Considerable work, part of it summarized in Sally's book [15], has been concerned with the number of generators needed for ideals in a commutative ring \( R \). If there is a fixed bound \( n \), valid for all ideals, on the number of generators needed, \( R \) is said to have the \( n \)-generator property. That means, each ideal of \( R \) is \( n \)-generated (i.e. can be generated by \( n \) elements). If \( \dim R > 1 \), no such bound exists. Considerable interest has been shown in rings with the \( n \)-generator property. See for example [4], [11], [15] and [16].

Let \( G \) be an abelian group. The group ring associated to \( R \) and \( G \), denoted by \( R[G] \), is the ring of elements of the form \( \sum_{g \in G} a_g X^g \), where \( \{a_g \mid g \in G\} \) is a family of elements of \( R \) which are almost all zero. We refer to [15] for elementary properties of group rings. Of particular interest is the study of the question of when \( R[G] \) has the \( n \)-generator property. This question, either in general or for specific choice of \( n \), has received further attention by several authors. See [1], [8], [9], [10], [13], [14] and [17].

From the restriction on Krull dimension, we have \( 1 \geq \dim R[G] = \dim R + \gamma \), where \( \gamma \) denotes the torsion free rank of \( G \). If \( \gamma \neq 0 \), then \( G \) must be a finite group. If \( \gamma = 1 \), then \( G \cong \mathbb{Z} \oplus H \), where \( H \) is a finite abelian group and \( \mathbb{Z} \) denotes the group of the integers. We will focus on the case in which \( R \) is Artinian and \( \gamma = 0 \), i.e. \( G \) is a finite abelian group, since the case \( \gamma = 1 \) was considered by Olson and Vicknair in [14, Theorem 5.1]. Furthermore, [1] is entirely devoted to \( n = 3 \). However, for \( n \geq 4 \) and under our...
assumptions, the problem of when \( R[G] \) has the \( n \)-generator property remains open.

In this note, we consider the problem of determining when a group ring \( R[G] \) has the 4-generator property, where \( R \) is an Artinian principal ideal ring and \( G \) is a finite group.

Throughout this note rings and groups are taken to be commutative and the groups are written additively. If \( p \) is a prime integer, then the \( p \)-Sylow subgroup of the finite abelian group \( G \) will be denoted \( G_p \). When \( I \) is an ideal of \( R \), we shall use \( \mu(I) \) to denote the number of generators in a minimal basis for \( I \). Finally, recall that if \( I \) is an \( n \)-generated ideal in a local ring, then the \( n \) generators of \( I \) may be chosen from elements of a given set of generators of \( I \) (cf. [12, (5,3), p. 14]).

**Proposition 1.** Assume that \( G \) is a nontrivial finite 2-group, \((R,M)\) is an Artinian local principal ideal ring which is not a field and \( 2 \in M \). Then \( R[G] \) has the 4-generator property if and only if

A. \( (i) \) \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) with \( i \geq 1 \)

(ii) when \( M^4 \neq 0 \), then \( I \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

B. \( (ii) \) is a cyclic group

(iii) When \( M^4 \neq 0 \), then

\( (a) \) \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) where \( 1 \leq i \leq 2 \), \( 2 \in M^2 \)

(b) \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) where \( 1 \leq i \leq 3 \), \( 2 \in M \setminus M^2 \).

Before proving this proposition we establish a lemma which will be used frequently in the sequel.

**Lemma 2.** Assume that \((R,M)\) is a local principal ideal ring and \( G \) is a finite cyclic group. Let \( N \) be the maximal ideal of the local ring \( R[G] \). Then \( R[G] \) has the 4-generator property if and only if \( N, N^2, N^3, N^4 \) and \( N^5 \) are 4-generated.

**Proof.** \( R[G] \) is local with maximal ideal \( N = (r, 1 - X^2) \), where \( r \) generates \( M \) in \( R \) and \( g \) is the generator of \( G \). Suppose that \( N, N^2, N^3, N^4 \) are 4-generated. We need to prove that only proper ideals \( I \) of \( R[G] \) is 4-generated. By [16, Corollary 4.2.1], it suffices to consider the case where \( I \not\cong N^3 \). Let \( x \in I \setminus N^3 \).

If \( x \notin N^3 \), then \( \lambda \notin N, \mu \notin M \), then \( \lambda + \mu \notin N \). Therefore \( N^2 = (x, (1 - X^2)^2, (1 - X^2)^3) \) or \( N^3 = (r, (1 - X^2)^2, (1 - X^2)^3) \). Hence, \( \mu(N^2(x)) = \mu(N^3(x)) \leq 2 \).

