Polynomial Solutions of Differential Equations

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Abstract

A new approach for investigating polynomial solutions of differential equations is proposed. It is based on elementary linear algebra. Any differential operator of the form $L(y) = \sum_{k=0}^{k=N} a_k(x)y^{(k)}$, where a_k is a polynomial of degree $\leq k$, over an infinite ground field F has all eigenvalues in F in the space of polynomials of degree at most n, for all n. If these eigenvalues are distinct, then there is a unique monic polynomial of degree n which is an eigenfunction of the operator L- for every non-negative integer n.

Specializing to the real field, the potential of the method is illustrated by recovering Bochner's classification of second order ODEs with polynomial coefficients and polynomial solutions, as well as cases missed by him - namely that of Romanovski polynomials, which are of recent interest in theoretical physics, and some Jacobi type polynomials. An important feature of this approach is the simplicity with which the eigenfunctions and their orthogonality and norms can be determined, resulting in significant reduction in computational complexity of such problems.

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1 Introduction

Polynomial solutions of differential equations is a classical subject, going back to Routh [9], Bochner [2] and Breneke [3] and it continues to be of interest in applications, as in e.g. [5] and [8]. The idea we wish to present in this paper is to conduct the discussion of differential equations with polynomial coefficients in a linear algebraic context. It is surprising that by such a change of view point, one can add more than what is available in the classical literature and, at the same time, recover classical results efficiently and in a unified manner.

We take this opportunity to correct a common misconception regarding Brenke's contributions in the classification of 2nd order ODEs that have polynomial solutions [7, p 508]. He first considers all 2nd order ODEs that have a polynomial solution in every degree and only subsequently classifies self-adjoint equations by an argument similar to that given in Section 3. He then returns to the general 2nd order equation and, for an inexplicable reason, does not carry through the argument to its logical conclusion and misses some important cases.

In this paper we investigate operators of the form $L(y) = \sum_{k=0}^{k=N} a_k(x)y^{(k)}$, where a_k is a polynomial of degree $\leq k$, with coefficients in an infinite ground field F. Clearly, any linear nth order differential operator, which has polynomial coefficients and eigen-polynomials of degrees 0 up to n, must be of this form, and, as shown in Section 2, such operators may not have eigenpolynomials in every degree. We show that these operators, operating on polynomials, have all their eigenvalues in the ground field and in case the eigenvalues are distinct, there is exactly one monic polynomial in every degree which is an eigenfunction of L.

Specializing to second order equations because of their importance in applications - and leaving in this paper the higher order case due to its technical complexity - the canonical forms of second order equations, their eigenvalues and geometric multiplicities are investigated. This includes the family of Romanovski polynomials and some Jacobi type polynomials, which are missing in the classification of Brenke and Bochner as well as in the latest books on the subject; the Romanovski polynomials are the main subject of some recent physics literature [8, 10].

Necessary and sufficient conditions for a second order operator to be self-adjoint are obtained and a reduction formula for computations of norms of eigenfunctions of these operators is also given, which avoids the customary case by case analysis found for example in [1, 6, 7].

A complete classification of second order operators which are self-adjoint with respect to some weight function is also given: amongst all polynomial solutions of differential equations, the classical Legendre, Hermite, Laguerre and Jacobi polynomials make their appearance as soon as one searches for self-adjoint operators. This classification is due originally to Brenke [3].

Although one normally assumes that the leading polynomial coefficient of a differential equation should never vanish, it is worth noting that it is precisely the singularities of the equation that encapsulate all the important information about the equation.

In the final section, the main examples of second order classical and some non-standard operators are given in detail; Part A of this section is largely of pedagogical interest.

2 Basic Results

Throughout, $\mathbb{P}$ is the space of all polynomials over an infinite field F and $\mathbb{P}_n$ is the subspace of polynomials with degree at most n, and for a fixed positive integer N, $L : \mathbb{P} \longrightarrow \mathbb{P}$ is the N-th order operator given by $Ly = \sum_{k=1}^{N} a_k(x)D^k y$, where D is the usual differential operator and where $a_k(x)$ is a polynomial of degree at most k $(1 \le k \le N)$. In this way, for each nonnegative integer n, $\mathbb{P}_n$ is L-invariant. Put $a_k(x) = \sum_{h\ge 0} a_{kh}x^h$, where $a_{kh} = 0$ if $k < h$. As $L(x^n)$ is a scalar multiple of x^n plus lower order terms, we see that the matrix representation of L, w.r.t. the standard basis $B_n = \{1, x, ..., x^n\}$, is upper triangular and the eigenvalues are the coefficients of x^n in $L(x^n)$. In more detail the $(n+1)\times(n+1)$ matrix of L operating on $\mathbb{P}_n$ is

$$A_n = \left[\sum_{k\ge 1}(j-k)_k a_{k,k+i-j}\right]_{1\le i,j\le n+1}$$

where $(j-k)_k = (j-1)(j-2)\cdots(j-k)$ and $a_{kh} = 0$ when $k < h$, so A_n is upper triangular (and where each row and each column has at most $N+1$ nonzero entries). Clearly, A_{n+1} is obtained from A_n by adding one row and one column at the end, and so all the eigenvalues of the operator L are in F and are given by

