3 Differentiation Rules

3.1 The Derivative of Polynomial and Exponential Functions

In this section we learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Recall first the definition of the derivative:

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},}$$

provided the limit exists.

Alternative Formula for the Derivative

$$\boxed{f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}.}$$

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

To drive this rule, just apply the definition as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

To drive the power rule, we need to use the binomial theorem which is:

$$
\begin{aligned}
(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
&= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \\
&= a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots + n a b^{n-1} + b^n
\end{aligned}
$$

Show it here:

Example 3.1. Find $\frac{df}{dx}$ of the following:

a) $f(x) = \pi^2$

b) $f(x) = \sqrt[3]{x^5}$

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

The Constant Multiple Rule, the Sum Rule, and the Difference Rule can be combined with the Power Rule to differentiate any polynomial.

Definition of the Number e

$$e \text{ is the number such that } \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Example 3.2. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

The Normal Line

The normal line to a curve C at a point P is the line through P that is perpendicular to the tangent line at P. i. e. if the slope of the tangent line is m then the slope of the normal line is $-\frac{1}{m}$.

Example 3.3. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

Exercise

1) The number of points at which the curve $y = x^4 - 8x^2 + 3$ has horizontal tangents is

   a) 4
   b) 2
   c) 3
   d) 0
   e) 1

2) An equation for the tangent line to the curve $y = \sqrt{x}$ that passes through the point $(-4, 0)$ is

   a) $3x - 4y = -12$
   b) $x + y = 2$
   c) $x + 4y = -4$
   d) $2x - y = -8$
   e) $x - 4y = -4$

3) The **number** of points at which the curve $y = x^3 - 3x^2 + 4$ has tangent lines parallel to the line $3x + y = 2$ is

   a) Two
   b) Three
   c) Four
   d) One
   e) Zero

4) If the normal line to the curve $x^2 - xy + y^2 = 1$ at $(1, 1)$ **intersects** the curve at another point (a, b), then $a + b =$

   a) 2
   b) -1
   c) 0
   d) 3
   e) -2

3.2 The Product and Quotient Rules

3.2.1 The Product Rule

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Example 3.4. Let $f(x) = xe^x$.

a) Find f'.

b) Find the nth derivative, $f^{(n)}(x)$

3.2.2 The Quotient Rule

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Table of Differentiation Formulas

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x$
$(cf)' = cf'$	$(f+g)' = f' + g'$	$(f-g)' = f' - g'$
$(fg)' = fg' + gf'$	$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$	

Example 3.5. Suppose $f(2) = -1$, $g(2) = 2$, $f'(2) = 3$, and $g'(2) = -4$. Find

a) $(fg)'(2)$

b) $\left(\frac{f}{g}\right)'(2)$

Example 3.6. Find $\frac{d}{dx}(x^{-4}e^x)$

Example 3.7. Find the equation of the normal line to the curve $y = \frac{x+\sqrt{x}}{x-\sqrt[3]{x}}$ at $(1, 2)$.

Exercise

1) If the line $y = 2x + 3$ is perpendicular to the tangent lines to the curve $y = \dfrac{x}{x-2}$ at the points (a, b) and (c, d), then $a + b + c + d =$

   a) 6
   b) 0
   c) 4
   d) 2
   e) 1

2) If the line $y = 2x + 8$ is a tangent line to the curve $y = \dfrac{c}{x+2}$, then $c^3 - 3c + 4 =$

   a) 5
   b) -3
   c) 1
   d) 0
   e) 2

3) The slope the tangent line to the curve with equation $y = \dfrac{x+3}{1-x}$ at $x = -2$ is

   a) $\dfrac{5}{9}$
   b) undefined
   c) $\dfrac{4}{9}$
   d) $\dfrac{1}{2}$
   e) 4

4) Let f and g be differentiable functions up to the third order and $h(x) = f(x)g(x)$, then $h^{(3)}(x) =$

   a) $f'''g + 3f''g' + 3f'g'' + fg'''$
   b) $f'''g' + 2f''g'' + f'g'''$
   c) $f'''g + 2f''g'' + fg'''$
   d) $f'''g'''$
   e) $f'''g + fg'''$

3.3 Derivatives of Trigonometric Functions

First, we introduce these formulas, which will be used to drive the derivative of the trigonometric functions:

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1} \qquad \boxed{\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0}$$

Example 3.8. Find

a) $\displaystyle\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right)$

b) $\displaystyle\lim_{x\to 0}\left(\frac{2\sin 3x+4\tan 5x}{5\tan 2xx}\right)$

Derivative of $\sin x$

$$(\sin x)' = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right] = \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right]$$

$$= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \lim_{h\to 0}\sin x\cdot\cancel{\lim_{h\to 0}\frac{\cos h-1}{h}}+\lim_{h\to 0}\cos x\cdot\boxed{\lim_{h\to 0}\frac{\sin h}{h}} = (\sin x)\cdot 0+(\cos x)\cdot 1$$

$$= \cos x$$

So, the first formula:

$$\boxed{\frac{d}{dx}(\sin x)=\cos x}$$

With similar steps, one can drive that

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivative of $\tan x$

Rewriting $\tan x$ as $\frac{\sin x}{\cos x}$ and applying the quotient rule we get

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule.

The differentiation formulas for trigonometric functions are presented all in the following table:

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Note 11. The above formulas are valid only when x is measured in radians.