By [11, Theorem 1, e 
\( 1 \rightarrow 1 \)], \( R[G] / (x) \) has the 2-generator property. Then \( \mu(\gamma(x)) \leq 2 \).

**Proof of Proposition 1.** \( \Rightarrow \) Assume \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \) where \( 0 < t_1 \leq t_2 \leq \cdots \leq t_n \). If \( R[G] \) has the 4-generator property, then the homomorphic image \( (R/\mathfrak{M}[G]) \) also does. By [14, Corollary 2.2.1, p. 3].

We now prove that if \( s = 3 \) does not hold. Indeed, if \( R[Z/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] \) has the 4-generator property, then the homomorphic image \( R[Z/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] \) also does. Since \( R \) is a local ring with residue field of characteristic 2, \( R[Z/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] \) is local with maximal ideal \( N := (r, 1 - X^2, 1 - X^3, 1 - X^4) \), where \( r \) generates \( M \) in \( R \) and \( g \notin M, g \notin M, g \notin M, g \notin M \).

Since \( g \notin M, g \notin M, g \notin M, g \notin M \), it is clear that \( r \notin M, (1 - X^3)^2 \) and \( (1 - X^3)^3 \) are required as generators of \( N^2 \). Furthermore, using arguments similar to those used above, we obtain that \( r(1 - X^3)^2 \) is a generator of \( N^3 \), and \( r(1 - X^3)^3 \) is a generator of \( N^4 \).

Since \( M^4 \neq 0 \), and \( g \notin M, g \notin M, g \notin M, g \notin M \), then \( N^2 \) needs more than four generators, a contradiction. Consequently, \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

**Group Rings \( R[G] \): An Artinian Principal Ideal Ring**

Let \( (R,M) \) be a local principal ideal ring with maximal ideal \( N := (r, 1 - X^2, 1 - X^3, 1 - X^4, 1 - X^5) \).

If \( r(1 - X^3)^2 \in R[G] \), then the augmentation map \( R[G] \rightarrow R \) is surjective. Therefore \( N^2 \) is a generator of \( N^2 \).

Since \( (1 - X^3)^2 \notin N^2 \), and \( g \notin M, g \notin M, g \notin M, g \notin M \), then \( N^2 \) is also a generator of \( N^2 \).

Therefore \( N^2 \) needs more than four generators, a contradiction. Consequently, \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

**Group Rings \( R[G] \): An Artinian Principal Ideal Ring**

Let \( R \) be a local principal ideal ring with maximal ideal \( N := (r, 1 - X^2, 1 - X^3, 1 - X^4) \).

If \( r(1 - X^3)^2 \in R[G] \), then the augmentation map \( R[G] \rightarrow R \) is surjective. Therefore \( N^2 \) is a generator of \( N^2 \).

Since \( (1 - X^3)^2 \notin N^2 \), and \( g \notin M, g \notin M, g \notin M, g \notin M \), then \( N^2 \) is also a generator of \( N^2 \).

Therefore \( N^2 \) needs more than four generators, a contradiction. Consequently, \( G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).
Let \( I \) be a proper ideal of \( R[G] \). Since \( N^3 = (1 - X^3)N^2 \), [11, Lemma 2] implies that
\[
\mu(I) \leq \mu(I + N^2).
\]
In order to show that \( I \) is 4--generated, we may assume \( N^3 \subset I \). Let
\( x \in I \setminus N^3 \), \( x \in N \). By [8, Theorem 159], \( \mu(N/(x)) = \mu(N) - 1 = 2 \). Let us show that
\[
\mu((N/(x))^2) \leq 2. \text{ Since } \mu(N/(x))^2 \leq 2, \text{ we have } N = (r, x, 1 - X^4), N = (r, x, 1 - X^4) \text{ or } N = (x, 1 - X^4, X^3). \]
If \( N = (r, x, 1 - X^4) \) then \( N/(x) = (r, 1 - X^4) \), where bars denote images under the natural map \( R[G] \twoheadrightarrow R[G]/(x) \). Then \( r^2 = 0 \) then \( (N/(x))^2 = (\bar{r}(1 - X^3), (1 - X^3)^2) \), and hence \( \mu((N/(x))^2) \leq 2 \). The argument for \( N = (r, x, 1 - X^4) \) is similar.

If \( N = (r, x, 1 - X^4, 1 - X^3) \), then \( (N/(x))^2 = (\bar{r}(1 - X^3), (1 - X^3)^2, (1 - X^3)^2) \).