$$\lambda_0 = 0,\ \lambda_n = na_{11} + n(n-1)a_{22} + \cdots + n!a_{nn} \text{ for } n \ge 1 \tag{2.1}$$

where $a_{nn} = 0$ if $n > N$. Each λ_n has, as an eigenfunction, a polynomial $y_n(x) = y_{n0} + y_{n1}x + \cdots + y_{nn}x^n$ whose vector representation $(y_{n0}, ..., y_{nn})^T$ in the standard basis B_n of $\mathbb{P}_n$ can be directly computed using the homogeneous triangular system

$$(A_n - \lambda_n I)(y_{n0}, ..., y_{nn})^T = 0$$

In particular, if the eigenvalues $\lambda_0, \lambda_1, ..., \lambda_n$ (for some n) are distinct, then $\mathbb{P}_n$ has a basis of eigenfunctions and, for reasons of degree, L has (up to a constant) a unique polynomial of degree r (for each r, $0 \leq r \leq n$) corresponding to λ_r as an eigenfunction. We summarize this in

Proposition 2.1 *Let $L : \mathbb{P} \longrightarrow \mathbb{P}$ be an operator given by $Ly = \sum_{k=1}^{N} a_k(x)D^k y$, where $a_k(x)$ is a polynomial of degree at most k. For each k ($1 \leq k \leq N$), let c_k be the coefficient of x^k in $a_k(x)$. Then all the eigenvalues of L are in the ground field F and are $\mathbb{Z}$-linear combinations of the c_k. If all the eigenvalues are distinct, then L has, up to a constant, a unique polynomial for each degree as an eigenfunction.*

Some observations concerning the eigenvalues and their multiplicity are in order. First, let

$$f(x) = c_1 x + c_2 x(x-1) + \cdots + c_N x(x-1)\cdots(x-N+1)$$

where, as in Proposition 2.1, $c_k = a_{kk}$ is the coefficient of x^k in $a_k(x)$. Then each eigenvalue λ_n of L is just $f(n)$ ($n \geq 0$). This immediately gives an $(N+1)$-term recurrence relation between the eigenvalues, for if E is the shift operator given by $Ef(x) = f(x+1)$, then $(E-1)^{N+1} f(n) = 0$. When all the c_k are zero (i.e. all eigenvalues are equal to zero), then f is identically zero and one can get the eigenfunctions of L by considering the $(N-1)$-st order operator obtained from L by replacing y by Dy. We therefore assume that f is not the zero polynomial. Suppose that an eigenvalue is repeated r times, say

$$\lambda_{n_1} = \lambda_{n_2} = \cdots = \lambda_{n_r}, \text{ where } 0 \leq n_1 < n_2 < \cdots < n_r.$$

In this case, f takes on the same value at r different nonnegative integers, and so $r \leq \deg(f) \leq N$. Moreover, if the field $F = \mathbb{R}$, it is clear that there is a positive integer u such that for each integer $v \geq u$, the set $\{n : f(n) = v\}$ is a singleton. This means that only finitely many eigenvalues λ_n of L have multiplicity greater than 1, and, if any exist, they must all lie between the largest local maximum and the smallest local minimum of f.

An interesting fact occurs when $N = 2$. Suppose, as before, that not both coefficients c_1 and c_2 are zero and that an eigenvalue has algebraic multiplicity 2, say $\lambda_n = \lambda_{n'}$ for some nonnegative integers $n < n'$. Then from equation (2.1), $nc_1 + n(n-1)c_2 = n'c_1 + n'(n'-1)c_2$, i.e. $\lambda_{n+n'} = (n+n')c_1 + (n+n')(n+n'-1)c_2 = 0$. Since the multiplicity of the eigenvalue zero cannot exceed 2, we obtain that for each integer $k > n+n'$, the eigenvalue λ_k has multiplicity 1. We also see that if $n_1 + n_2 = n + n'$, where $n_1 < n_2$, then $\lambda_{n_1} = \lambda_{n_2}$. this means that the number of eigenvalues that have multiplicity 2 is $\left\lceil \frac{n+n'}{2} \right\rceil$. We thus have

Proposition 2.2 *Let the ground field be* $\mathbb{R}$. *Then, with the above notation, either all eigenvalues of the* N*-th order operator* L *are equal to* 0 *or all have multiplicity* 1 *except for finitely many of them which will then have multiplicity at most* N. *In case* $N = 2$, *there will be eigenvalues with multiplicity* 2 *precisely when a nonnegative integer* k *exists for which* $c_1 + kc_2 = 0$, *and then the number of such eigenvalues is* $\left\lceil \frac{k+1}{2} \right\rceil$.

It is perhaps tempting to think that the eigenvalues of the operator L, although not necessarily distinct, are still semisimple. As proposition 2.3 below shows, this is not always the case.

