

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2007-2008 (072)

Monday, March 17, 2007

Time: 6:00- 7:45 PM

Name: _____

ID Number: _____

Solution Key

Section Number: _____

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 9 different problems (6 pages + cover page)

Problem No	Grade	Maximum Points
1		33
2		7
3		12
4		6
5		8
6		9
7		8
8		8
9		9
Total		100

1. Evaluate the limit if it exists.

(a) $\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$ $\frac{0}{0}$, undefined. (4 points)

By Factoring, we get

$$\lim_{x \rightarrow 4} \frac{x(x-4)}{(x+1)(x-4)} \quad (2 \text{ pts})$$

$$= \lim_{x \rightarrow 4} \frac{x}{x+1} = \frac{4}{5} \quad (2 \text{ pts})$$

(b) $\lim_{x \rightarrow 1} \frac{\sqrt{x^2+1} - \sqrt{2}}{1-x}$ $\frac{0}{0}$, undefined (6 points)

We multiply by the conjugate:

$$\lim_{x \rightarrow 1} \frac{\sqrt{x^2+1} - \sqrt{2}}{1-x} \cdot \frac{\sqrt{x^2+1} + \sqrt{2}}{\sqrt{x^2+1} + \sqrt{2}} = \lim_{x \rightarrow 1} \frac{x^2-1}{(1-x)(\sqrt{x^2+1} + \sqrt{2})} \quad (2)$$

$$= \lim_{x \rightarrow 1} \frac{-(x+1)}{\sqrt{x^2+1} + \sqrt{2}} \quad (2) = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} \quad (2)$$

(c) $\lim_{x \rightarrow 0^+} \frac{3}{x} \left(\frac{1}{4+x} - \frac{1}{4-x} \right)$ $+\infty \cdot 0$, undefined (6 points)

$$= \lim_{x \rightarrow 0^+} \frac{3}{x} \left(\frac{(4-x) - (4+x)}{(4+x)(4-x)} \right) = \lim_{x \rightarrow 0^+} \frac{3}{x} \left(\frac{-2x}{(4+x)(4-x)} \right)$$

$$= \lim_{x \rightarrow 0^+} \frac{-6}{(4+x)(4-x)} = \frac{-6}{16} = \frac{-3}{8} \quad (2)$$

(4)

(d) $\lim_{x \rightarrow 2^-} ([x-1] - x^2)$, where $[\cdot]$ denotes the greatest integer function. (3 points)

(2) $\lim_{x \rightarrow 2^-} [x-1] = 0$ (as $x \rightarrow 2^-$, then $x < 2$)

(1) So $\lim_{x \rightarrow 2^-} ([x-1] - x^2) = 0 - 4 = -4$

(e) $\lim_{x \rightarrow +\infty} \frac{\cos^2 x}{x^3}$.

(6 points)

Since $x \rightarrow +\infty$, then $x > 0$.

$$\Rightarrow \begin{aligned} 0 &\leq \cos^2 x \leq 1 \\ 0 &\leq \frac{\cos^2 x}{x^3} \leq \frac{1}{x^3} \quad (\text{as } x > 0) \end{aligned} \quad \left. \begin{array}{l} \textcircled{1} \\ \textcircled{1} \end{array} \right\}$$

Since $\lim_{x \rightarrow +\infty} 0 = 0$ & $\lim_{x \rightarrow +\infty} \frac{1}{x^3} = 0$, $\left. \begin{array}{l} \textcircled{1} + \textcircled{1} \end{array} \right\}$

then by the Squeeze Theorem

$$\lim_{x \rightarrow +\infty} \frac{\cos^2 x}{x^3} = 0 \quad \left. \begin{array}{l} \textcircled{1} \\ \textcircled{1} \end{array} \right\}$$

(f) $\lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 2x})$. $-\infty + \infty$, undefined