If \( r \in M^2 = (0) \), we're finished. Otherwise, \( M = (r) = (2) \). Clearly \( 2 = \lambda x + \mu(1 - X^4) + \delta(1 - X^3) \) for some \( \lambda, \mu, \delta \in R[G] \). Furthermore, we may assume that \( \mu \) and \( \delta \) are not invertible. So \( \lambda, \delta \neq 0, \lambda \in R[G] \). Hence \( 2 = \lambda x + \mu(1 - X^4)(1 - X^3) + \delta(1 - X^3)^2 + \beta(1 - X^4)^2 \). For some \( \lambda, \mu, \beta, \delta \in R[G] \). Then \( 2(1 - \beta(1 - X^3)) = \lambda x + \mu(1 - X^4)(1 - X^3) + \delta(1 - X^3)^2 \).

Since \( 1 - \beta(1 - X^3) \) is a unit in \( R[G] \), \( 2 \in \left( \bar{1}(X^3), (X^3)^2 \right) \) and so does \( \bar{2}(X^3) \). Consequently, \( (N/(x))^2 = (\bar{r}(1 - X^3), (1 - X^3)^2, (1 - X^3)^2) \), and hence \( \mu((N/(x))^2) \leq 2 \).

By [11, Theorem 6], \( R[G]/(x) \) has the 2--generator property. Then \( I/(x) \) is 2--generated, and hence \( I \) is 4--generated. This completes the proof of (i).

\( \Rightarrow \) (ii) Assume \( G \cong Z/2 \oplus Z/2 \) and \( M^2 \neq 0 \). Then \( R[Z/2 \oplus Z/2] \) is a local ring with maximal ideal \( N = (r, 1 - X^4, 1 - X^3) \), where \( r \) generates \( M \) in \( R \) and \( a \) is redundant. Then passing to the homomorphic images \( R/(r) \), we get \( \mu(I) = \mu(I + N^2) \). As before, we may assume that \( N^2 \subset I \). Let \( x \in I \setminus N^2 \), \( x \in N \). By [8, Theorem 159], \( \mu(N/(x)) = \mu(N) - 1 = 2 \). Thus \( N = (r, x, 1 - X^4) \) or \( N = (r, x, 1 - X^4, 1 - X^3) \). It is easily seen that the two first cases we have \( \mu(N/(x))^2 \leq 2 \). Now let consider the remaining case, i.e. \( N = (x, 1 - X^4, 1 - X^3) \).

Then \( (N/(x))^2 = (\bar{x}(1 - X^3), (1 - X^3)^2, (1 - X^3)^2) \).

If \( 2 \in M^2 = (r^2) \), since \( r \in N/(x) \), then \( 2 = \lambda h(1 - X^3) + \mu(1 - X^4)(1 - X^3) + \delta(1 - X^3)^2 \). This completes the proof of (ii).

B) Suppose that \( G \) is a cyclic group \( (s = 1) \). Let \( g \) be the generator of \( G \). We have
\[
N = (r, 1 - X^4); \quad N^2 = (r^2, r(1 - X^4), (1 - X^3)^2); \quad N^3 = (r^3, r^2(1 - X^4), r(1 - X^3)^2, (1 - X^3)^2); \quad N^4 = (r^4, r^3(1 - X^4), r^2(1 - X^3)^2, r(1 - X^3)^2, (1 - X^3)^4).
\]

(i) Assume \( M^4 = 0 \). Applying Lemma 2, we conclude that \( R[G] \) has the 4--generator property.

(ii) (a) Assume \( M^4 \neq 0 \) and \( 2 \in M^2 \). In order to conclude, it suffices to show that \( R[Z/2 \oplus Z/2] \) has the 4--generator property while \( R[Z/2 \oplus Z/2] \) does not. Suppose that \( R[Z/2 \oplus Z/2] \) has the 4--generator property. Then \( N^4 \) is 4--generated.

Since \( M^4 \neq 0 \) and \( 1 < g > 4 \), it is easily seen that \( r^4 \) and \( (1 - X^3)^4 \) are generators of \( N^4 \).

If \( r^2(1 - X^3) \) is a redundant generator of \( N^4 \), then passing to the homomorphic image \( R/(r^2) \), we get \( r^2(1 - X^3)^2 = a(1 - X^3)^2 \) with \( a \in R/r^2 \). It follows that \( r^2 = 0 \) in \( R/(r^2) \), a contradiction.