We now concentrate on second order operators. Let $L(y) = a(x)y'' + b(x)y'$, where $\deg(a) = 2$, $\deg(b) \leq 1$. Following Bochner [2], by scaling and translation, we may assume that $a(x) = x^2 - 1, x^2 + 1$ or x^2. We then have the following result.

Proposition 2.3 *(i) The equation* $(x^2 + \epsilon)y'' + (\alpha x + \beta)y' + \lambda y = 0, \quad \epsilon = 0, 1, -1$ *has unique monic polynomial solutions in every degree if* $\alpha > 0$ *or if* $\alpha < 0$ *and* *it is not an integer.*

If $\alpha = -(n+m-1)$ *for* $0 \le m \le (n-1)$, *then the eigenvalue* $\lambda = n(n-1)+\alpha n = -nm$ *has algebraic multiplicity* 2 *and eigenpolynomials can only be of degree* n *or* m. *An eigenpolynomial* $y = \sum_{k=0}^{n} a_k x^k$ *is of degree* n *if and only if*

$$\epsilon a_{m+2}(m+2)(m+1) + \beta a_{m+1}(m+1) = 0$$

in which case the λ *eigenspace in* $\mathbb{P}_n$ *is 2-dimensional; otherwise the* λ *eigenspace is 1-dimensional.*

(ii) The equation $xy'' + (\alpha x + \beta)y' + \lambda y = 0$ *has unique monic polynomial solutions in every degree if* $\alpha \neq 0$.

(iii) The equation $y'' + (\alpha x + \beta)y' + \lambda y = 0$ *has unique monic polynomial solutions in every degree if* $\alpha \neq 0$.

Proof. Let $L(y) = (x^2 + \epsilon)y'' + (\alpha x + \beta)y'$, where $\epsilon = 0, 1, -1$. By Proposition 2.1 or noticing that the eigenvalues are given by the coefficients of x^n in $L(x^n)$, these eigenvalues are $\lambda = n(n-1) + \alpha n$. Suppose this eigenvalue is a repeated eigenvalue. Then $L(x^m) = \lambda x^m +$ lower degree terms, where $m \neq n$. Therefore $n(n-1) + \alpha n = m(m-1) + \alpha m$ gives $\alpha = -(n+m-1)$. This means that if α is not an integer then the operator L has distinct eigenvalues. Similarly if α is a positive integer the operator L has distinct eigenvalues. Therefore in both these cases there is up to a scalar only one polynomial in every degree which is an eigenfunction of L.

Now suppose $\alpha = -(n+m-1)$ for distinct non-negative integers n, m and $\lambda = n(n-1) + \alpha n = -nm$. We may assume that $n > m$. Suppose $L(x^k) = \lambda x^k +$ lower degree terms, with $k \neq n$. Then $\alpha = -(n+m-1) = -(n+k-1)$ gives $k = m$. Therefore if there is a repeated eigenvalue, it is of multiplicity 2 and eigenpolynomials can only be of degrees m and n. Moreover if $\alpha = -(n+m-1) = -(i+j-1)$ then the eigenvalue $-ij$ is also repeated.

Also, if an eigenvalue is not repeated and there is a polynomial of degree, say, n for this eigenvalue and there is a polynomial of degree k which is also an eigenpolynomial, then n must equal k. If there are two linearly independent polynomials of degree n which are eigenpolynomials for the same eigenvalue λ, we may suppose that they are monic. Their difference is then an eigenfunction of lower degree, which implies that λ is a repeated eigenvalue.

Therefore, for a non-repeated eigenvalue, there is exactly one monic polynomial which is an eigenfunction of L.

Let us now determine the geometric multiplicities of all the eigenvalues in case α is a non-positive integer.

Let $\alpha = -(n+m-1)$ where $n > m$ and $n > 0$. As seen above the eigenvalue $\lambda = n(n-1) + \alpha n = -nm$ is of algebraic multiplicity 2 and the corresponding eigenpolynomials can only be in degrees n and m. If $y = \sum_{k=0}^{n} a_k x^k$ then

$$\begin{aligned} L(y) &= \sum_{k=0}^{n-2} \left[a_k k(k-1) + \epsilon a_{k+2}(k+2)(k+1) + \alpha a_k k + \beta a_{k+1}(k+1)\right] x^k \\ &\quad + \left[a_{n-1}(n-1)(n-2) + \alpha a_{n-1}(n-1) + \beta a_n n\right] x^{n-1} + \left[n(n-1) + \alpha n\right] a_n x^n \end{aligned}$$

The solutions of $L(y) = \lambda y = (n(n-1) + \alpha n)y = -(nm)y$ are therefore given by

$$a_k k(k-1) + \epsilon a_{k+2}(k+2)(k+1) + \alpha a_k k + \beta a_{k+1}(k+1) = \lambda a_k, \quad (k = 0, \ldots, n) \tag{2.2}$$

where $a_{n+1} = 0, a_{n+2} = 0$ and $a_n \neq 0$. From equation (2.2) we can solve for a_k in terms of $a_{k+1},\ a_{k+2}$ provided $(k(k-1) + \alpha k - \lambda) \neq 0$. Therefore we can solve for all a_k with $k > m$ in terms of a_n. The equation (2.2) for $k = m$ reads

$$\epsilon a_{m+2}(m+2)(m+1) + \beta a_{m+1}(m+1) = \left[\lambda - m(m-1) - \alpha m\right] a_m = 0. \tag{2.3}$$

If equation (2.3) holds then a_m can be arbitrary and every a_k for $k < m$ is determined in terms of a_m and a_n. In this case the λ eigenspace is 2-dimensional. If equation (2.3) does not hold then there is no eigenpolynomial of degree n. In this case there will be a unique monic polynomial of degree m.

