The next example illustrates this fact:

Example 3. Let \( (A, \lambda_A) \) and \( (B, \lambda_B) \) be \( R \)-algebras such that \( A \) is an AF-ring and \( \text{Ann}(A \otimes R B) \neq 0 \). Let \( p \) be a prime ideal of \( R \) and let \( \pi : R \to R/p \) be the canonical ring homomorphism. Let \( \lambda_A : R \times R \times R \to R/p \times A \) and \( \lambda_B : R \times R \times R \to R/p \times B \) be the ring homomorphisms defined respectively by \( \lambda_A(x, y, z) = (\pi(x), \lambda_A(y)) \) and \( \lambda_B(x, y, z) = (\pi(z), \lambda_B(y)) \). It is an easy matter to verify that \( \text{Max}(R/p \times A, R/p \times B) = \{(0, \lambda_A) \} \}
\]

Hence via Theorem 2.4, it is easy to check that the dimension of the tensor product of \( (R/p \times A, \lambda_A) \) and \( (R/p \times B, \lambda_B) \) is zero. On the other hand, let \( \lambda_A : R \times R \times R \to R/p \times B \) be the ring homomorphism defined by \( \lambda_A(x, y, z) = (\pi(x), \lambda_B(y)) \); now by Theorem 2.4 we obtain that the dimension of the tensor product of \( (R/p \times A, \lambda_A) \) and \( (R/p \times B, \lambda_B) \) is equal to \( \dim(A \otimes R B) \). Thus, it suffices to choose \( A \) and \( B \) such that \( \dim(A \otimes R B) > 0 \) (for instance, when \( R \) is a field and \( A, B \) are non trivial \( R \)-algebras). Therefore the two values are different according to the \( (R \times R \times R) \)-module structure of \( R/p \times A \) and \( R/p \times B \).

With the further assumption that \( A \) is an AF-domain, we obtain the following:

**Corollary 2.12.** Let \( (A, \lambda_A) \) be an AF-domain and let \( (B, \lambda_B) \) be any \( R \)-algebra such that \( A \otimes R B \neq 0 \). Then
\[
\dim(A \otimes R B) = D(t(A : R), \text{Ann}(A \otimes R B)),
\]
where \( p_A = \ker \lambda_A \). Furthermore, if \( B \) is an integral domain, then
\[
\dim(A \otimes R B) = D(t(A : R), \text{Ann}(A \otimes R B)).
\]

**Proof.** Since \( A \) is an integral domain, for any prime ideal \( P \) of \( A \), \( \lambda_A^{-1}(P) = p_A \) and \( t(P) = t(A : R) \); so Theorem 2.4 implies that \( \dim(A \otimes R B) = \sup \{ D(t(A : R), \text{ht}(P, B/pA) \mid P \in \text{Spec}(A) \} \). Since \( D(s, d, A) \) is a nondecreasing function of the second argument, then \( \dim(A \otimes R B) = D(t(A : R), \text{Ann}(A \otimes R B)) \), as asserted. \( \square \)

Next, we state a technical result that allows us to determine a necessary and sufficient condition under which the dimension of the tensor product of AF-rings over a zero-dimensional ring satisfies the formula of Wadsworth's Theorem 3.8.

**Proposition 2.13.** Let \( (A_1, \lambda_1), \ldots, (A_n, \lambda_n) \) be tensorially compatible AF-rings. Then
\[
\dim(A_1 \otimes R \cdots \otimes R A_n) = \sup \{ \text{ht}(M_1 + \cdots + M_n, \ldots, t_m + \cdots + t_m) \mid M_i \in \text{Max}(A_i) \}
\]

**Proof.** It is deduced from the fact that \( \dim(A_1 \otimes R \cdots \otimes R A_n) = \sup \{ \text{ht}(M_1 + \cdots + M_n, \ldots, t_m + \cdots + t_m) \mid M_i \in \text{Max}(A_i) \} \}
\]

for any \( i = 1, 2, \ldots, n \). Now we conclude via [BGK1, Lemma 1.6 and Remark 1.7]. \( \square \)