If \( r^2(1 - X^3)^2 \) is redundant, then by passing to the homomorphic image \( R/(r^2) \), we obtain \( r^2(1 - X^3)^2 = a(1 - X^3)^2 \) with \( a = \sum a_i X_i \), where \( a_i \in R/r^2 \). After setting corresponding terms equal, we obtain the following equations:

\[
\begin{align*}
X^0 & \quad a_0 - a_3 + a_0 = 0 \\
X^1 & \quad -3a_0 + a_2 - a_0 = 0 \\
X^2 & \quad 3a_0 - 3a_1 + a_2 - a_0 = 0 \\
X^3 & \quad -a_0 + 3a_3 - a_0 + a_3 = 0 \\
X^4 & \quad a_1 + a_3 - a_0 + a_3 = 0 \\
X^5 & \quad -a_0 + 3a_4 - a_0 + 3a_2 = 0 \\
X^6 & \quad 3a_0 - 3a_1 - a_0 - a_4 = 0
\end{align*}
\]

This yields \( r^2 = 0 \) in \( R/r^2 \). Hence \( R^2 = R/r^2 \), a contradiction.

If \( r(1 - X^3)^2 \) is redundant, then by passing to the homomorphic image \( R/(r^2) \), we get \( r(1 - X^3)^2 = (1 - X^3)^2)(R/(r^2)) \) \( < g > \). Since \( 2 \in M^2 = (r^2) \), \( r(1 - X^3)^2 = 0 \) in \( R/(r^2) \), a contradiction. Consequently, \( N^4 \) needs more than four generators, contradicting the fact that \( N^4 \) is 4--generated.

Now let us show that \( R[Z/2 \oplus Z/2] \) has the 4--generator property. If \( 2 \in M^2 \setminus M^3 \) then
\[
M^2 = (r^2) = (2).
\]
We have
\[
1 = (1 - X^2 + X^4)^2 \\
= 1 + 4(1 - X^4)X^2 + 6(1 - X^2)X^2 + 4(1 - X^2)^2X^2 + (1 - X^2)^4
\]
Then $2(1 - X^y)^2 \in (4(1 - X^y), 1 - X^y)^2 \subseteq (r^4, 1 - X^y)^2$. Therefore $2(1 - X^y)^2 \in (r^4, 1 - X^y)^2$. Consequently, $N^4$ is 4-generated. If $2 \in M^2$, we get

$$
(1 - X^y)^4 = 1 - 4X^y + 6X^{2y} - 4X^{3y} + X^{4y}
$$

$$
= 2 - 4X^y + 6X^{2y} - 4X^{3y}
$$

$$
= 2 - 2X^y + 2X^{2y} + 4X^{3y} - 4X^{2y}
$$

$$
= 2(1 - X^y) - 2X^y(1 - X^y) + 4X^{3y}(1 - X^y)
$$

$$
= 2(1 - X^y)(1 - 2X^{2y})
$$

Then $(1 - X^y)^4 \in \langle (1 - X^y) \rangle \subseteq (r^4, 1 - X^y)$. Hence $N^4$ is 4-generated. Lemma 2 completes the proof.

b) Assume $M^4 \neq 0$ and $2 \in M \setminus M^2$. It suffices to prove that $R[Z/16Z]$ has the 4-generator property while $R[Z/16Z]$ does not. Clearly $M = \langle r \rangle$ and $N^4 = \langle (1 - X^y), 4(1 - X^y)^2, 2(1 - X^y)^3, (1 - X^y)^4 \rangle$.

Assume $g > 0$ and $S \subseteq \mathbb{Z}/16Z$. We have

$$
1 = (1 - X^y + X^{2y})^g
$$

$$
= \sum_{i=0}^{g} \binom{g}{i} (1 - X^y)^{g-i}X^{i(8-9y)}
$$

$$
= 1 + 8(1 - X^y)X^{8y} + 28(1 - X^y)^2X^{16y} + 56(1 - X^y)^3X^{24y} + \ldots
$$

$$
+ (1 - X^y)^g \left( \sum_{i=0}^{2g/3} \binom{g}{i} (1 - X^y)^{g-i}X^{i(8-9y)} \right)
$$

Then $8(1 - X^y)^2 \in \langle (1 - X^y)^2, 1 - X^y \rangle$, and hence $N^4$ is 4-generated. Thus $R[Z/16Z]$ has the 4-generator property.

Assume $g > 0$ and $S \subseteq \mathbb{Z}/16Z$. We prove that $N^4$ is not 4-generated. It is clear that 16 and $(1 - X^y)^2$ are required as generators of $N^4$.