Example 3.9. Find the slope of the tangent line to the graph of the curve $y = \dfrac{1+\sin x}{x+\cos x}$ at $\frac{\pi}{3}$

Example 3.10. If $f(x) = e^x \cos x$, find $f''(\pi)$

Exercise

1) If $f(x) = \sin(x) + \cos(x)$, then $f^{(20)}(0) + f^{(21)}(0) =$

   a) -2
   b) 2
   c) 1
   d) -1
   e) 0

2) If $y = \dfrac{1}{\sec x + \tan x}$ then $\dfrac{dy}{dx}|_{x=0} =$

   a) 1
   b) 1/2
   c) 0
   d) -1/2
   e) -1

3) The equation of the tangent line to the curve $y = 2\tan\left(\dfrac{\pi\, x}{4}\right)$ at $x = 1$ is

   a) $y = x + \dfrac{\pi}{4}$
   b) $y = \pi x + 2 - \pi$
   c) $y = -\pi x + 2 + \pi$
   d) $y = \dfrac{\pi}{4}x + 2 - \dfrac{\pi}{4}$
   e) $y = 3\pi x + 2 - 3\pi$

4) If $y = \dfrac{\cos x}{e^x}$ then $y'' + 2y' + 3y =$

   a) $\dfrac{\cos x}{e^x}$
   b) $\dfrac{-\sin x}{e^x}$
   c) $\dfrac{\sin x + \cos x}{e^x}$
   d) $\dfrac{\sin x}{e^{2x}}$
   e) $\dfrac{\sin x - \cos x}{e^{2x}}$

3.4 The Chain Rule

Suppose you are asked to differentiate the function $\cos(e^x)$ which is of the form $f(g(x))$ where $f(x) = \cos x$ and $g(x) = e^x$. The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate composite functions.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g. It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the Chain Rule.

Definition 3.1. If g is differentiable at x and f is differentiable at $g(x)$, the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and can be written either in the prime notation

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

or (Leibniz notation), if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Example 3.11. Find the derivative of the following:

a) $\cos(e^{7x})$

b) $(x^3 - 2x + 1)^{\frac{3}{2}}$

c) $\sec^5(4x)$

d) $\sqrt{x + \sqrt{1 + 3^x}}$

Note 12. We have the following general formulas:

a) $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\frac{d}{dx}[f(x)]$ "the power rule"

b) $\frac{d}{dx}\left[a^{f(x)}\right] = \left[a^{f(x)}\right]\frac{d}{dx}[f(x)]\ln a$ "the exponential function"

Exercise

(1) If $f(x) = \sin(\sin^2 x)$, then $f'(x) =$

a) $2\sin x\, \cos x\, \cdot \sin(\sin^2 x)$

b) $2\cos x \cdot \cos(\sin^2 x)$

c) $2\sin^2(x) \cdot \cos(\sin^2 x)$

d) $\sin(2x) \cdot \sin(\cos^2 x)$

e) $\sin(2x) \cdot \cos(\sin^2 x)$

(2) If $z = \left(\dfrac{u-1}{u+1}\right)^2$ and $u = \dfrac{1}{x^2} - 1$, then $\dfrac{dz}{dx}|_{x=-1}$ is

a) 4

b) -8

c) $\frac{1}{2}$

d) 1

e) -2

(3) If $y = \cos(2\ln x)$, then $x^2y'' + xy' =$

a) $2y$

b) $5y$

c) 0

d) $-4y$

e) $\frac{y}{x}$

(4) If f is a differentiable function of x and $g(x) = e^x f(e^{-x})$, then $g'(x) =$

a) $g(x) + f'(e^{-x})$

b) $e^x f'(e^{-x})$

c) $e^x(f'(e^{-x}) + f(e^{-x}))$

d) $f'(e^{-x})$

e) $g(x) - f'(e^{-x})$

(5) If $y = \sin 2x - \cos 2x,$ then $y^{(4)}(0) =$

a) -16

b) -8

c) 32

d) 24

e) 0

(4) If $y = \cot^3(x^2),$ then $y'\left(\dfrac{\sqrt{\pi}}{2}\right) =$

a) $-2\sqrt{3\pi}$

b) $3\sqrt{\pi}$

c) $\sqrt{2\pi}$

d) $\sqrt{3\pi}$

e) $-6\sqrt{\pi}$

3.5 Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable. For example, $y = e^{x^2+1}$ or $y = x \sin x$ or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between x and y such as

- $x^2 + y^2 = 25$ (it is possible to solve such equation for y as an explicit function of x.) or
- $x^3 + y^3 = 6xy$ (It's not easy to solve for y explicitly as a function of x by hand).

By using implicit differentiation, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can differentiate both sides of the equation with respect to x and then solving the resulting equation for y'.

In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Implicit differentiation steps:

1. differentiate both sides of the equation with respect to x treating y as a function of x.
2. collect the terms with $\frac{dy}{dx}$ (or y') on one side of the equation.
3. solve for $\frac{dy}{dx}$ (or y').

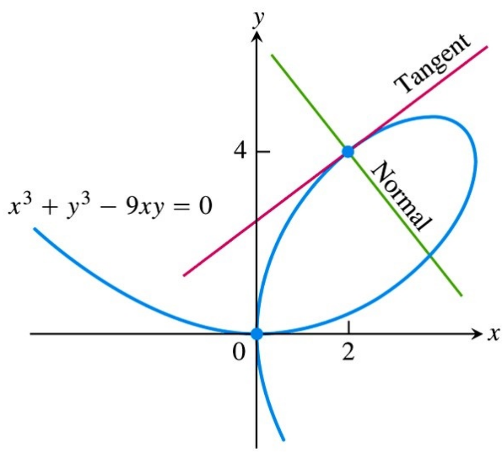
Example 3.12. Find $\frac{dy}{dx}$ (or y') of the following:

a) $x^2 + y^2 = 25$

b) $x^3 + y^3 = 6xy$

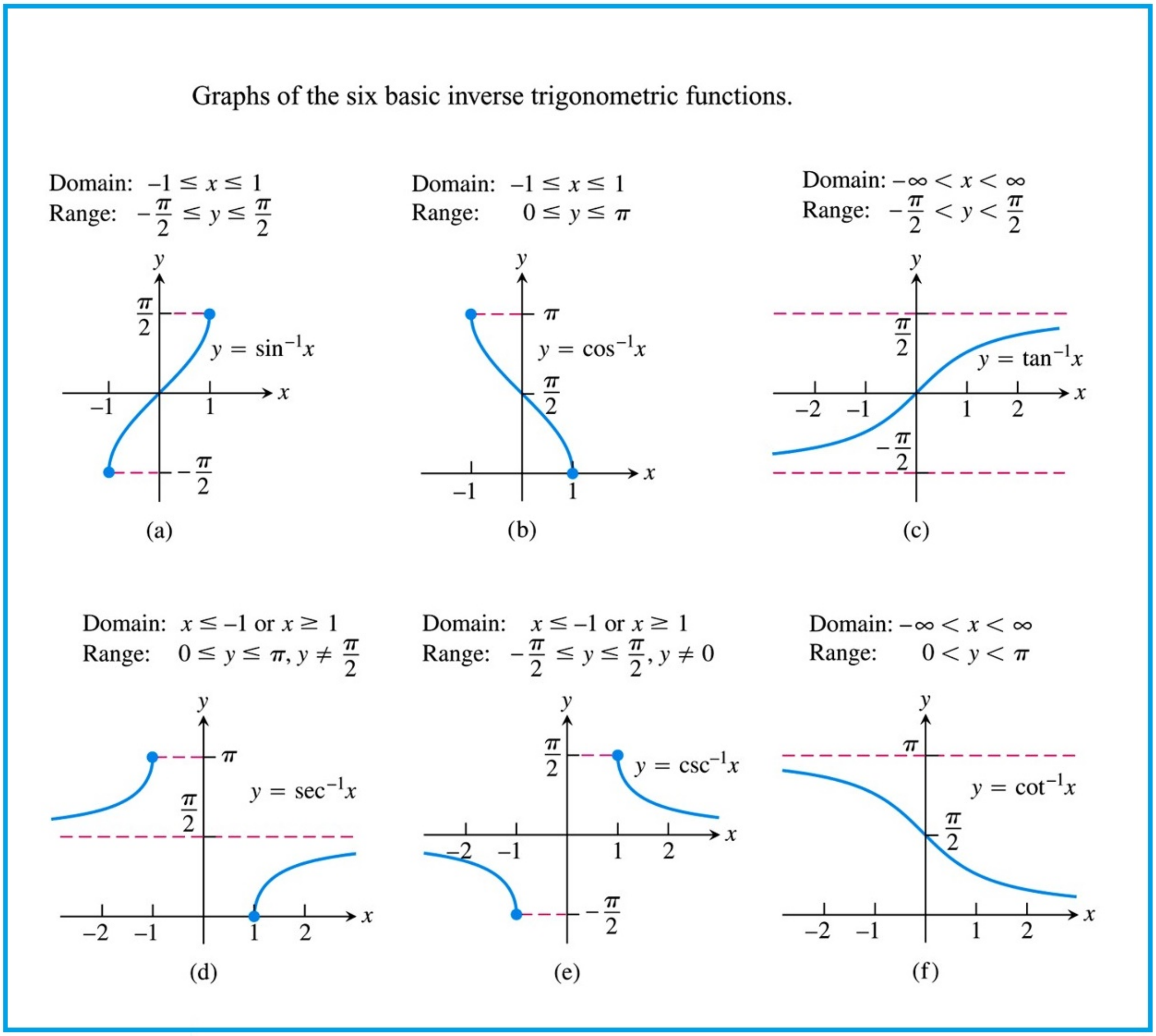
c) $e^{\frac{y}{x}} = \sin xy + 1$

Example 3.13. If $x^3 + y^3 - 9xy = 0$, then find the equation of the tangent line and normal line at $(2, 4)$.



Derivatives of Inverse Trigonometric Functions

Before we start the derivation of the derivatives of the inverse of trigonometric functions, we need to review the following (which is covered in Math 002):



We know that the definition of the arcsine function:

$y = \arcsin x$ means $\sin y = x$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$, then differentiating implicitly with respect to x, we obtain

$$(\cos y).\frac{dy}{dx} = 1$$

solving for $\frac{dy}{dx}$ we get,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

So,

$$\frac{dy}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}.$$

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \arctan x$ means $\tan y = x$ Differentiating implicitly with respect to x, we have

$$(\sec^2 y).\frac{dy}{dx} = 1$$

solving for $\frac{dy}{dx}$ we get,

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

So,

$$\frac{dy}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}.$$

The derivatives of the remaining four are given in the following table

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Example 3.14. Find $\frac{dy}{dx}$ (or y') of the following:

a) $y = \sin^{-1}((3/4)x)$

b) $y = \tan^{-1}\sqrt{3\cot(2x)}$

Formula in exercise no.77

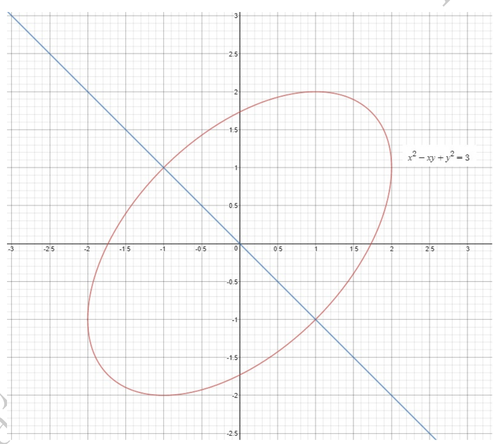
77. (a) Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}} \quad **$$

provided that the denominator is not 0.

(b) If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.

Example 3.15. The equation $x^2 - 2xy + y^2 = 3$ represents a "rotated ellipse," that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.



