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RECOVERY OF MEDIUM PROPERTIES IN THREE-DIMENSIONS WITH THREE-DIMENSIONAL PARAMETERS VARIABILITY

K. MASOOD [1] and F. D. ZAMAN [2]

[1] **Hafr Al-Batin Community College**
King Fahd University of Petroleum and Minerals
P.O. Box 5087, Dhahran 31261, Saudi Arabia
e-mail: masood@kfupm.edu.sa

[2] **Department of Mathematical Sciences**
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia
e-mail: fzaman@kfupm.edu.sa

We consider a problem of identification of physical properties of the Earth using the damped wave equation, based on the linearized inversion associated with Born's inversion theory. We assume that damping and sound speed are well approximated by the background plus the perturbation. Application of the method leads to a linear integral equation involving variations in sound speed and damping. Our aim is to recover these variations in velocity and damping, what in turn yields a map of the interfaces in the interior of the Earth. We consider the three-dimensional inverse problem of determining three-dimensional variations in the propagation speed and damping by considering the damped wave equation. We exploit the high-frequency character of seismic data to simplify the problem.

1. Introduction

The inverse problems are important in seismic exploration and underground geology. The frequently used Earth model for such a purpose is that of a homogeneous and isotropic medium. This may not be an accurate description of many practical situations. We consider a model that incorporates the effects of damping in the medium and develop an inversion procedure for this case. The objective of seismic inversion is to estimate the Earth parameters, such as velocity and damping from seismic data. In a seismic experiment, a source is set off at a point on the surface of Earth, and the upward propagating wave is then measured at an array of receivers near the source. The image of a subsurface so constructed is important, either for theoretical purposes or for identification by a geologist of possible subsurface regions during resource extraction. These inverse problems form a class of problems in which unknown coefficients of wave equation represent internal parameters of a medium and the known information consists of boundary measurements.

An inverse problem can be simplified if the scattered field is approximated as a linear functional of the object. The linearization used in derivation of the inversion procedure is often referred to as the Born approximation (Morse and Feshbach [1]). The frequently used Earth model for such a purpose is that of a homogeneous and isotropic medium. While this provides a good comparison model, it may not be an accurate description of many practical situations. We consider a model that incorporates the effects of damping in the medium and develop an inversion procedure in this case, see Zaman and Masood [2]. The damped medium is assumed to contain an inhomogeneity that causes variation in the material parameters with the depth. It is hoped that the results based upon this model will prove to be more realistic in some situations of interest. The objective here is to study the problem of mapping the interior of the Earth as an inverse problem and to develop methods which yield increasingly more accurate results of that inverse problem.

The methods we use are classical, employing the perturbation techniques, integral transformation methods and asymptotic analysis to get information about the interior of the Earth. We assume that the perturbation in wave speed and damping have parallel forms and it is this perturbation we seek to recover. One or more signals are introduced on the surface of the Earth in a region of interest and responses from irregularities in the interior of the Earth are recorded. Since the signals are introduced on the surface of the Earth, like explosive or vibrator, so only the wave propagation in the Earth is of interest. Under the assumption of constant density, an approximate solution to such inverse problems for the velocity without damping was demonstrated by Bleistein *et al.* [3, 4] and Cohen *et al.* [5].

In this article we introduce a damping term in the three-dimensional wave equation (Stakgold [6]) and study its effect on inversion. The damping may be caused due to impurities in the medium, the presence of fluid-saturated rocks in the medium, distributed boundary frictions or small viscous effects. In the second Section, the problem is formulated in three dimensions and Born approximation is applied to simplify the problem. The constant background zero-offset geometry is considered in the third Section. In the fourth Section the recovery of one-dimensional variation in velocity is described. The recovery of damping and an iterative procedure to improve the recovered velocity and damping profiles is described in the fifth Section. Finally, in the last Section the results are summarized.

2. Formulation of the Problem

We introduce a three-dimensional coordinate system, $\mathbf{x} = (x_1, x_2, x_3)$, with x_3 positive in the downward direction. The propagation speed and damping is assumed to be known in some portion of the region $x_3 \geq 0$ and unknown outside that portion of the region. We consider a band-limited impulsive point source at $\mathbf{x}_s$ and response to this source will be observed by one or more geophones $\mathbf{x}_g$. The objective is to obtain information about the propagation speed, $v(\mathbf{x})$, and damping of the medium, $\gamma(\mathbf{x})$, from observations of the wavefield. The displacement $u(\mathbf{x}, \mathbf{x}_s, \omega)$ is governed by the three-dimensional Helmholtz equation

$$\pounds u(\mathbf{x}, \mathbf{x}_s, \omega) = \left[\nabla^2 + \frac{\omega^2 + i\omega\gamma(\mathbf{x})}{v^2(\mathbf{x})}\right] u(\mathbf{x}, \mathbf{x}_s, \omega) = -F(\omega)\,\delta(\mathbf{x} - \mathbf{x}_s)\,, \tag{2.1}$$

where $F(\omega)$ is some frequency domain filter. The total field $u(x, x_s, \omega)$ satisfies the standard Sommerfeld radiation conditions:

$$ru \text{ is bounded}, \qquad r\left(\frac{\partial u}{\partial r} - \frac{i\sqrt{\omega^2 + i\omega\gamma}}{v}u\right) \to 0 \qquad \text{as} \qquad r \to \infty, \qquad r = |\mathbf{x}|\,. \tag{2.2}$$

This radiation condition is assumed to ensure that the solutions to Eq. (2.1) will be unique in unbounded media. Physically, these conditions ensure that there are no waves propagating inwards from infinity.