If $8(1 - X^y)^2$ is redundant, then passing to the homomorphic image $(R/(16))[< g >]$, yields $8(1 - X^y)^2 \in \langle (1 - X^y)^2 \rangle[R/(16)][< g >]$. By Lemma 1.5, $8 = 16\alpha$ for some $\alpha$ in $R/(16)$. Hence $S = 0$ in $R/(16)$, a contradiction.

If $2(1 - X^y)^3$ is redundant, then passing to the homomorphic image $(R/(4))[< g >]$, we obtain that

$$
2(1 - X^y)^3 = a(1 - X^y)^4
$$

where $a \in \langle 2 \rangle$.

After setting corresponding terms equal, we obtain among other equations the following:

$$
X^0 = \alpha_0 + \alpha_{12} + 2\alpha_{14} = 2
$$

$$
X^{2y} = 2\alpha_0 + \alpha_{12} + \alpha_{14} = 2
$$

$$
X^{4y} = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

$$
X^{6y} = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

$$
X^{8y} = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

$$
X^0 y^0 = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

$$
X^{12y} = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

$$
X^{14y} = \alpha_0 + 2\alpha_{20} + \alpha_{24} = 0
$$

After resolving this system, we obtain $2 = 0 \in R/(4)$, then $2 \in M^2$, a contradiction.

If $4(1 - X^y)^2$ is redundant, then passing to the homomorphic image $(R/(8))[< g >]$, yields $4(1 - X^y)^2 = (a(1 - X^y))^2$ where $a \in \langle R/(8) \rangle[< g >]$. As before, we obtain a system of 16 linear equations in 16 unknowns. After resolving this system, we obtain

$$
4 = 0 \in R/(8),
$$

a contradiction $M^4 \neq 0$.

It follows that $N^4$ needs more than four generators. Hence $R[Z/16Z]$ does not have the 4-generator property. This completes the proof of Proposition 1.

**PROPOSITION 3** Assume that $G$ is a nontrivial finite 3-group, $(R, M)$ is an Artinian local principal ideal ring which is not a field and $3 \in M$. Then $R[G]$ has the 4-generator property if and only if

A) $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $S \subseteq M \setminus M^2$ and $M^2 = 0$.

B) $(G)$ is a cyclic group.

(i) When $M^4 \neq 0$, then

(a) $G \cong \mathbb{Z}/3\mathbb{Z}$, $3 \notin M^2$.

(b) $G \cong \mathbb{Z}/3\mathbb{Z}$, where $1 \leq i \leq 3, 3 \notin M \setminus M^2$.

**LEMMA 4** Let $(R, M)$ be a local ring such that $M^0$ is $n$-generated, where $n$ is a positive integer. Then for each ideal $I$ of $R$, $\mu(I) \leq \mu(I + M)/n$.

Proof. We may assume that $R$ has an infinite residue field (see [15, p.10]). Since $M^0$ is $n$-generated, then [1, 2.3, p.30] implies that $M^0 = \mu(M^0)$ for some $\mu \in M$. By [11, Lemma 2], $\mu(I) \leq \mu(I + M)/n$ for each ideal $I$ of $R$.

**LEMMA 5** Let $(R, M)$ be a local ring such that $M^0$ is 3-generated, $I$ a proper ideal of $R$ and $x \equiv 1 \pmod{M^2}$ such that $x \in M$. Then $\mu(I/(x)) \leq \mu(I + M)/3$.

Proof. $M^0$ is 3-generated and $x \in M \setminus M^2$ implies that $\mu((I/(x)) = \mu(M^0) = 1$. By applying Lemma 4 to $R/(x)$, we get $\mu(I/(x)) \leq \mu(I + M/(x)) = \mu(M/(x)) = 1$.

Proof of Proposition 3. By hypothesis, $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/3\mathbb{Z}$, where $0 < \ell_0 < \ell_1 < \cdots < \ell_a$. Suppose that $R[G]$ has the 4-generator property, then the homomorphic image $(R[M])G$ does also. By [14, Corollary 2.3], $s = 2$.

A) $\Rightarrow$). If $s = 2$, [14, Proposition 2.1(a)] implies that $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ with $t \geq 1$.