Exercise

(1) If $\cot\left(\dfrac{x}{y}\right) = x + y^2$, then using implicit differentiation we get $y' =$

a) $\dfrac{x\csc^2\left(\frac{x}{y}\right) + y^2}{y\csc^2\left(\frac{x}{y}\right) - x^2}$

b) $\dfrac{\csc^2\left(\frac{x}{y}\right) + y}{\csc^2\left(\frac{x}{y}\right) - 2y}$

c) $\dfrac{y - \csc^2\left(\frac{x}{y}\right)}{x + \csc^2\left(\frac{x}{y}\right)}$

d) $\dfrac{xy - \csc\left(\frac{x}{y}\right)}{y^2 + \csc\left(\frac{x}{y}\right)}$

e) $\dfrac{y\csc^2\left(\frac{x}{y}\right) + y^2}{x\csc^2\left(\frac{x}{y}\right) - 2y^3}$

(2) If $f(x) = (\ln x)^{\tan x}$, then

a) $f'(x) = \tan x(\ln x)^{\tan x} - (\sec^2 x)\ln x$

b) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{x\ln x} + (\sec^2 x)\ln x$

c) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{x\ln x} + (\sec^2 x)\ln(\ln x)(\ln x)^{\tan x}$

d) $f'(x) = \tan x(\ln x)^{\tan x} - (\sec^2 x)\ln(\ln x)$

e) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{\ln x} + (\sec^2 x)\ln(\ln x)(\ln x)^{\tan x}$

(3) Let $f(x) = 1 + 2x - x^2$, $x \leq 1$. Then $\dfrac{df^{-1}}{dx}|_{x=-2} =$

a) $\dfrac{1}{4}$

b) $\dfrac{1}{6}$

c) $\dfrac{1}{3}$

d) $-\dfrac{1}{4}$

e) -1

(4) The slope of the tangent line to the graph of the curve $x^2 + y^2 = \left(2x^2 + 2y^2 - x\right)^2$ at the point $(0, \frac{2}{2})$ is

a) 1

b) $\frac{1}{6}$

c) $\frac{1}{2}$

d) $\frac{1}{4}$

e) 2

(5) If $y = \tan^{-1}\left(\frac{1-\cot x}{1+\cot x}\right)$, then $y' =$

a) 2

b) $\csc^2 x$

c) 1

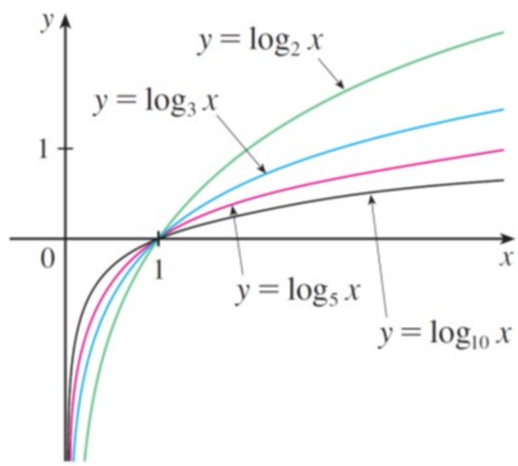
d) $\frac{2}{1+\csc^2 x}$

e) 0

3.6 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivatives of the logarithmic functions $y = \log_b x$ and, in particular, the natural logarithmic function $y = \ln x$.

It can be proved that logarithmic functions are differentiable; this is clear from their graphs. See the Figure below.



Formulas

1. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$
2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$. In general, if we combine this formula with the Chain Rule, we get
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$.

Example 3.16. Find $\frac{dy}{dx}$ (or y') of the following:

a) $y = \ln\left(\sqrt{\sin x}\right)$

b) $y = \log_3 \frac{x+1}{\sqrt{x-2}}$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

Steps of Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the laws of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve for y'.

Example 3.17. Find $\frac{dy}{dx}$ (or y') of the following:

a) $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

b) $y = \frac{(\sin 3x)^{5/2}(\cot 2x)^{4/3}}{\sqrt{5x^3+2x+4}}$

c) $y = x^{\sqrt{x}}$

d) $y = (2x+1)^{\tan x}$

The Number e as a Limit

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

and we have also this form,

$$e = \lim_{x\to\infty}(1+1/x)^x$$

Exercise

(1) If $y = \sqrt[3]{\dfrac{x(x+1)(x+2)}{(x+3)(x+4)(x+5)}}$, then $y'|_{x=1} =$

a) $\dfrac{73}{60\ \sqrt[3]{20}}$

b) $\dfrac{61}{170\ \sqrt[3]{20}}$

c) $\dfrac{52}{61\ \sqrt[3]{23}}$

d) $\dfrac{43}{3\ \sqrt[3]{30}}$

e) $\dfrac{73}{180\ \sqrt[3]{20}}$

(2) If $f(x) = (\ln x)^{\tan x}$, then

a) $f'(x) = \tan x(\ln x)^{\tan x} - (\sec^2 x)\ln x$

b) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{x\ln x} + (\sec^2 x)\ln x$

c) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{x\ln x} + (\sec^2 x)\ln(\ln x)(\ln x)^{\tan x}$

d) $f'(x) = \tan x(\ln x)^{\tan x} - (\sec^2 x)\ln(\ln x)$

e) $f'(x) = \dfrac{\tan x(\ln x)^{\tan x}}{\ln x} + (\sec^2 x)\ln(\ln x)(\ln x)^{\tan x}$

(3) If $y = (\sin x)^{\sqrt[3]{x}}$, then $\dfrac{y'}{y} =$

a) $\dfrac{x\tan x - \ln(\sin x)}{\sqrt[3]{x^2}}$

b) $\dfrac{3x\cot x + \ln(\sin x)}{3\sqrt[3]{x^2}}$

c) $\dfrac{x\cot x + 3\ln(\sin x)}{3\sqrt[3]{x}}$

d) $\sqrt[3]{x}(\sin x)^{\sqrt[3]{x}-1} \cdot \cos x$

e) $\sqrt[3]{x}\tan x + \ln(\sin x)$

3.7 Rates of Change in the Natural and Social Sciences

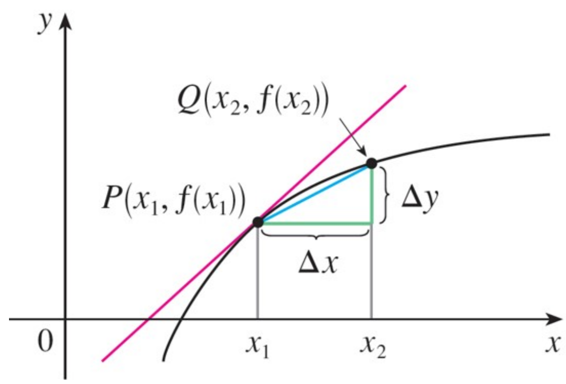
We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

We know that the basic idea behind rates of change. If x changes from x_1 to x_2, then the change in x is $\triangle x = x_2 - x_1$ and the corresponding change in y is $\triangle y = f(x_2) - f(x_1)$.