(8 points)

$$\lim_{x \rightarrow -\infty} x + \sqrt{x^2 + 2x} \cdot \frac{x - \sqrt{x^2 + 2x}}{x - \sqrt{x^2 + 2x}} \quad \left. \begin{array}{l} \textcircled{1} \\ \textcircled{1} \end{array} \right\}$$

$$= \lim_{x \rightarrow -\infty} \frac{-2x}{x - \sqrt{x^2 + 2x}}$$

$$= \lim_{x \rightarrow -\infty} \frac{-2x}{x - |x| \sqrt{1 + \frac{2}{x}}} \quad \textcircled{2}$$

$$= \lim_{x \rightarrow -\infty} \frac{-2x}{x + x \sqrt{1 + \frac{2}{x}}} \quad \textcircled{2} \quad (\text{as } x \rightarrow -\infty, |x| = -x)$$

$$= \lim_{x \rightarrow -\infty} \frac{-2}{1 + \sqrt{1 + \frac{2}{x}}} \quad \textcircled{1}$$

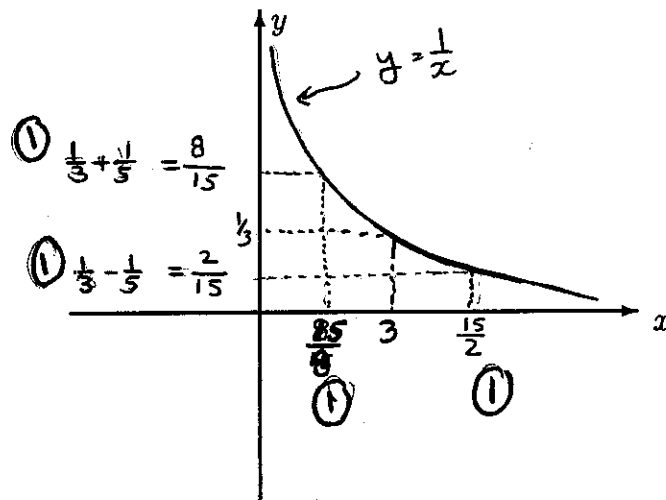
$$= \frac{-2}{1+1} = -1 \quad \textcircled{1}$$

2. Use the graph of $f(x) = \frac{1}{x}$ to find a number δ such that

$$\left| \frac{1}{x} - \frac{1}{3} \right| < \frac{1}{5} \quad \text{whenever} \quad |x - 3| < \delta.$$

(7 points)

$$\begin{aligned} \delta &= \text{minimum} \left(3 - \frac{85}{8}, \frac{15}{2} - 3 \right) \quad (2) \\ &= \text{minimum} \left(\frac{9}{8}, \frac{9}{2} \right) \\ &= \frac{9}{8} \quad (1) \\ &\text{(or any smaller positive number)} \end{aligned}$$



3. Consider the function

$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ e - \ln x & \text{if } x > 1 \end{cases}$$

(a) Is f continuous from the left at 0. Justify.

(6 points)

We check if $\lim_{x \rightarrow 0^-} f(x) = f(0)$.

$$\bullet f(0) = e^0 = 1 \quad (2)$$

$$\bullet \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x+2) = 2 \quad (2)$$

Since $\lim_{x \rightarrow 0^-} f(x) \neq f(0)$, then f is not continuous from the left at 0. (2)

(b) Is f continuous at 1. Justify.

(6 points)

We check if $\lim_{x \rightarrow 1} f(x) = f(1)$

(1)

$$\bullet f(1) = e^1 = e$$

$$\bullet \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} e^x = e$$

$$\bullet \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (e - \ln x) = e - 0 = e \quad \Rightarrow \lim_{x \rightarrow 1} f(x) = e$$

(1)+(1)+(1)

(2) Since $\lim_{x \rightarrow 1} f(x) = f(1)$, then f is continuous at 1.

4. Where is the function $f(x) = \frac{\sin\left(\frac{1}{x}\right)}{e^x - 2}$ continuous. (6 points)

f is continuous in its domain

• $\sin\left(\frac{1}{x}\right)$ is undefined when $x=0$ (2)

• $e^x - 2 = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$ (2)

Thus f is continuous at every point in

$$(-\infty, 0) \cup (0, \ln 2) \cup (\ln 2, +\infty)$$
 (2)

(OR, f is conts when $x \neq 0, \ln 2$)

5. Show that the equation $e^x = -1 - 2x$ has a root in the interval $(-1, 0)$. (8 points)

We apply the Intermediate Value Theorem by letting

(2) $f(x) = e^x + 2x + 1$, $[-1, 0]$, $N=0$.