Assume $3 \in M^2$. Let $N = \langle r, 1 - X^y, 1 - X^y \rangle$, where $r$ generates $M$ in $R$ and $< g > \neq \langle h \rangle$. Let $N = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. We have

$$
N^2 = (r^2, r(1 - X^y), r(1 - X^y), (1 - X^y)(1 - X^y), (1 - X^y)^2, (1 - X^y)^3).
$$

Using arguments similar to ones above used above it is easy to check that $\langle r^2, r(1 - X^y), (1 - X^y)(1 - X^y), (1 - X^y)^2 \rangle$ and $(1 - X^y)^3$ are required as generators of $N^2$. Thus $R[\mathbb{Z}/3\mathbb{Z}]$ does not have the 4-generator property, a contradiction. Consequently, $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ with $t \geq 1$.

Assume $(G) \neq 0$. Let $N = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ be the maximal ideal of $R[\mathbb{Z}/3\mathbb{Z}]$. Consider the ideal $I = (9, N^2)$. Then $I = \langle 9, 3(1 - X^y), 3(1 - X^y), (1 - X^y)^2(1 - X^y), (1 - X^y)^3(1 - X^y)^2 \rangle$. Group Rings $R[G]$. An Artinian Principal Ideal Ring
It is easily seen that all these elements are required as generators of $N$. Thus $R[\mathbb{Z}/3\mathbb{Z}]$ does not have the $4$-generator property, a contradiction. Consequently, $M^4 = 0$.

We claim that $R[\mathbb{Z}/3\mathbb{Z}]$ does not have the four generator property. Let $N$ be its maximal ideal and $g,h$ the generators of $R[\mathbb{Z}/3\mathbb{Z}]$, respectively. Then we have

$$N^2 = (3(1 - X^3), (1 - X^3)(1 - X^3), (1 - X^3)^2), (1 - X^3)^3)$$

$$N^3 = (3(1 - X^3), (1 - X^3)^3, (1 - X^3)^2(1 - X^3), (1 - X^3)^3(1 - X^3)^2, (1 - X^3)^3)^3).$$

If $(1 - X^3)^2$ is a redundant generator of $N^2$, then by applying the augmentation map $R[< h >] \rightarrow R[< g >]$, we get $3(1 - X^3)^3 \in (1 - X^3)^2 R[< h >]$. By [1, Lemma 1.7], $3 \in N_3$ for some $\lambda \in R$. Then $M = (3(1 - X^3))$, a contradiction. The arguments for $3(1 - X^3), (1 - X^3)^3, (1 - X^3)^2(1 - X^3)$ and $(1 - X^3)^3(1 - X^3)^2$ are similar to ones used above. Hence $\mu(N^3) > 4$.

Assume $G = \mathbb{Z}/3\mathbb{Z}, 3 \in M \setminus M^2$ and $M^2 = 0$. Let us show that $R[G]$ has the 4-generator property. Let $N$ be the maximal ideal of $R[G]$ and $< g > \subset < h > = \mathbb{Z}/3\mathbb{Z}$. We have

$$N = (3, 1 - X^3, 1 - X^4)$$

$$N^3 = ((1 - X^3)^2(1 - X^3), (1 - X^3)^3(1 - X^3))$$

$$N^3 = (3(1 - X^3), (1 - X^3)^3, (1 - X^3)^2(1 - X^3), (1 - X^3)^3(1 - X^3)^2)$$

$$N^4 = (3(1 - X^3)^2, (1 - X^3)^3, (1 - X^3)^2(1 - X^3), (1 - X^3)^3(1 - X^3)^2).$$

Let $I$ be an proper ideal of $R[G]$, we need to prove that $I$ is 4-generated. Applying Lemma 4 to $N^4$, yields $\mu(I) \leq \mu(I + N^3)$. Since $N^2$ is 4-generated, we may assume $N^3 \subset I$. Let $x \in I \setminus N^3$. If $x \in N^2$, since $N^2$ is 3-generated, Lemma 5 implies the desired conclusion. If $x \notin N^2$, by [8, Theorem 15b], it follows that $N = (3, 1 - X^3)$ or $N = (3, 1 - X^3)$ and $N^3 = (1 - X^3)^2(1 - X^3)^2)$, where bars denote images under the natural map $R[G] \rightarrow R[G](x)$. As in the proof of Lemma 2, we conclude via part (6) of [11, Theorem 1]. Likewise for $N = (3, 1 - X^3)$. If $N = (3, x, 1 - X^3)$ and $(N^3)^2 = (1 - X^3)^2(1 - X^3)^2)$, where bars denote images under the natural map $R[G] \rightarrow R[G](x)$. As in the proof of Lemma 2, we conclude via part (6) of [11, Theorem 1].