The proofs of statements (ii) and (iii) are straightforward. ■

Proposition 2.3 shows in particular, that also for Jacobi-type differential equations, there are cases where the algebraic and geometric multiplicities are equal to 2 and cases where the algebraic multiplicity is 2 and the geometric multiplicity is 1. (Cf. [2].)

Corollary 2.4 *Let $L(y) = x^2 y'' + (\alpha x + \beta) y'$.*

(i) If α is not a non-positive integer then all the eigenvalues of L are simple.

(ii) If $\alpha = -(n+m-1)$ where $n > m$ then all eigenvalues λ except $\lambda = -nm$, are simple and the eigenvalue $-nm$ has multiplicity 2.
In this case if $\beta = 0$ then all the eigenvalues are semisimple with eigenpolynomials x^k $(k = 0, 1, \dots)$; and if $\beta \neq 0$ then the repeated eigenvalue $-nm$ is defective with eigenpolynomial $\sum\limits_{i=0}^{m} \binom{n}{i}\binom{m}{i} i!(-x/\beta)^i$.
If $\alpha = -(n+m-1) = -(r+s-1)$ where $r > s$ then the eigenvalue $-rs$ is also defective with eigenpolynomial $\sum\limits_{i=0}^{s} \binom{r}{i}\binom{s}{i} i!(-x/\beta)^i$.

Proof. The proof follows by using equations (2.2), (2.3) with $\epsilon = 0$. In this case if there is an eigenpolynomial of degree n then $a_{m+1} = 0 = \cdots = a_n$. Therefore the eigenpolynomial must be of degree m. The form of the eigenpolynomial follows by using equation (2.2). ■

In proposition 2.3 there is no claim to any kind of orthogonality properties. Nevertheless, the non-classical functions appearing here are of great interest in Physics and their properties and applications are investigated in [4, 8, 10].

Now the equations $a(x)y'' + b(x)y' + c(x)y = \lambda x$ can be written as 2nd order Sturm-Liouville equations in the sense of [6, p. 291] by multiplying by a suitable weight function [1, p. 45] and for suitable boundary conditions. A natural question is:
What is the explanation for the weight function and the particular form of the boundary conditions?
The following proposition shows that both the weight and general boundary conditions

are forced upon us as soon as we demand that the operator $L(y) = a(x)y'' + b(x)y' + c(x)y$ should be self-adjoint for some weight function p.

Proposition 2.5 *Let L be the operator defined by $Ly = a(x)y'' + b(x)y' + c(x)y$ on a linear space C of functions which are at least twice differentiable on a finite interval I. Define a bilinear function on C by $(y, u) = \int_I pyudx$, where $p \in C$ is nonnegative and does not vanish identically in any open subinterval of I. Then*

$$(Ly, u) - (y, Lu) = pa(uy' - u'y)|_I \text{ for all } y, u \in C \text{ if and only if } (pa)' = pb$$

Proof. We have

$$(Ly, u) - (y, Lu) = \int_I (pa)\,(uy' - u'y)' + \int_I pb(uy' - u'y)$$

If $(pa)' = pb$, then $(Ly, u) - (y, Lu) = \int_I \big((pa)\,(uy' - u'y)' + (pa)'\,(uy' - u'y)\big) = (pa)\,(uy' - u'y)|_I$.

For the converse, assume that $(Ly, u) - (y, Lu) = (pa)\,(uy' - u'y)|_I$, then, since

$$(pa)\,(uy' - u'y)|_I - \int_I (pa)\,(uy' - u'y)' = \int_I (pa)'\,(uy' - u'y)$$

we get $\int_I (pa)'\,(uy' - u'y) = \int_I pb(uy' - u'y)$, i.e.

$$\int_I \big((pa)' - pb\big)\,(uy' - u'y) = 0$$

Putting $w = (pa)' - pb$, $u = 1$, and choosing y so that $y' = (pa)' - pb$, we get

$$\int_I w^2 = 0.$$

Suppose first that I is an open interval (α, β). Then, from $\lim\limits_{\substack{s \to \alpha^+ \\ t \to \beta^-}} \int_s^t w^2 = 0$, we get for each subinterval $[\sigma, \tau]$ of I, $\int_\sigma^\tau w^2 = 0$. This implies that w is identically zero on $[\sigma, \tau]$, and since this interval is an arbitrary subinterval of I, we have $w = 0$ on I. The case when one or both endpoints of I are in I is similarly dealt with. ∎

Norms of eigenfunctions

The norms of the eigenfunctions relative to the weight p can be obtained by using the well-known three term recurrence relation for orthogonal polynomials. We include a proof for the readers' convenience and because it is the main point in computation of norms of eigenfunctions.