**Theorem 2.14.** Let \( (A_1, \lambda_1), \ldots, (A_n, \lambda_n) \) be tensorially compatible AF-rings with \( t_i = t(A_i : R) \) and \( d_i = \text{dim}(A_i) \). Then \( \dim(A_1 \otimes R \cdots \otimes R A_n) = t_1 + \cdots + t_n - \sup \{ t_i - d_i \mid 1 \leq i \leq n \} \) if and only if there exist maximal ideals \( M_1, \ldots, M_n \) belonging respectively to \( A_1, \ldots, A_n \) such that \( \lambda_i^{-1}(M_i) = \cdots = \lambda_n^{-1}(M_n) \) and \( \text{ht}(M_r, A_i) = d_i \) for \( 1 \leq r \leq n \) and \( A_i \neq A_r \) for \( i 
eq r \) in \( \{ 1, \ldots, n \} \), in which case equality holds if and only if \( A_i \neq A_r \) for \( i 
eq r \) in \( \{ 1, \ldots, n \} \).

**Proof.** It is deduced from the fact that \( \dim(A_1 \otimes R \cdots \otimes R A_n) = \sup \{ \text{ht}(M_1 + \cdots + M_n, \ldots, t_m + \cdots + t_m) \mid M_i \in \text{Max}(A_i) \} \}
\]

and \( \dim(A_1 \otimes R \cdots \otimes R A_n) = \sup \{ \text{ht}(M_1 + \cdots + M_n, \ldots, t_m + \cdots + t_m) \mid M_i \in \text{Max}(A_i) \} \). \( \square \)

**Corollary 2.15.** Let \( (A_1, \lambda_1), \ldots, (A_n, \lambda_n) \) be tensorially compatible AF-rings with \( t_i = t(A_i : R) \) and \( d_i = \text{dim}(A_i) \). If one of the following conditions is satisfied:

1. There exist maximal ideals \( M_1, \ldots, M_n \) belonging respectively to \( A_1, \ldots, A_n \) such that \( \lambda_i^{-1}(M_i) = \cdots = \lambda_n^{-1}(M_n) \) and \( \text{ht}(M_i) = d_i \) for \( i = 1, 2, \ldots, n \).
2. If \( M_1, \ldots, M_n \) are maximal ideals belonging respectively to \( A_1, \ldots, A_n \) such that \( \lambda_i^{-1}(M_i) = \cdots = \lambda_n^{-1}(M_n) \), then \( t_M = t_r \) for \( i = 1, 2, \ldots, n \).
(3) If $P_1, \ldots, P_n$ are minimal prime ideals belonging respectively to $A_1, \ldots, A_n$ such that $\lambda^{i_1}(P_1) = \cdots = \lambda^{i_n}(P_n)$, then $t(A_t/P_t : R) = t_i$ for $i = 1, \ldots, n$.

(4) $A_1, \ldots, A_n$ are equidimensional.

then

$$\dim(A_1 \otimes_R \cdots \otimes_R A_n) = t_1 + \cdots + t_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}.$$ 

The proofs of (1), (2), (3) and (4) are similar to those of [BGK1, Corollaries 1.10, 1.11, 1.13, and 1.14], respectively.

**Corollary 2.16.** Let $(A_1, \lambda_1), \ldots, (A_n, \lambda_n)$ be tensorially compatible AF-domains with $t_i = t(A_i : R)$ and $d_i = \dim(A_i)$. Then

$$\dim(A_1 \otimes_R \cdots \otimes_R A_n) = t_1 + \cdots + t_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}.$$ 

**Proof.** Since $A_1 \otimes_R \cdots \otimes_R A_n \neq 0$, by Proposition 1.4 we have $p_{A_1} = p_{A_2} = \cdots = p_{A_n} = p$; then $A_1 \otimes_R \cdots \otimes_R A_n \cong A_1 \otimes_{R/p} \cdots \otimes_{R/p} A_n$. The result follows from [W, Theorem 3.8].

Now we consider the special case in which $(A_1, \lambda_1)$.