If $N = (x, 1 - X^3, 1 - X^4)$, then $(N^3)^2 = (N^3 + (x, 1 - X^3))^2 = I(x)$, We consider separately two cases:

- If $(N^3)^2 \subset I(x)$, choose $x \in I$ such that $x_1 \in I \setminus (N^3)^2$. Assume $y \in (N^3)^2$.

- Since $(N^3)^2$ is 3-generated, Lemma 5 yields

$$\mu(I(x)) = \mu(N(x)) \leq \mu(y) \leq 2.$$
the homomorphic image \((R/(r^4))[[g >]]\) and applying [1, Lemma 1.5], yields \(r^3 = 0\) in \(R/(r^4)\), a contradiction. If \((r - X^3)^2\) is redundant, then passing to the homomorphic image \([R/(r^2)]<[g >]\), we get \(r - X^3\) is \((1 - X^3)^2(R/(r^2))<[g >]\), a contradiction. If \(r^2(1 - X^3)^3 = 0\) in \([R/(r^2)]<[g >]\), with \(r\) and \((1 - X^3)^2\) as generators of \([R/(r^2)]<[g >]\) and applying [1, Lemma 1.5], yields \(r^2 = \lambda p\) for some \(\lambda \in R/(r^4)\). Since \(p \neq 1\), \(r^2 = 0\) in \(R/(r^4)\), a contradiction. If \(r^2(1 - X^3)^3\) is redundant, then passing to the homomorphic image \([R/(r^2)]<[g >]\) and applying [1, Lemma 1.7], yields \(r^2 = \lambda p\) for some \(\lambda \in R/(r^2)\). Since \(p \neq 1\), \(r^2 = 0\) in \(R/(r^2)\), a contradiction. Finally, if \(r - X^3\) is redundant, then by passing to the homomorphic image \([R/(r^2)]<[g >]\), we get \(r - X^3\) is \((1 - X^3)^2(R/(r^2))<[g >]\), and \((1 - X^3)^2\) as required as generators of \(N^4\). Since \(p \neq 1\), \(r - X^3 = 0\) in \(R/(r^2)\), a contradiction. Therefore \(N^4 = 4\)-generated. Let us prove that \(N^4 = 4\)-generated.

PROPOSITION 6 Let \((R, M)\) be a local Artinian principal ideal ring which is not a field, \(p\) a prime integer such that \(p > 3\) and \(p \in M\). Let \(G\) be a nontrivial finite \(p\)-group. Then \(R[G]\) has the \(4\)-generator property if and only if

(i) \(G\) is a cyclic group

(ii) \(M^4 \neq 0\), then \(p \notin M^4\) and

(a) \(G \cong \mathbb{Z}/p\mathbb{Z}\), if \(p \notin M^2\)

(b) \(G \cong \mathbb{Z}/p^2\mathbb{Z}\), if \(p \in M^2\) and \(p \leq 3\), if \(p \in M^2\) \(- 3\).

Proof. If \(R[G]\) has the \(4\)-generator property, by [14, Proposition 3.5], \(G\) is a cyclic group, and if in addition \(M^4 = 0\) then \(G \cong \mathbb{Z}/p^2\mathbb{Z}\) with \(p \leq 3\).

Let \(g\) be the generator of \(G\) and \(N = \langle g, 1 - X^3 \rangle\) the maximal ideal of \(R[G]\). As before, to show that \(R[G]\) has the \(4\)-generator property, by Lemma 2 it suffices to prove that \(N^4 = 4\)-generated. We have

\[N^4 = \langle r^4, r^3(1 - X^3), r^2(1 - X^2)^2, r(1 - X)^2, (1 - X)^2 \rangle.\]
We have
\[ 1 = (1 - X^q + X^q)p^2 = \sum_{i=0}^{p^2-2} \left( \binom{p^2}{i} (1 - X^q)^i X^{(p^2-2)\cdot i} \right) \]
\[ = 1 + \left( \binom{p^2}{1} (1 - X^q)X^{(p^2-1)\cdot 1} + \binom{p^2}{2} (1 - X^q)^2 X^{(p^2-2)\cdot 2} \right) \]
\[ + \binom{p^2}{3} (1 - X^q)^3 X^{(p^2-3)\cdot 3} + (1 - X^q)^4 \left( \sum_{i=0}^{p^2-4} \binom{p^2}{i} X^{(p^2-4)\cdot i} \right) \]  

It is straightforward that \( p^2(1 - X^q) \in \langle p^2(1 - X^q)^3, p^2(1 - X^q)^2, (1 - X^q)^4 \rangle \subset \langle p^2(1 - X^q)^3, p(1 - X^q)^3, (1 - X^q)^4 \rangle \). Hence \( N^2 \) is 4-generated. This completes the proof of Proposition 6.