The difference quotient

$$\frac{\triangle y}{\triangle x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change** of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in the below figure.

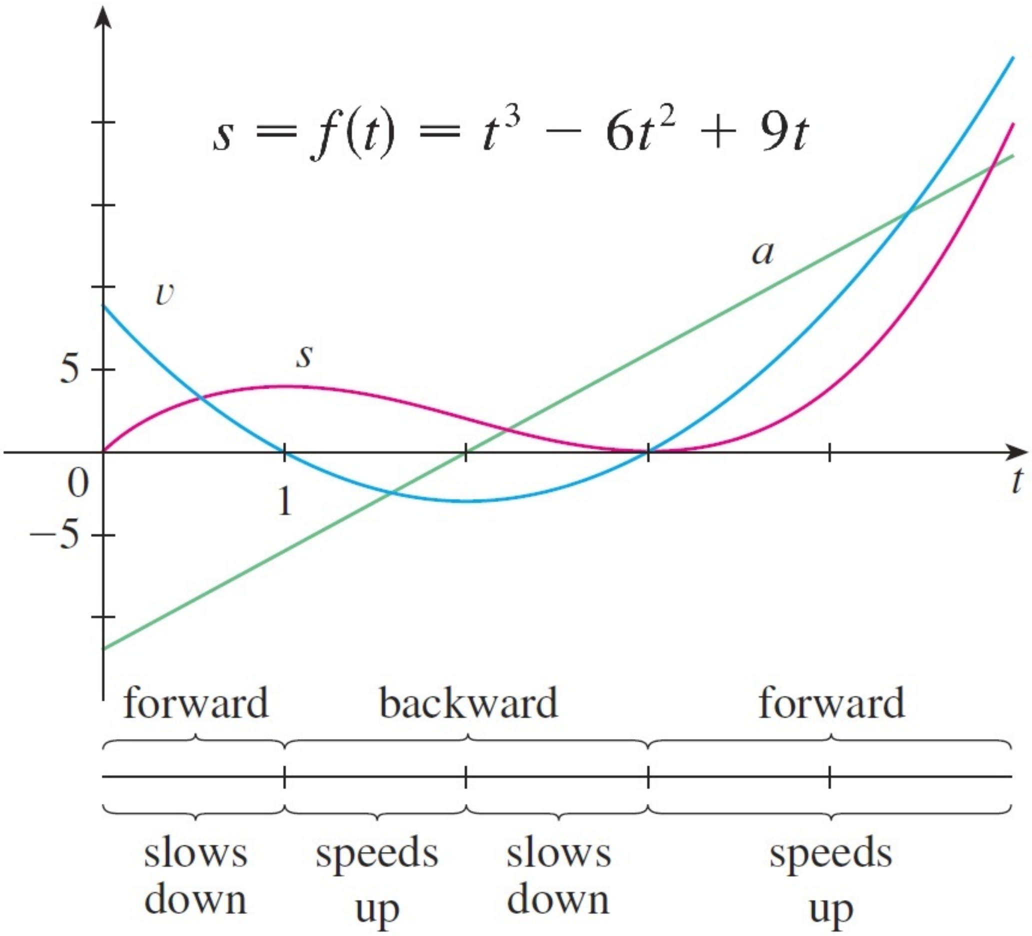


Its limit as $\triangle x \to 0$ is the derivative $f(x_1)$, which can therefore be interpreted as **the instantaneous rate of change** of y with respect to x or the slope of the tangent line at $P(x_1, f(x_1))$.

Using Leibniz notation, we write the process in the form

$$\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$$

In Physics, if $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\frac{\triangle s}{\triangle t}$ represents the average velocity over a time period $\triangle t$, and $v = \frac{ds}{dt}$ represents the instantaneous velocity (the rate of change of displacement with respect to time). The instantaneous rate of change of velocity with respect to time is acceleration: $a(t) = v'(t) = s''(t)$.



Note 13. The particle speeds up (slows down) when both the velocity and the acceleration have the same sign (opposite sign).

Example 3.18. The position of a particle is given by the equation

$$s(t) = (-t^2 + 1 - 1)\, e^{-t}$$

where t is measured in seconds and s in meters.

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (that is, in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first five seconds.

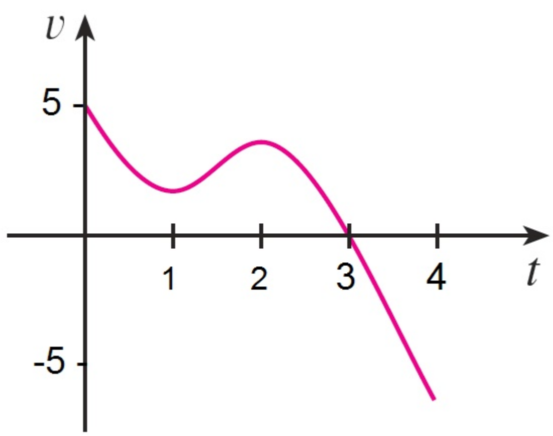
(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

Exercise

(1) Graphs of the velocity function of a particle is shown in the figure,where t is measured in seconds. The particle is slowing down on



a) $(0, 1) \cup (2, 3)$
b) $(0, 1) \cup (3, 4)$
c) $(1, 2) \cup (3, 4)$
d) $(2, 3)$
e) $(0, 1)$