Proposition 2.6 [1, p.306] *If $\{P_n\}_{n=0,1,2...}$ is a sequence of orthogonal polynomials, then in the expression*

$$xP_n = \sum_{j=0}^{j=n+1} k_j P_j,$$

all the coefficients are 0 *except for* $j = n+1, n, n-1$.

Proof. Denoting the inner product by round brackets, we have

$$k_j(P_j, P_j) = (xP_n, P_j) = (P_n, xP_j) = 0,$$

if $j + 1 \leq n - 1$ that is, if $j \leq n - 2$, which is what we wanted to show. ■

Now

$$xP_n = k_{n+1}P_{n+1} + k_n P_n + k_{n-1}P_{n-1} \tag{2.4}$$

Let us rewrite this equation as

$$xP_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}$$

As there is only one monic eigenpolynomial in every degree, the differential equation must determine all the coefficients. We assume that all eigenfunctions are normalized to be monic.

So $P_n = xP_{n-1} + q(x)$, where $deg(q) \leq n-1$. Therefore $(P_n, P_n) = (xP_{n-1}, P_n) = (P_{n-1}, xP_n) = (P_{n-1}, c_n P_{n-1}) = c_n(P_{n-1}, P_{n-1})$ - using equation (2.4) and orthogonality of eigenfunctions of different degrees.

Now, from the differential equation, determining the leading three coefficients of every P_n and using equation (2.4) leads to the determination of c_n, taking into account that $a_n = 1$. This gives $(P_n, P_n) = c_n c_{n-1} \dots c_1 (P_0, P_0)$ and (P_0, P_0) is the integral of the weight function p over an appropriate interval.

The values of $a_n,\ b_n,\ c_n$ for classical orthogonal polynomials are given in the table below.

Polynomial	a_n	b_n	c_n
Legendre	1	0	$\frac{n^2}{(2n+1)(2n-1)}$
Hermite	1	0	$\frac{n}{2}$
Laguerre	1	$2n+1$	n^2
Chebychev	1	0	$\frac{1}{4}$ (for $n \geq 2$)
Jacobi	1	$\frac{-\beta(2+\alpha)}{(2n-2-\alpha)(2n-\alpha)}$ $b_1 = \frac{\beta(2+\alpha)}{\alpha(2-\alpha)}$	$\frac{n(n-\alpha-2)(2n-(\beta+\alpha+2))(2n+(\beta-\alpha-2))}{(2n-\alpha-3)(2n-2-\alpha)^2(2n-\alpha-1)}$ (for $n \geq 2$) $c_1 = \frac{(\alpha-\beta)(\alpha+\beta)}{(1-\alpha)\alpha^2}$: here $\alpha < \beta < -\alpha$

3 Canonical Forms of Self-adjoint Second Order Equations with Polynomial Coefficients

Let us now determine the operators for which there is a basis of orthogonal eigen-polynomials for the weight function determined by the operator. The results of this section were arrived at independently; however the authors found later that such a classification was done first by Brenke [3].

From proposition 2.5, the operator L would be self adjoint if there is no contribution from the boundary terms: this is ensured if the product $a(x)p(x)$ vanishes at the end points of the interval$-$finite or infinite$-$on which the natural weight function $p(x)$ associated to L is integrable on the entire interval.

The integrability of the weight function determines the differential equation and finiteness of the norm of polynomials ensures that manipulations as in proposition 2.5 are legitimate. The operator L will then be self-adjoint and it will operate on the vector space of all polynomials of degree at most n for every non-negative integer n.

As L has a basis of eigenvectors in any finite dimensional subspace on which it operates, we see that there will be monic polynomial of degree n, which will be an eigenfunction of L, and the corresponding eigenvalues would therefore be determined from the form of the equation. If these eigenvalues are distinct for different degrees, these polynomials would automatically be orthogonal.

We now determine the operators L from the requirements that

(1) the leading term $a(x)$ is non-zero and of degree at most 2, the degree of $b(x)$ is at most 1 and $c(x)$ is a constant

(2) the natural weight function associated to L is integrable on the interval I determined by roots of $a(x)$

(3) $a(x)p(x)$ vanishes at the end points of I and, in case there is an end point at infinity, the product $a(x)p(x)P(x)$ should vanish at infinity for all polynomials $P(x)$

(4) all polynomials should have finite norm on the interval I with the weight $p(x)$.

Case I: The polynomial $a(x)$ has two distinct real roots.