The previous propositions were steps to state the following theorem.

**Theorem**

Let \( R \) be an Artinian principal ideal ring and \( G \) a nontrivial finite abelian group. Then \( R[G] \) has the 4-generator property if and only if \( R = R_1 \oplus \cdots \oplus R_n \), where, for each \( j \), \((R_j, M_j)\) is a local Artinian principal ideal ring subject to:

(I) Assume \( R_j \) is a field of characteristic \( p \).

(a) when \( p = 2 \), then \( G_p \) is a homomorphic image of \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) or \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) where \( i \geq 0 \).

(b) when \( p = 3 \), then \( G_p \) is a homomorphic image of \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) where \( i \geq 0 \).

(II) Assume \((R_j, M_j)\) is a principal ideal ring which is not a field and \( p \) a prime integer such that \( p \) divides \( \text{Ord}(G) \) and \( p \in M_j \).

(a) Assume \( p = 2 \).

A) (i) \( G_p \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) with \( i \geq 1 \)

(ii) when \( M_j^2 \neq 0 \), then \( G_p \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \).

B) (i) \( G_p \) is a cyclic group

(a) When \( M_j^2 \neq 0 \), then

\( \text{a} \) \( G_p \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), \( 1 \leq i \leq 2 \), if \( 2 \notin M_j^2 \)

\( \text{b} \) \( G_p \cong \mathbb{Z}/2\mathbb{Z} \), \( 1 \leq i \leq 2 \), if \( 2 \notin M_j \setminus M_j^2 \).

(II) Assume \( p = 3 \).

A) \( G_p \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) if \( i \notin M_j^2 \) and \( M_j^2 = 0 \).

B) (i) \( G_p \) is a cyclic group

(a) When \( M_j^2 \neq 0 \), then

\( \text{a} \) \( G_p \cong \mathbb{Z}/3\mathbb{Z} \), if \( i \notin M_j^2 \)

\( \text{b} \) \( G_p \cong \mathbb{Z}/3\mathbb{Z} \), \( 1 \leq i \leq 2 \), if \( 3 \notin M_j \setminus M_j^2 \).

(II) Assume \( p > 3 \).

A) \( G_p \) is a cyclic group

(a) If \( M_j^3 \neq 0 \), then \( p \notin M_j^3 \) and

\( \text{a} \) \( G_p \cong \mathbb{Z}/p\mathbb{Z} \), if \( p \notin M_j^3 \)

\( \text{b} \) \( G_p \cong \mathbb{Z}/2\mathbb{Z} \), \( 1 \leq i \leq 2 \), if \( p \notin M_j \setminus M_j^3 \).

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**References**

For a commutative ring $R$ with identity, let $T(R)$ denote its total quotient ring and $U(R)$ its group of units. For an extension of commutative rings $R \subseteq S$ we can form $U(S)/U(R)$, the quotient of the unit groups. In the case where $R$ is an integral domain with quotient field $K$, then $U(K)/U(R) = K^*/U(R)$ is the group of divisibility of $R$ and is denoted by $G(R)$. Here $K^* = \mathbb{K} - \{0\}$ is the multiplicative group of $K$.

We will be particularly interested in the following two questions. 

(1) When is $U(S)/U(R)$ finite or finitely generated? 

(2) When does $U(S)/U(R)$ finite or finitely generated imply that $S$ is a finitely generated $R$-module?

First, suppose that $K = R \subseteq S = F$ are both fields. Brandis' Theorem [4] or [8, Theorem 4.3.11] answers both questions.

**BRANDIS' THEOREM.** Let $K \subseteq F$ be a field extension. Then $F^*/K^*$ is finitely generated if and only if (1) $K = F$ or (2) $K$ is finite and $[F : K] < \infty$.

Actually, a stronger result due to L. Avramov and Davis and Maroscia [6] is true. Let $K \subseteq F$ be a field extension and let $r_0(F^*/K^*) = \dim_{\mathbb{Q}}((F^*/K^*) \otimes \mathbb{Q})$ be the torsion-free rank of $F^*/K^*$. Then the following statements are equivalent:

(a) $r_0(F^*/K^*) < \infty$, (b) $r_0(F^*/K^*) = 0$, (c) char $K = p > 0$ and either $F$ is algebraic over $\mathbb{F}_p$ or $F$ is purely inseparable over $K$. For a simpler proof of this result and for a discussion of the group $F^*/K^*$, the reader is referred to [5].