**Corollary 2.17.** Let $(A, \lambda)$ be an AF-ring. Then $\dim(A \otimes_R A) = \dim(A) + t(A : R)$ if and only if there exist maximal ideals $M$ and $N$ in $A$ such that $\lambda^M(A) = \lambda^N(A)$, $\lambda^M = \lambda^N$, $t(M) = \dim(A)$, $t(N : R) = t(A : R)$ and $t(A/N : R) \leq t(A/M : R)$.

### 3. The valuative dimension of tensor products and Jaffard rings

[BGK1, Theorem 2.1] establishes that if $A$ is an AF-ring over a field $k$ and $B$ is a locally Jaffard ring, then $A \otimes_k B$ is a locally Jaffard ring. We next extend this result to AF-rings over a zero-dimensional ring.

**Theorem 3.18.** Let $(A, \lambda)$ be an AF-ring and $(B, \mu)$ a locally Jaffard ring such that $A \otimes_R B \neq 0$. Then $A \otimes_R B$ is a locally Jaffard ring.

**Proof.** It is sufficient to prove that for each prime ideal $Q$ of $A \otimes_R B$ and for each nonnegative integer $n$, $ht(Q[X_1, \ldots, X_n]) = ht(Q)$ (see [ABDPEF] and [C]). Let $P = \mu^{1}(Q), I = \mu^{n+1}(Q)$ and $p = \lambda_1^m\lambda_n^{b=1}(Q)$; according to Lemma 2.2, $A/pA$ is an AF-ring over the field $R/p$; moreover, by Lemma 1.2 $B/pB$ is a locally Jaffard ring; so we can apply Theorem 2.1 of [BGK1] to the $(R/p)$-algebras $A/pA$ and $B/pB$ obtaining that $(A/pA) \otimes_{R/p} (B/pB)$ is a locally Jaffard ring. Since $(A/pA) \otimes_{R/p} B/pB \cong (A \otimes_R B)/(pA \otimes_R B)$, then for each nonnegative integer $n$, it results that $ht((Q/p)(A \otimes_R B))[X_1, \ldots, X_n] = ht(Q/p(A \otimes_R B))$; so according to Lemma 1.2, $ht Q = ht Q[X_1, \ldots, X_n]$, as desired.

**Remark 3.** Let $(A, \lambda_A)$ be an AF-ring and $(B, \lambda_B)$ any $R$-algebra such that $A \otimes R B \neq 0$. Let $Q \in \text{Spec}(A \otimes_R B)$. $P = \mu_1^1(Q)$ and $I = \mu_1^{n+1}(Q)$. We obtain from [BGK 1, Lemma 2.2] the following result:

$$ht Q + t((A \otimes_R B)/Q : R) = t_D + ht I[X_1, \ldots, X_{n}],$$

Let us recall that the valuative dimension of tensor products of algebras over a field does not seem to be effectively computable in general. However, [G1, Proposition 3.1] states that provided $A_1$ and $A_2$ are two algebras over a field $k$, then

$$\dim_v(A_1 \otimes_k A_2) \leq \min(\dim_v A_1 + t(A_2 : k), t(A_1 : k) + \dim_v A_2).$$

The next result establishes the analogue of this result for the zero-dimensio­nal case.

**Proposition 3.19.** Let $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ be tensorially compatible $R$-algebras. Then

$$\dim_v(A_1 \otimes_R A_2) \leq \min(\dim_v A_1 + t(A_2 : R), t(A_1 : R) + \dim_v A_2).$$

**Proof.** Let $Q$ be any prime ideal of $A_1 \otimes_R A_2$; let $P_1 = \mu_1^1(Q)$, $P_2 = \mu_2^1(Q)$ and $p = \lambda_1^1(P_1) = \lambda_2^1(P_2)$. Let $T = A_1 \otimes_R A_2$. Then

$$\dim_v(T/Q) \leq \dim_v (T/(\text{Im}(j_1 \otimes id_{A_2}) + \text{Im}(id_{A_1} \otimes j_2))).$$