(2) The position function of a particle moving along a line is

$$s(t) = \cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right)$$

where t is measured in seconds and s in meters. The **total distance** traveled by the particle in the interval $[0, 1]$ is

a) $2\sqrt{2} - 2$ meters

b) $2\sqrt{2}+2$ meters

c) 4 meters

d) $2\sqrt{2}$ meters

e) 2 meters

(3) A body is moving along a straight line with position function $s(t) = -t^3 + 3t^2 - 1,\ t \geq 0$. The **total distance** traveled by the body from $t = 1$ to $t = 3$ is equal to (s in meters, t in seconds)

a) 8 m

b) 14 m

c) 10 m

d) 12 m

e) 6 m

(4) The position function of a particle moving along a straight line is $s(t) = 2t - t^2$ for t in $[0, 5]$, where t is measured in seconds and s in meters. The particle is speeding up when

a) $0 < t < 1$

b) $1 < t < 5$

c) $0 < t < 5$

d) $0 < t < 2$

e) $0 < t < 3$

3.8 Exponential Growth and Decay

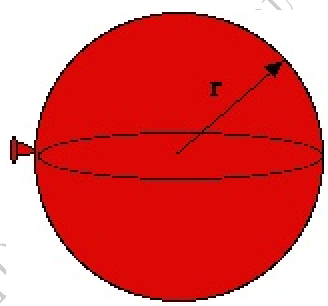
3.9 Related Rates

Suppose that two variables x and y are functions of another variable t, say $x = f(t)$ and $y = g(t)$. Then, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the rates of change of x and y with respect to t and they are called "related rates".

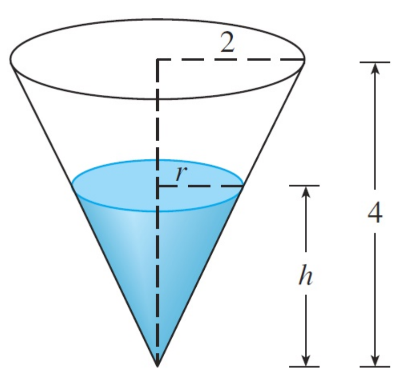
Related Rates Problem Strategy

1) Draw a picture of the physical situation.
2) Write an equation that relates the quantities of interest.
3) Take the derivative with respect to t of both sides of your equation.
4) Solve for the quantity you are looking for.

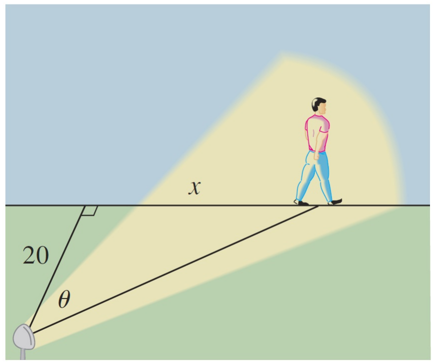
Example 3.19. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm? (volume of a sphere: $V = \frac{4}{3}\pi r^3$)



Example 3.20. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep. (volume of a circular cone $V = \frac{1}{3}\pi r^2 h$)



Example 3.21. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Exercise

(1) A street light is mounted at the top of a 5-meter-tall pole. A man $2m$ tall walks away from the pole with a speed of $\frac{3}{2}\, m/s$ along a straight path. How fast is **his shadow** moving when he is $10\, m$ from the pole?

a) 5 m/s
b) 4 m/s
c) 1 m/s
d) 3 m/s
e) 2 m/s

(2) A ladder $15\, ft$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall horizontally at a rate of $4\, ft/sec$. How fast is the ladder sliding down the wall when the top of the ladder is $12\, ft$ from the ground?

a) $-\frac{3}{2}\, ft/sec$
b) $3\, ft/sec$
c) $\dfrac{3}{2}\, ft/sec$
d) $-3\, ft/sec$
e) $-6\, ft/sec$

(3) One side of a rectangle is increasing at a rate of 3 cm/sec and the other side is decreasing at a rate of 4 cm/sec. How fast is the area of the rectangle changing when the increasing side is 12 cm long and the decreasing side is 10 cm long?

a) $-18 \quad cm^2/sec$
b) $18 \quad cm^2/sec$

c) $78 \quad cm^2/sec$
d) $-12 \quad cm^2/sec$
e) $12 \quad cm^2/sec$

(4) The coordinates of a particle in the xy-plane are differentiable functions of time t with $\frac{dx}{dt} = -1 \ m/sec$ and $\frac{dy}{dt} = -5 \ m/sec$. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$

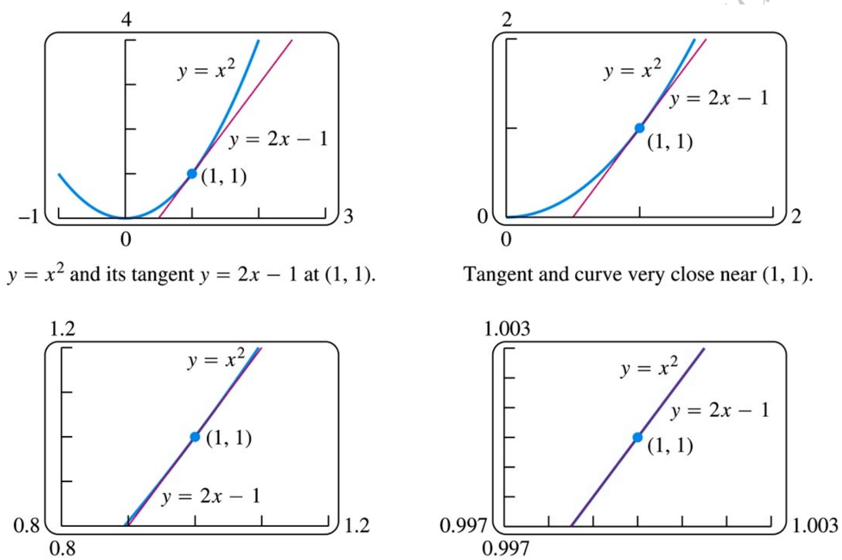
a) $\frac{-3}{7}$
b) 4
c) -5
d) $\frac{2}{13}$
e) $\frac{-5}{13}$

3.10 Linear Approximations and Differentials

3.10.1 Linear Approximations

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See the below figure)



$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).