By a linear change of variables and scaling we may assume that the roots are 1 and -1. Assuming that $a(x)$ is non-negative in the interval $[-1,1]$, we have $a(x) = 1 - x^2$. Let $b(x) = \alpha x + \beta$ so

$$\frac{b(x)}{a(x)} = \frac{\alpha x + \beta}{(1-x)(1+x)} = \frac{\frac{\beta+\alpha}{2}}{1-x} + \frac{\frac{\beta-\alpha}{2}}{1+x}.$$

So the weight $p(x)$ is

$$p(x) = \frac{1}{1-x^2}\, e^{\int (\frac{\frac{\beta+\alpha}{2}}{1-x} + \frac{\frac{\beta-\alpha}{2}}{1+x})dx} = \frac{(1+x)^{\frac{\beta-\alpha-2}{2}}}{(1-x)^{\frac{\beta+\alpha+2}{2}}}$$

The weight is obviously finite in the interval $(-1,1)$. For $p(x)$ to be integrable we must have $\beta - \alpha > 0$ and $\beta + \alpha < 0$. Thus $\alpha < \beta < -\alpha$, so $\alpha < 0$.

Case II: The polynomial $a(x)$ has repeated real roots. In this case we can assume that $a(x) = x^2$. Let $b(x) = \alpha x + \beta$. The weight function is now $p(x) = \frac{1}{x^2} e^{\int \frac{\alpha x + \beta}{x^2} dx} = \frac{|x|^\alpha}{x^2} e^{-\beta/x}$. We may take the interval $I = (0, \infty)$.

In this case a necessary condition for the integrability of the weight is that $\deg(b) - \deg(a) + 1 > 0$, so this case does not arise.

Case III: The polynomial $a(x)$ is linear.

In this case we can take $a(x) = x$. Let $b(x) = \alpha x+\beta$. In this case the weight function is $p(x) = \dfrac{1}{|x|}e^{\int \frac{\alpha x+\beta}{x}dx} = |x|^{\beta-1}e^{\alpha x}$. This is integrable near zero if and only if $\beta \geq 1$. Since $\displaystyle\int_0^\infty e^{\alpha x}x^\epsilon dx$, where $\epsilon > 0$, is finite only if $\alpha \leq 0$ we see that we cannot take the interval I from $-\infty$ to ∞. Without loss of generality we can take this to be the interval $[0, \infty)$. So the weight function is now $p(x) = x^{\beta-1}e^{\alpha x}$ with $\alpha < 0$ and $\beta \geq 1$. All polynomials have finite norm with respect to this weight and for all polynomials $p(x)$ the product $P(x)p(x)$ vanishes at 0 and ∞. Therefore the equation $xy'' + (\alpha x + \beta)y' + \lambda y = 0$ has polynomial solutions for every degree n. The corresponding eigenvalue is $\lambda = -\alpha n$.

Case IV: $\alpha(x) = 1$

In this case $L(y) = y'' + (\alpha x + \beta)y' + \gamma y$. The weight is $p(x) = e^{\frac{\alpha x^2}{2}}e^{\beta x}$. So α must be negative, for the product $P(x)p(x)$ to vanish at the end points of the interval I for all polynomials $P(x)$, and therefore I must be $(-\infty, \infty)$.

Remark The case of a second degree $a(x)$ with no real roots does not arise, because of the requirements (3) and (4) above which a weight function must satisfy.

4 Examples

Examples of classical orthogonal polynomials and some non-classical polynomials with finite orthogonality properties in the sense of [8, 10] are discussed below using ideas of the previous sections.

A) Classical orthogonal polynomials

1. **Legendre polynomials:**

   Consider the eigenvalue problem

$$(1 - x^2)y'' - 2xy' = \lambda y.$$

   Let L be the operator defined by $L(y) = (1 - x^2)y'' - 2xy'$. The weight function here is $p(x) = \left(\dfrac{1}{|1 - x^2|}\right) e^{\int \frac{-2x}{1-x^2}dx} = 1$. So in the interval $[-1, 1]$, $p(x)a(x)$ is $1 - x^2$.

For every non-negative integer n the operator L maps the vector space $\mathbb{P}_n$ of polynomials of degree at most n into itself. As $p(x)a(x)$ vanishes at the boundary points, there are no boundary conditions required to make L self-adjoint. So L must have a basis of eigenvectors in $\mathbb{P}_n$.

As L maps every $\mathbb{P}_m$ to itself for all $m \leq n$ there must be a polynomial of every degree n which is an eigenfunction of L. We normalize this polynomial so that the coefficient of its leading term is 1. The corresponding eigenvalue is therefore given by the coefficient of x^n in the expression

$$(1-x^2)(n(n-1)x^{n-2}x\cdots) - 2x(nx^{n-1}+\cdots).$$

So, it is equal to $-n(n-1) - 2n = -n(n+1)$.

These eigenvalues are all distinct. Therefore, for every non- negative integer n, there is a unique polynomial of degree n which is an eigenfunction of L. Because the corresponding eigenvalues are distinct, these polynomials are orthogonal.