Moreover, using the canonical isomorphism

$$T/(\text{Im}(j_1 \otimes id_{A_2}) + \text{Im}(id_{A_1} \otimes j_2)) \cong (A_1/P_1) \otimes_{R/p} (A_2/P_2)$$

and [G1, Proposition 3.1], yields

$$\dim_v(T/Q) \leq \dim_v ((A_1/P_1) \otimes_{R/p} (A_2/P_2)) \leq \dim_v A_1/P_1 + t(A_2/P_2 : R), \dim_v A_2/P_2 + t(A_1/P_1 : R) \leq \dim_v A_1 + t(A_2 : R), \dim_v A_2 + t(A_1 : R).$$

The next result handles the case where one of two $R$-algebras is an AF-ring.

**Proposition 3.20.** Let $(A, \lambda)$ and $(B, \lambda_B)$ be tensorially compatible $R$-algebras and $A$ an AF-ring. Then, for any $r > \dim_v B - 1$,

$$\dim_v(A \otimes_R B) = \sup(D(tp + r, ht P + r, B/pB) \mid P \in \text{Spec}(A), p = \lambda_1^1(P) \text{ and } pb \neq B) - r = \sup(ht I[X_1, \ldots, X_n]) + \min (t_P, \text{ht } P + t(B/I : R)) \mid (P, I) \in \Gamma(A, B).$$

Proof. Let $r \geq \dim_v B - 1$. Then, by [C, Proposition 1, ii)], $B[X_1, \ldots, X_n]$ is a locally Jaffard ring. So, according to Theorem 3.1, $A \otimes_R B[X_1, \ldots, X_n]$ is a locally Jaffard ring and hence a Jaffard ring. Therefore, by Corollary 2.5, $$\dim_v(A \otimes_R B[X_1, \ldots, X_n]) = \dim(A \otimes_R B[X_1, \ldots, X_n]) = \sup \{D(tp, \dim_v(B[p/B]) | P \in \Spec(A) \text{ with } \lambda^n_0(P) = p \text{ and } pB \neq B\}.$$ Hence, according to [BGK1, Lemma 2.3], $$\dim_v(A \otimes_R B) = \sup \{D(tp + \dim_v(B[p/B]) | P \in \Spec(A) \text{ with } \lambda^n_0(P) = p \text{ and } pB \neq B\} - r = \sup \{htI[X_1, \ldots, X_n] + \min(t, \dim_v(B/I : R)) | (P, I) \in \Gamma(A, B)\}. \quad \Box$$

We conclude this section with two results on AF-domains.

Corollary 3.21. Let $(A, \lambda_A)$ be an AF-domain and $B$ any $R$-algebra such that $A \otimes_R B \neq 0$. Then for any $r \geq \dim_v B - 1$, $$\dim_v(A \otimes_R B) = D(t + r, d + r, B[p/A]) - r = \sup \{ht Q[X_1, \ldots, X_n] + \min(t, d + (B/I : R)) | I \in \Spec(B) \text{ and } \lambda^n_0(I) = pA\},$$
where $t = t(A : R)$ and $d = \dim_v A$.

Corollary 3.22. Let $(A, \lambda_A)$ and $(B, \lambda_B)$ be $R$-algebras such that $A$ is an AF-domain and $A \otimes_R B \neq 0$. If $\dim_v B \leq t(A : R) + 1$, then $A \otimes_R B$ is a Jaffard ring.

Remark 4. We thank the Referee for the following observation. Let $A_{\text{red}}$ be the reduced ring associated to a ring $A$. Then $t(A : R) = t(A_{\text{red}} : R_{\text{red}})$ for any $R$-algebra $(A, \lambda_A)$; moreover, if $(A_1, \lambda_1)$ and $(A_2, \lambda_2)$ are $R$-algebras, then $(A_1 \otimes_R A_2)_{\text{red}} = (A_1 \otimes_{R_{\text{red}}} A_2)_{\text{red}}$ [GD, Corollary 4.5.12]. One may therefore assume that $R$ is absolutely flat and $(A_1, \lambda_1)$, $(A_2, \lambda_2)$ are reduced $R$-algebras.

REFERENCES