We introduce variations in damping and propagation speed which have parallel form, that is the background plus the perturbation, and therefore these profiles have the following representations:

$$\gamma(\mathbf{x}) = \gamma_0(\mathbf{x}) + \epsilon\gamma_1(\mathbf{x})\,, \tag{2.3}$$

$$v(\mathbf{x}) = v_0(\mathbf{x}) + \epsilon v_1(\mathbf{x})\,, \tag{2.4}$$

where we have retained only the first-order terms of ϵ. These representations are substituted into (2.1) and only linear terms in ϵ are retained. The resulting equation is

$$\begin{aligned} \pounds_0 u(\mathbf{x}, \mathbf{x}_s, \omega) &= \left[\nabla^2 + \frac{\omega^2 + i\omega\gamma_0(\mathbf{x})}{v_0^2(\mathbf{x})}\right] u(\mathbf{x}, \mathbf{x}_s, \omega) = -F(\omega)\,\delta(\mathbf{x} - \mathbf{x}_s) \\ &+ \left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\,v_1(\mathbf{x})}{v_0(\mathbf{x})}\right]\frac{u(\mathbf{x}, \mathbf{x}_s, \omega)}{v_0^2(\mathbf{x})}. \end{aligned} \tag{2.5}$$

The wavefield can be decomposed into a reference field $u_I(\mathbf{x}, \mathbf{x}_s, \omega)$ and a scattered field $u_S(\mathbf{x}, \mathbf{x}_s, \omega)$:

$$u(\mathbf{x}, \mathbf{x}_s, \omega) = u_I(\mathbf{x}, \mathbf{x}_s, \omega) + u_S(\mathbf{x}, \mathbf{x}_s, \omega)\,. \tag{2.6}$$

Now we substitute (2.6) into (2.5) and require that $u_I(\mathbf{x}, \mathbf{x}_s, \omega)$ should be a solution of the unperturbed equation

$$\pounds_0 u_I(\mathbf{x}, \mathbf{x}_s, \omega) = -F(\omega)\,\delta(\mathbf{x} - \mathbf{x}_s)\,, \tag{2.7}$$

subject to the radiation condition (2.2). It now follows that $u_S(\mathbf{x}, \mathbf{x}_s, \omega)$ satisfies the following equation:

$$\begin{aligned} \pounds_0 u_S(\mathbf{x}, \mathbf{x}_s, \omega) = &\left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\,v_1(\mathbf{x})}{v_0(\mathbf{x})}\right] \\ &\frac{(u_I(\mathbf{x}, \mathbf{x}_s, \omega) + u_S(\mathbf{x}, \mathbf{x}_s, \omega))}{v_0^2(\mathbf{x})}. \end{aligned} \tag{2.8}$$

We write $u_S(\mathbf{x}, \mathbf{x}_s, \omega)$ in terms of the Green function

$$\begin{aligned} u_S(\mathbf{x}_g, \mathbf{x}_s, \omega) = \int\limits_D &\left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\,v_1(\mathbf{x})}{v_0(\mathbf{x})}\right] \\ &\frac{[u_I(\mathbf{x}, \mathbf{x}_s, \omega) + u_S(\mathbf{x}, \mathbf{x}_s, \omega)]\,g(\mathbf{x}, \mathbf{x}, \omega)}{v_0^2(\mathbf{x})} d^3x. \end{aligned} \tag{2.9}$$

Here, the domain D of integration must contain the support of $v(\mathbf{x})$ and $\gamma(\mathbf{x})$ assumed to be in some finite subdomain of $x_3 > 0$.

Now we have to apply the Born approximation to the integral equation (2.9). For small perturbations ($\epsilon v(\mathbf{x})$ and $\epsilon\gamma(\mathbf{x})$), we would like to argue that the scattered field $u_S(\mathbf{x}, \mathbf{x}_s, \omega)$ is small. Thus we can neglect the products of $\epsilon v(\mathbf{x})$ and $\epsilon\gamma(\mathbf{x})$ with $u_S(\mathbf{x}, \mathbf{x}_s, \omega)$ as compared to the products with $u_I(\mathbf{x}, \mathbf{x}_s, \omega)$ under the integral sign of the Eq. (2.9). Thus, Eq. (2.9) takes the following form

$$u_S(\mathbf{x}_g, \mathbf{x}_s, \omega) = \int_D \left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\, v_1(\mathbf{x})}{v_0(\mathbf{x})} \right] \frac{u_I(\mathbf{x}, \mathbf{x}_s, \omega)\, g(\mathbf{x}, \mathbf{x}_g, \omega)}{v_0^2(\mathbf{x})} d^3x. \tag{2.10}$$

The Eq. (2.10) is the fundamental integral equation for acoustic inversion and we will use it as a starting point for generating the inversion formulas.