We have therefore recovered the differential equation for the Legendre polynomial: namely $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

2. **Laguerre polynomials**:

   Here the eigenvalue problem is

   $$xy'' + (1-x)y' = \lambda y.$$

   The operator L is given by $L(y) = xy'' + (1-x)y'$. The weight function is $p(x) = \left(\frac{1}{|x|}\right) e^{\int \frac{1-x}{x} dx} = e^{-x}$. Here the inner product is defined for all functions on $[0, \infty)$ for the weight $p(x) = e^{-x}$. As $\int_0^\infty x^n e^{-x} dx = n!$, all polynomials have finite norm. As $xp(x)$ vanishes at $x = 0$, the operator L will be self adjoint on the space of all functions with finite norm which are such that $xe^{-x}(uy' - u'y)$ vanishes at ∞ for any two functions u, y in this space. As in the first example, for every non-negative integer n, there must be a polynomial of degree n which is an eigenfunction of L. The corresponding eigenvalue is given by the coefficient of x^n in $x\left(n(n-1)x^{n-2}+\cdots\right)+(1-x)(nx^{n-1}+\cdots)$. So, it is equal to $-n$ The differential equation for Laguerre polynomial is therefore $xy''+(1-x)y'+ny=0$.

3. **Hermite polynomials**:

   The Hermite differential equation is $y''-2xy'+\lambda y=0$. The weight function is $p(x)=e^{\int -2x\,dx}=e^{-x^2}$. Thus all polynomials have finite norm relative to this weight. By the same consideration as in the previous examples there must be a unique monic polynomial of every degree n which is an eigenfunction of L with eigenvalue $-2n$.

4. **Confluent hypergeometric equation:**

   This is the equation $xy''+(c-x)y'-\lambda y=0$. The operator here is $L(y)=xy''+(c-x)y'$. The weight function is therefore $p(x)=|x|^{c-1}e^{-x}$. L maps polynomials to polynomials and all polynomials have finite norm relative to $p(x)$ if $c>1$. As above L maps the space $\mathbb{P}_n$ to itself, so by similar considerations as in the previous examples, there must be a polynomial of every degree n which is an eigenfunction of L. The corresponding eigenvalue is $-n$. These polynomials are therefore solutions of the differential equation $xy''+(c-x)y'+ny=0$.

5. **Chebyschev polynomials:**

   These are eigenfunctions of the equations

$$L(y)=(1-x^2)y''-xy' \text{ and } L(y)=(1-x^2)x''-3xy'.$$

   We will discuss only the first case, as the other is similar. The weight function is $p(x)=\dfrac{1}{\sqrt{1-x^2}}$. The singularities at $x=1,-1$ are not essential singularities. The reason is that for any continuous function f, the integral $\displaystyle\int_{-1}^{1}\frac{f(x)dx}{\sqrt{1-x^2}}$ is finite, as one sees by the substitution $x=\cos(\theta)$.

Also, the product of the leading term and the weight is $\sqrt{1-x^2}$, so the operator L is self-adjoint on the interval $[-1,1]$ as the contribution from the boundary terms vanishes. Thus, exactly as for the case of Legendre polynomials, there is, up to a constant, exactly one polynomial of degree n which is an eigenfunction of the operator L. The corresponding eigenvalue is $-n^2$ and these polynomials are solutions of the equation $(1-x^2)y''-xy'+n^2y=0$.

6. Jacobi polynomials:

First note that for any differentiable function f with f' continuous, the integral $\int_0^\epsilon \frac{f(x)}{x^\alpha}\,dx$ is finite if $\alpha < 1$–as one sees by using integration by parts.

Consider the equation $(1-x^2)y'' + (\alpha x+\beta)y' + \lambda y = 0$. As above, the weight function $p(x)$ for the operator

$$L(y) = (1-x^2)y'' + (\alpha x+\beta)y'$$

is

$$p(x) = \frac{1}{1-x^2}\, e^{\left(\frac{\frac{\beta+\alpha}{2}}{1-x}+\frac{\frac{\beta-\alpha}{2}}{1+x}\right)dx} = \frac{(1+x)^{\frac{\beta-\alpha-2}{2}}}{(1-x)^{\frac{\beta+\alpha+2}{2}}} = \frac{1}{(1-x)^{\frac{\beta+\alpha+2}{2}}(1+x)^{\frac{-\beta+\alpha+2}{2}}}$$

So $\int_{-1}^1 p(x)f(x)dx$ would be finite if $\beta+\alpha<0$ and $-\beta+\alpha<0$, that is, if $\alpha<\beta<-\alpha$.

The weight is not differentiable at the end points of the interval. So, first consider L operating on twice differentiable functions on the interval $[-1+\epsilon, 1-\epsilon]$. If u, v are functions in this class then by Proposition 2.5,

$$\int_{-1+\epsilon}^{1-\epsilon} p(x)L(u(x))v(x)dx - \int_{-1+\epsilon}^{1-\epsilon} p(x)u(x)L(v(x))dx$$
$$= p(x)a(x)(u(x)v'(x)-u'(x)v(x))|_{-1+\epsilon}^{1-\epsilon}$$

Moreover, $(1-x^2)p(x) = (1-x)^{\frac{-(\beta+\alpha)}{2}}(1+x)^{\frac{\beta-\alpha}{2}}$ is continuous on the interval $[-1,1]$ and vanishes at the end-points -1 and 1. Therefore, if we define

$(u,v) = \lim_{\epsilon \to 0} \int_{-1+\epsilon}^{1-\epsilon} p(x)u(x)v(x)dx$ then L would be a self-adjoint operator on all polynomials of degree n and so, there must be, up to a scalar, a unique polynomial which is an eigenfunction of L for eigenvalue $-n(n-1)+n\alpha$.

