Each question worth 10 points

King Fuhd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 101-3,5
First Exam , Semester 042
Time:6:00-7:10 pm, Monday, March 14, 2005

80

Name :--ID # :--

Q1. Use the definition of derivative to find $f'(x)$ where $f(x) = \dfrac{1}{x^2+1}$

$$f'(x) = \lim_{h \to 0} \frac{1}{(x+h)^2+1} - \frac{1}{x^2+1} = \frac{\cancel{x^2}\cancel{+1} - (\cancel{x^2}+2xh+h^2\cancel{+1})}{h\,((x+h)^2+1)(x^2+1)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}[(x+h)^2+1][x^2+1]} = \frac{-2x}{(x^2+1)^2}$$

Q2. Find values for the constants k that will make the following function continuous

$$f(x) = \begin{cases} kx^2+2 & x \ge 2 \\ \dfrac{x-2}{\tan 2(x-2)} & x < 2 \end{cases}$$

If $k = 2$ what kind of discontinuity we have?

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} kx^2+2 = 4k+2$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x-2}{\tan 2(x-2)} = \lim_{t \to 0^-} \frac{t}{\tan 2t} = \lim_{t \to 0^-} \frac{2t}{2\sin 2t} \cdot \cos 2t$$

$$= \frac{1}{2}(1)(1) = 1/2 \Rightarrow$$ f(x) continuous if $4k+2 = 1/2$ $\quad k = -3/8$

If $k=2$
then $\lim_{x \to 2^+} f(x) = 10 \ne \frac{1}{2} = \lim_{x \to 2^-} f(x)$ Jump discontinuity.

Q3. If $\sqrt{x+\sqrt{x}}-\sqrt{x} \le f(x) \le \dfrac{x^2-12}{4x+2x^2}$ find the value of $\lim\limits_{x\to\infty} f(x)$

$$\lim_{x\to\infty} \frac{x^2-12}{4x+2x^2} = \lim_{x\to\infty} \frac{\cancel{x^2}}{2\cancel{x^2}} = 1/2$$

$$\lim_{x\to\infty} \sqrt{x+\sqrt{x}}-\sqrt{x} \cdot \frac{\sqrt{x+\sqrt{x}}+\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}} = \lim_{x\to\infty} \frac{\cancel{x}+\sqrt{x}-\cancel{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\cancel{\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{2\sqrt{x}} = 1/2.$$

$$\Rightarrow \lim_{x\to\infty} f(x) = 1/2.$$

Q4. . If $f(x) = \begin{cases} 4\sqrt{x} & x<4 \\ x & x \ge 4 \end{cases}$ then find $f'(4)$

$$\left.\begin{aligned} \lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-} 4\sqrt{x} = 8 \\ \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} x = 4 \end{aligned}\right\} \text{Jump dicontinuity at } x=4$$

Since $f(x)$ not cont. at $x = \cancel{4}$

then $f'(\cancel{4})$ DNE

Q5. Show that there exists some $c \in (1, \ 2)$ such that $\frac{c}{2} = 2 - c^2$.

$c = 4 - 2c^2 \Rightarrow \quad 2c^2 + c - 4 = 0$

Let $f(x) = 2x^2 + x - 4$

$f(1) = -1$ while $f(2) = 6$

Since $f(x)$ is contiuous becouse $f(x)$ is polynomial and $f(1)$ & $f(2)$ have oppsize sign then from IMVT $\exists \ c \in (1, 2)$

s.t $f(c) = 0$

$\Rightarrow \quad 2c^2 + c - 4 = 0$

$\Rightarrow \quad c/2 = 2 - c^2$

Q6. A car moves in the positive direction along straight line so that after t minutes its distance is $s(t) = 3t^2$ feet from the origin. Find the average velocity of the car over the interval $[1,2]$. Then find the instantaneous velocity at $t = 1$.

$$\text{Velocity} = \frac{S(2) - S(1)}{2 - 1} = \frac{3(4) - 3(1)}{1} = 9 \text{ ft/sec}$$

$$\text{Ins. velocity} = \lim_{t \to 1} \frac{S(t) - S(1)}{t - 1} = \lim_{t \to 1} \frac{3(t^2 - 1)}{t - 1}$$

$$= \lim_{t \to 1} \frac{3\cancel{(t-1)}(t+1)}{\cancel{t-1}} = \lim_{t \to 1} 3(t+1) = 3(2) = 6 \text{ ft/sec.}$$

Q7. Find $\lim_{x\to 1} \frac{\sin(\pi x)}{x-1}$.

$t = x-1 \qquad x = t+1$

$= \lim_{t\to 0} \frac{\sin \pi(t+1)}{t}$

$= \lim_{t\to 0} \frac{\sin \pi t \underbrace{\cos \pi}_{=-1} + \cos \pi t \underbrace{\sin \pi}_{=0}}{t}$

$= \lim_{t\to 0} \frac{-\pi \sin \pi t}{\pi t} = -\pi(1)$

$= -\pi$

Q8. $\lim_{x\to 7} \frac{x-7}{49-x^2}$

$= \lim_{x\to 7} \frac{x-7}{(7-x)(7+x)}$ $=-1$

$= \lim_{x\to 7} \frac{-1}{7+x} = \frac{-1}{14}$