2.8 The derivative as a function

The derivative of \( f(x) \) with respect to \( x \) is \( f'(x) \)

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h}
\]

provided the limit exists.

Also

\[
\frac{d^2 f(x)}{dx^2} = \lim_{{x \to x}} \frac{f(z) - f(x)}{z - x}
\]

Ex. \( f(x) = x+1 \) \( f'(x) = \lim_{{h \to 0}} \frac{x+h+1 - (x+1)}{h} = 1 \)

Ex. \( f(x) = \frac{x+2}{x} \) \( f'(x) = \lim_{{h \to 0}} \frac{x+h+2 - x}{x(x+h)} = \lim_{{h \to 0}} \frac{x(x+h+2) - x(x+2)}{h(x)(x+h)} = \lim_{{h \to 0}} \frac{x(x+h+1) - x(x+1)}{h} = \lim_{{h \to 0}} \frac{x(x+h+1) - x(x+1)(x+h)}{h(x)(x+h)} = \lim_{{h \to 0}} \frac{x(x+h+1) - x(x+1)x}{h(x)(x+h)} = \lim_{{h \to 0}} \frac{-2h}{h(x)(x+h)} = \frac{-2}{x^2} \)

Def. A function \( f \) is differentiable at \( a \) if \( f'(a) \) exists. It is differentiable on \((a,b)\) if it is differentiable at every point.

Ex. Where is \( f(x) = |x| \) differentiable \( \forall x \in \mathbb{R} \setminus \{0\} \)

why not 0?
Ex. \( f(x) = x^3 + x \)

\[
f'(x) = \lim_{{h \to 0}} \frac{{(x+h)^3 + (x+h) - x^3 - x}}{h} = 3x^2 + 1
\]

\[\text{Eq.} \]

Find the tangent to the curve at \( x = 1 \)

Point \( (x_0, y_0) \) \( y = mx + b \) or \( (y - y_0) = m(x - x_0) \)

Point \( x = 1 \Rightarrow y = 2 \) \( \text{the slope} \) \( \frac{\text{d}}{\text{d}x} x = 1 \Rightarrow f'(1) = 4 \)

then the tangent \( \text{Eq.} \)

\[
y - 2 = 4(x - 1)
\]

\[
y = 4x - 2
\]

Ex. \( f(x) = \sqrt{x+1} \) find the Eq. of tangent \( \lim \) at \( (2,3) \)

\[
f'(x) = \lim_{{h \to 0}} \frac{{\sqrt{x+h+1} - \sqrt{x+1}}}{h} \cdot \frac{\sqrt{x+1} + \sqrt{x+1}}{\sqrt{x+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}
\]

\[
f'(2) = \frac{1}{6}
\]

Ex. \( y - 3 = \frac{1}{6}(x - 8) \)

Ex. \( f'(x) = x^2 - 8x + 9 \) find \( f'(a) \)

\[
f'(a) = \lim_{{h \to 0}} \frac{{(a+h)^2 - 8(a+h) + 9 - a^2 + 8a - 9}}{h} = 2a - 8.
\]
Notations for derivative

\[ f'(x), \quad \frac{dy}{dx} = y' = \frac{df}{dx}, \quad D(f)(x), \quad D_x(f) \]

\[ f'(x_0) = \frac{df}{dx}_{x=x_0} \]

Graphing the derivative

Ex.

\[ f'(1) = 1 \]
\[ f'(2) = 0 \]
\[ f'(3) = -1 \]
\[ f'(4) = 0 \]

Ex. Match the graph of \( f \) with their derivatives A-F

One-sided derivatives

\[ f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \]
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

**Ex.**

\[ f(x) = \begin{cases} 
  x^2 + 2 & \text{ if } x \leq 1 \\
  2x & \text{ if } x > 1
\end{cases} \]

continuous but not differentiable.

\[ f(1) = 3 = \lim_{x \to 1^+} x^2 + 2 = \lim_{x \to 1^-} x^2 + 2 = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) \]

So \( f(x) \) is continuous.

Now

\[ f'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1+h^2 + 2 - 1+2}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 1 \]

while

\[ f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 + 2 - (1^2 + 2)}{h} = \lim_{h \to 0^-} \frac{h(2+h)}{h} = 2 \]

Not differentiable.

**Ex.** \( f(x) = |x| \) continuous but not differentiable at \( x = 0 \).

\[ f'(0) = 1 \quad \text{while} \quad f'_-(0) = -1 \]
Differentiability

points of non-differentiability when the curve does not have a tangent line

Type
- Corners
  Ex. \( f(x) = |x| \)
- Vertical tangent / cusp
  Ex. \( f(x) = \sqrt[3]{x} \)
  \( f'(x) = \frac{1}{3\sqrt[3]{x^2}} \)

\* points of discontinuity

Ex. \( f(x) = [x] \)

** Th. If \( f(x) \) is diff then it is Cont.

Proof:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Now

\[ \lim_{x \to a} f(x) = \lim_{x \to a} (f(x) + f(x) - f(x)) \]

\[ = \lim_{x \to a} f(x) + \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot (x-a) \]

\[ = f(a) + f'(a) \cdot (0) = f(a). \]
Ex. Find the equation of the tangent line to the graph of \( y = f(x) \) at the point \( x = 3 \).

Given \( f(3) = -1, \ f'(3) = 5 \)

\[
f'(3) = 5 = m \quad \text{(the slope)}
\]

\[
y + 1 = 5(x - 3)
\]

Ex. Find \( f'(x) \) and \( a \) if \( f'(a) = \lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h} \)

Now \( \cos \pi = -1 \) so \( f'(a) = \lim_{h \to 0} \frac{\cos(\pi + h) - \cos(\pi)}{h} \)

\[
\Rightarrow a = \pi \quad \text{and} \quad f(x) = \cos x
\]

Ex. Sketch the graph of a function \( f \) for which

\( f(0) = 0, \ f'(0) = 0 \)

\( f'(x) > 0 \) if \( x > 0 \)

\( f'(x) < 0 \) if \( x < 0 \)

Higher Derivatives

\[
y', \quad y'', \quad y''', \quad y^{(4)}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \cdot \frac{dy}{dx}
\]

Ex. Find \( s'' \) if \( x = x^2 - x \). Use limit then find \( f^{(n)} \)