Tangent and curve very close throughout entire x-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this x-interval.

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

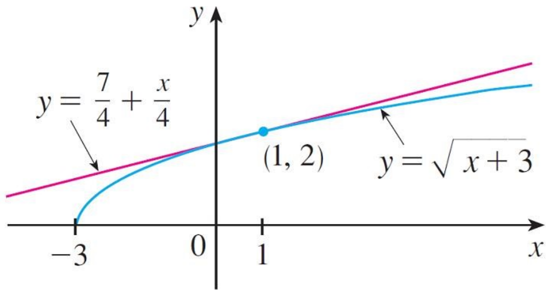
is called the linear approximation or tangent line approximation of f at a.

The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 3.22. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3,98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?



	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176...
$\sqrt{3.98}$	0.98	1.995	1.99499373...
$\sqrt{4}$	1	2	2.00000000...
$\sqrt{4.05}$	1.05	2.0125	2.01246117...
$\sqrt{4.1}$	1.1	2.025	2.02484567...
$\sqrt{5}$	2	2.25	2.23606797...
$\sqrt{6}$	3	2.5	2.44948974...

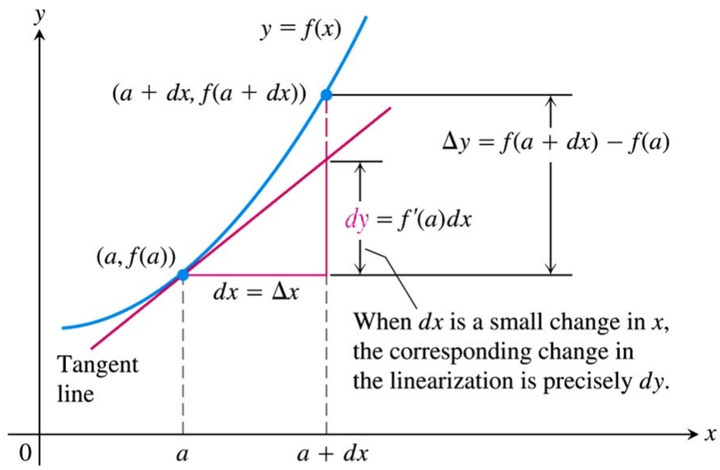
Example 3.23. Find the linearization of the function $f(x) = \cos x$ at $x = \frac{\pi}{2}$ and use it to approximate the numbers $\cos 91°$. **Answer**: $\cos x \approx \pi/2 - x$ and $\cos 91° \approx -\pi/180$.

3.10.2 Differentials

If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number.The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)\,dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined. The geometric meaning of differentials is shown in the figure below.



The corresponding change in y is

$$\triangle y = f(x + \triangle x) - f(x)$$

The slope of the tangent line is the derivative $f'(x)$. Thus the directed distance $dy = f'\,dx$.

Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas $\triangle y$ represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Summary of above terminologies

If x is the independent variable and y is the dependent variable where $y = f(x)$, then

i) $\triangle x = dx$ is the **exact change** in x. (Or the exact error in x)

ii) $\triangle y = f(x + \triangle x) - f(x)$ is the **exact change** in y. (Or the exact error in y)

iii) $dy = f'(x)\,dx$ is the **approximated change** in y. (Or the approximated error in y)i.e. $\triangle y \approx dy$.

iv) In general, $f(a + \triangle x) \approx f(a)+$ (the differential of f as x changes from a to x). Or simply, $f(a + \triangle x) \approx f(a) + dy$.

Example 3.24. Find the differential of $f(x) = x^3 \tan(2x)$.

Example 3.25. Use the differential to estimate $(2.001)^5$.

Example 3.26. (Using of differentials in estimating the errors)
The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere? (The sphere volume is $V = \frac{4}{3}\pi r^3$.)

Note 14. Although the possible error may appear to be rather large, a better measure for the error is given by the relative error, which is computed by dividing the error by the total volume (the relative error in the volume $= \frac{\Delta V}{V} \approx \frac{dV}{V}$). The errors could also be expressed as percentage errors (the percentage error $\approx \frac{dV}{V} \times 100$).

Example 3.27. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

Exercise

(1) Approximating $\tan^{-1}(1.01)$ using a linearization of $f(x) = \tan^{-1}(x)$ at a suitably chosen integer near 1.01 is equal to

a) 0.005

b) $\frac{\pi}{4} + 0.01$

c) $\frac{\pi}{4} + 0.005$

d) $\frac{\pi}{2} + 0.005$

e) $\frac{\pi}{2} + 0.01$

(2) If $L(x)$ is the linearization of $f(x) = 1 + \ln(1 - 2x)$ near $a = 0$, then $L(-1) =$

a) 3

b) -3

c) -2

d) 2

e) 0

(3) The radius of a sphere is measured to be 3 cm with a maximum error in measurement of 0.1 cm. Using differentials, the maximum error in calculating volume of the sphere is

a) $\frac{9\pi}{10}$ cm^3

b) $\frac{\pi}{10}$ cm^3

c) $\frac{36\pi}{10}$ cm^3

d) 9π cm^3

e) 36π cm^3

(4) The radius of a sphere was measured to be $20\,cm$ with a possible error in measurement of at most $0.05\,cm$. The maximum error in the computed volume of the sphere is approximately equal to

a) $10\,\pi\ cm^3$

b) $20\,\pi\ cm^3$

c) $60\,\pi\ cm^3$

d) $40\,\pi\ cm^3$

e) $80\,\pi\ cm^3$

(5) The radius of a circular disk is measured with possible percentage error 2.5%. If we use differentials, then the estimated percentage error in the calculated circumference of the disk is

a) 2.5
b) 5
c) 7.5
d) $5\,\pi$
e) 1.5

(6) A surveyor, standing $50ft$ from the base of a building, measures the angle of elevation to the top of the building to be $45°$. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 3%?