(2) • f is continuous on $[-1, 0]$

(1) • $f(-1) = e^{-1} - 2 + 1 = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$

(1) $f(0) = e^0 + 0 + 1 = 2 > 0$

So $N=0$ is between $f(-1)$ & $f(0)$

By the Intermediate Value Theorem, there is a number c in $(-1, 0)$ such that

$$f(c) = 0,$$

(2)

that is, $e^c + 2c + 1 = 0$

or $e^c = -1 - 2c.$

Thus the given equation has a root in the interval $(-1, 0)$.

6. Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{4x-1}{x^3-8x^2}. \text{ Explain.}$$

(9 points)

Horizontal Asymptotes

$$\lim_{x \rightarrow +\infty} \frac{4x-1}{x^3-8x^2} = \lim_{x \rightarrow +\infty} \frac{\frac{4}{x^2} - \frac{1}{x^3}}{1 - \frac{8}{x}}$$

$$= \frac{0}{1} = 0$$

①

$$\lim_{x \rightarrow -\infty} \frac{4x-1}{x^3-8x^2} = \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^2} - \frac{1}{x^3}}{1 - \frac{8}{x}}$$

$$= \frac{0}{1} = 0$$

①

Thus f has one H.A.: $y=0$ ①

Vertical Asymptotes

$$f(x) = \frac{4x-1}{x^3-8x^2} = \frac{4x-1}{x^2(x-8)}$$

$$x^2(x-8)=0 \Rightarrow x=0, x=8.$$

$$\text{Since } \lim_{x \rightarrow 0} \frac{4x-1}{x^2(x-8)} = +\infty, \text{ then}$$

②

$x=0$ is a V.A.

①

$$\text{Since } \lim_{x \rightarrow 8^+} \frac{4x-1}{x^2(x-8)} = +\infty$$

②

$$(\text{or } \lim_{x \rightarrow 8^-} \frac{4x-1}{x^2(x-8)} = -\infty),$$

then $x=8$ is a V.A.