So these polynomials satisfy the equation

$$(1-x^2)y'' + (\alpha x + \beta)y' + (n(n-1) - n\alpha)y = 0$$

and this equation has unique monic polynomial eigenfunctions of every degree, which are all orthogonal.

Although the Legendre and Chebyschev polynomials are special cases, corresponding to the values $\alpha = -1, 2, -3$ and $\beta = 0$, we have included them because of their specific importance.

B. Some non-standard examples:

1. **The equation** $t(1-t)y'' + (1-t)y + \lambda y = 0$.

   This equation is investigated in [5] and the eigenvalues determined experimentally, by machine computations. Here, we will determine the eigenvalues in the framework provided by Propositions 2.1 and 2.5.

   Let $L(y) = t(1-t)y'' + (1-t)y'$. Let $\mathbb{P}_n$ be the space of al polynomials of degree at most n. As L maps $\mathbb{P}_n$ into itself, the eigenvalues of L are given by the coefficient of x^n in $L(x^n)$. The eigenvalues turn out to be $-n^2$. As these eigenvalues are distinct, there is, up to a constant, a unique polynomial of degree n which is an eigenfunction of L.

   The weight function is $p(t) = \dfrac{1}{|1-t|} = \dfrac{1}{1-t}$ on the interval $[0,1]$ and it is not integrable. However, as $L(y)(1) = 0$, the operator maps the space V of all polynomials that are multiples of $(1-t)$ into itself. Moreover, $\int_0^1 p(t)((1-t)\psi(t))^2 dt$ is finite.

   The requirement for L to be self-adjoint on V is $t(\xi\eta' - \xi'\eta)|_0^1 = 0$ for all ξ, η in V. As ξ, η vanish at 1, the operator L is indeed self- adjoint on V.

Let $V_n = (1-t)\mathbb{P}_n$, where $\mathbb{P}_n$ is the space of all polynomials of degree at most n. As the codimension of V_n in V_{n+1} is 1, the operator L must have an eigenvector in V_n for all the degrees from 1 to $n+1$. If $y = (1-t)\psi$ is an eigenfunction and $\deg(\psi) = n$, then, by the argument as in the examples above, we see that the corresponding eigenvalue is $\lambda = -(n+1)^2$.

Therefore, up to a scalar, there is a unique eigenfunction of degree $n+1$ which is a multiple of $1-t$ and all these functions are orthogonal for the weight $p(t) = \dfrac{1}{1-t}$. Using the uniqueness up to scalars of these functions, the eigenfunctions are determined by the differential equation and can be computed explicitly.

2. Romanovski Polynomials:

These polynomials are investigated in [10, 8] and their finite orthogonality is also proved there. Here, we establish this in the framework of Proposition 2.5.

The Romanovski polynomials are eigenfunctions of the operator $L(y) = (1+x^2)y'' + (\alpha x + \beta)y'$. For $\alpha > 0$, or $\alpha < 0$, α not an integer, there is only one monic polynomial in every degree which is an eigenfunction of L; for α a non-positive integer, the eigenspaces can be 2 dimensional for certain values of β (Proposition 2.3).

The formal weight function is $p(x) = (x^2+1)^{\left(\frac{\alpha-2}{2}\right)} e^{\beta \tan^{-1}(x)} = (x^2+1)^{\frac{\gamma}{2}} e^{\beta \tan^{-1}(x)}$, where $\gamma = \alpha - 2$. Therefore, a polynomial of degree N is integrable over the reals with weight p if and only if $N + \gamma + 1 < 0$; and if the product of two polynomials P, Q is integrable, then the polynomials are themselves integrable for the weight p.

Arguing as in the proof of Proposition 2.5, we find that $(LP, Q) - (P, LQ) = (x^2+1)p(x)(PQ' - P'Q)|_{-\infty}^{\infty}$, because if $\deg(P) \neq \deg(Q)$ then the product $(x^2+1)p(x)(PQ' - P'Q)$ is asymptotic to $x^{2+\gamma+\deg(P)+\deg(Q)-1} = x^{\deg(P)+\deg(Q)+\gamma+1}$

and $\deg(P) + \deg(Q) + \gamma + 1 < 0$.

Therefore, if P, Q are integrable eigenfunctions of L with different eigenvalues and $\deg(P) + \deg(Q) + \gamma + 1) < 0$, then P, Q are orthogonal.

For several non-trivial applications to problems in Physics, the reader is referred to [8].

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