a) 1.5
b) 2
c) 3
d) 2.5
e) 1

(7) If the radius r of a circle is measured with a possible percentage error of $\pm 2\%$, then the estimated percentage error in calculating the area of the circle is

a) $\frac{4}{r}$
b) $2\,\pi$
c) 2
d) 4
e) $\frac{\pi}{r}$

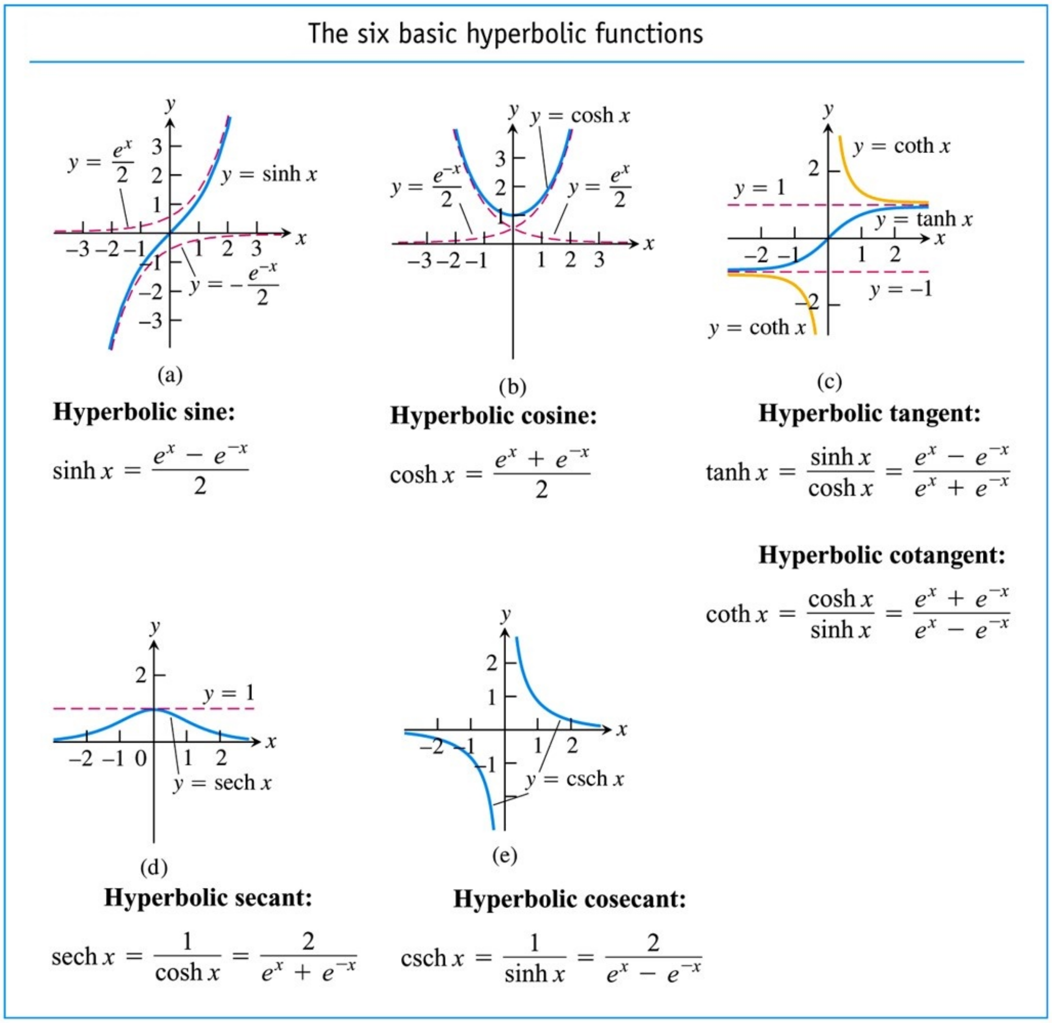
(8) The height h of a right circular cone is 30 cm whereas the radius r of the cone is 10 cm. There is no error in measurement of the height, but the radius of the cone is known to be accurate to within 0.1 cm. Using differentials, the maximum possible error in computing the volume of the cone is

$$\text{Hint: Volume of a cone} = \frac{1}{3}\pi r^2 h$$

a) $10\,\pi \quad cm^3$
b) $15\,\pi \quad cm^3$
c) $30\,\pi \quad cm^3$
d) $20\,\pi \quad cm^3$
e) $\pi \quad cm^3$

3.11 Hyperbolic Functions

Combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics, so they deserve to be given special names. For this reason, hyperbolic functions are introduced, and individually called hyperbolic sine, hyperbolic cosine, and so on.



Identities for Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Example 3.28. Find the value of the following:

a) $\sinh(3\ln 2)$

b) $\cosh(-3\ln 2)$

c) $\tanh(\ln 2)$

d) $\lim_{x\to-\infty}(\tanh x)$

Example 3.29. If $\tanh x = \frac{3}{5}$, find the values of the hyperbolic functions?

Example 3.30. Find the derivative of the following:

a) $y = (x\ \coth(1+x^2))$

b) $y = \frac{1-\cosh 3x}{1+\cosh 3x}$

Exercise

1) If $\tanh x = \dfrac{12}{13}$, then $5\ \sinh x + 13\ \text{sech}\ x\ =$

   a) 17
   b) 18
   c) 60
   d) 25
   e) 22

2) $\tanh(\ln\ x) =$

   a) $\dfrac{x^2-1}{x^2+1}$
   b) $\dfrac{x^2+1}{x^2-1}$
   c) $\dfrac{\ln\ x^2-1}{\ln\ x^2+1}$
   d) $\dfrac{x-1}{x+1}$
   e) 0

3) a)
   b)
   c)
   d)
   e)

4) a)
   b)
   c)
   d)
   e)