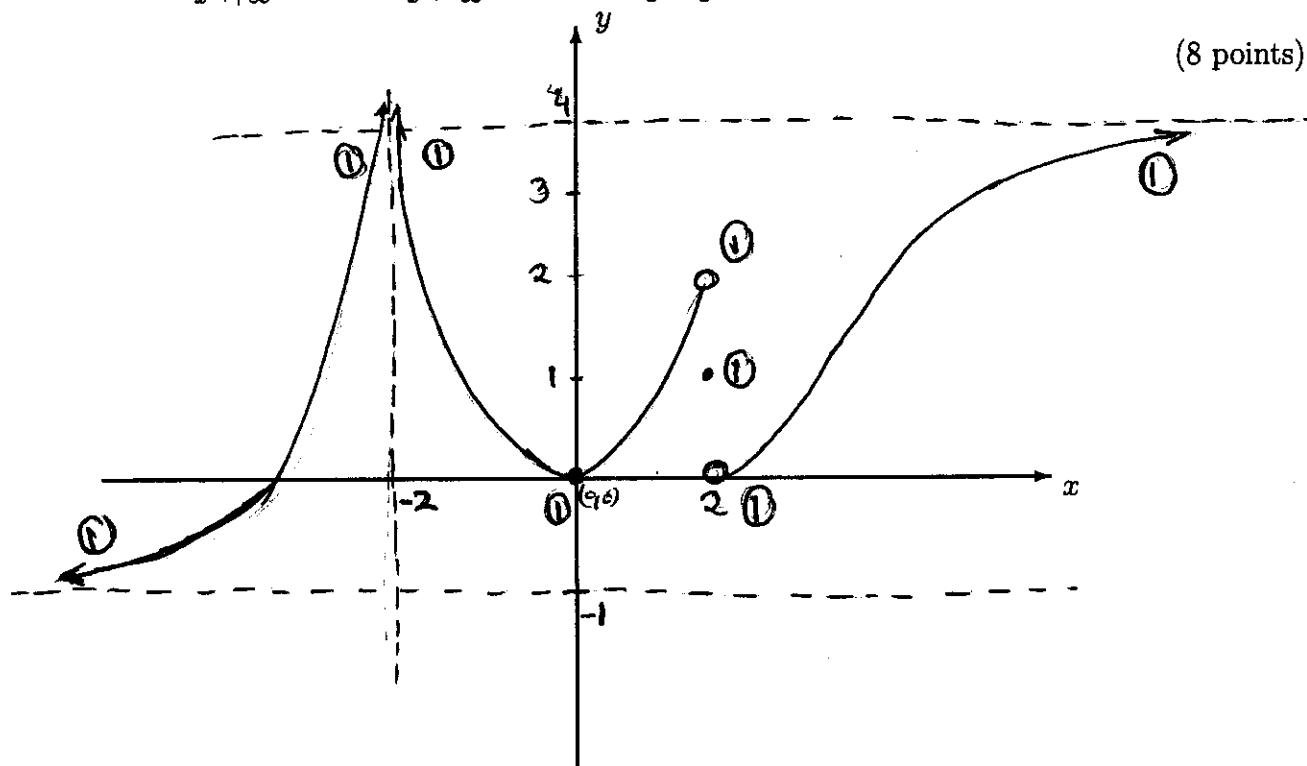
①

Thus f has two V.A.: $x=0$ & $x=8$

7. Sketch the graph of a function f that satisfies all of the given conditions:

$$f(0) = 0, f(2) = 1, \lim_{x \rightarrow 2^+} f(x) = 0, \lim_{x \rightarrow 2^-} f(x) = 2, \\ \lim_{x \rightarrow +\infty} f(x) = 4, \lim_{x \rightarrow -\infty} f(x) = -1, \lim_{x \rightarrow -2} f(x) = +\infty.$$

(8 points)



8. Find an equation of the tangent line to the curve $y = \frac{1}{x-2}$ at the point $\left(4, \frac{1}{2}\right)$.
 [You must use limits] (8 points)

$$\begin{aligned} \text{Slope: } m &= \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} \quad (1) \\ &= \lim_{x \rightarrow 4} \frac{\frac{1}{x-2} - \frac{1}{2}}{x-4} \\ &= \lim_{x \rightarrow 4} \frac{\frac{4-x}{2(x-2)}}{x-4} = \lim_{x \rightarrow 4} \frac{-1}{2(x-2)} \quad (2) \\ &= \frac{-1}{4} \quad (2) \end{aligned}$$

The equation of the tangent line is

$$y - \frac{1}{2} = -\frac{1}{4} \left(x - \frac{1}{2}\right) \quad (2)$$

$$\Rightarrow y = -\frac{1}{4}x + \frac{3}{2} \quad (1)$$

9. The position function of a particle moving in a straight line is given by the equation of motion $s = t^3 - 2t$, where t is measured in seconds and s in meters.

- (a) Find the average velocity of the particle over the time interval $[1, 3]$. (3 points)

$$\begin{aligned} \text{Average velocity} &= \frac{S(3) - S(1)}{3 - 1} \quad (2) \\ &= \frac{21 - (-1)}{2} = 11 \text{ m/s} \quad (1) \end{aligned}$$

- (b) Use limits to find the instantaneous velocity of the particle when $t = 2$. (6 points)

$$\begin{aligned} v(a) &= \lim_{h \rightarrow 0} \frac{S(2+h) - S(2)}{h} \quad (2) \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 2(2+h) - 4}{h} \\ &= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 4 - 2h - 4}{h} \quad (2) \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 6h^2 + 10h}{h} \\ &= \lim_{h \rightarrow 0} h^2 + 6h + 10 \\ &= 10 \text{ m/s} \quad (2) \end{aligned}$$