3. The Constant Background Zero-Offset Equation

The simplest problem to deal with is the one in which the source and receiver are coincident, that is $\mathbf{x}_s = \mathbf{x}_g$ on a flat surface $x_3 = 0$, the background speed and damping are constant, $v_0(\mathbf{x}) = v_0$, $\gamma_0(\mathbf{x}) = \gamma_0$. In this case it is convenient to introduce

$$\zeta = (\zeta_1, \zeta_2, 0) = \mathbf{x}_s = \mathbf{x}_g. \tag{3.1}$$

Assuming a high frequency, we can make the following approximation:

$$\sqrt{\frac{\omega^2 + i\omega\gamma_0}{v_0^2}} \approx \frac{\omega}{v_0}\left(1 + \frac{i\gamma_0}{2\omega}\right). \tag{3.2}$$

So, we would have the following representations for the Green function and the incident field

$$g(\zeta, \mathbf{x}, \omega) = \frac{\exp\left[\dfrac{i\omega\left(1 + \dfrac{i\gamma_0}{2\omega}\right) r}{v_0}\right]}{4\pi r}, \tag{3.3}$$

$$u_I(\zeta, \mathbf{x}, \omega) = F(\omega) \frac{\exp\left[\dfrac{i\omega\left(1 + \dfrac{i\gamma_0}{2\omega}\right) r}{v_0}\right]}{4\pi r}, \qquad r = |\mathbf{x} - \zeta|. \tag{3.4}$$

Therefore we can rewrite Eq. (2.10) as

$$u_S(\zeta,\omega) = \frac{F(\omega)}{(4\pi v_0)^2} \int\limits_{x_3>0} \left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\, v_1(\mathbf{x})}{v_0(\mathbf{x})} \right] \frac{\exp\left[\dfrac{2i\omega\left(1+\dfrac{i\gamma_0}{2\omega}\right)r}{v_0}\right]}{r^2}\, d^3x. \tag{3.5}$$

where $x_3 > 0$ corresponds to the half-space which in our case is the Earth.

4. Recovery of the Three-Dimensional Variation in $v(\mathbf{x})$ in the Presence of Damping

We again consider the integral equation (3.5) and assume that the vector ζ ranges over the entire upper surface. When $F(\omega) = 1$, this equation admits an exact solution and this solution will be derived in this Section. So first consider Eq. (3.5) with $F(\omega)$ replaced by unity

$$u_S(\zeta,\omega) = \frac{1}{(4\pi v_0)^2} \int\limits_{x_3>0} \left[-i\epsilon\omega\gamma_1(\mathbf{x}) + \frac{2\epsilon\omega^2 v_1(\mathbf{x})}{v_0(\mathbf{x})} + \frac{2i\epsilon\omega\gamma_0(\mathbf{x})\, v_1(\mathbf{x})}{v_0(\mathbf{x})} \right] \frac{1}{r^2} \exp\left[\frac{2i\omega r}{v_0}\right] \exp\left[\frac{-\gamma_0 r}{v_0}\right] d^3x, \tag{4.1}$$

where $\zeta = \mathbf{x}_s = \mathbf{x}_g$, and $r = |\mathbf{x} - \zeta|$. Since the source and the receiver are coincident, in the last expression we have modified the notation and set $u_S(\mathbf{x}_g, \mathbf{x}_s, \omega) = u_S(\zeta, \omega)$. We exploit the high-frequency nature of the data and keep the leading order terms of ω in the last integral equation. The result is

$$u_S(\zeta,\omega) = \frac{\omega^2}{8\pi^2 v_0^3} \int\limits_{x_3>0} \frac{\epsilon v_1(\mathbf{x})}{r^2} \exp\left[\frac{(2i\omega - \gamma_0)\, r}{v_0}\right] d^3x. \tag{4.2}$$

The integral Eq. (4.2) is of convolution form, the function $\epsilon v(\mathbf{x})$ being convolved with the function $r^{-2} \exp[(2i\omega - \gamma_0)\, r/v_0]$. Thus, a Fourier transform will replace the integration in x_1 and x_2 by multiplication of the two transformed functions. Unfortunately, the Fourier transform of $r^{-2} \exp[(2i\omega - \gamma_0)\, r/v_0]$ is unknown. On the other hand, if there were only one power of r in the denominator, that is $r^{-1} \exp[(2i\omega - \gamma_0)\, r/v_0]$, then the Fourier transform would be known in a closed form. Thus to get one power of r in the denominator, we differentiate the integral Eq. (4.2) with respect to ω. The result is

$$\int\limits_D \epsilon v_1(\mathbf{x})\, g_1(\zeta - \mathbf{x}, \omega)\, d^3x = -i\pi v_0^4 \frac{\partial}{\partial\omega} \left[\frac{u_S(\zeta,\omega)}{\omega^2}\right], \tag{4.3}$$

where g_1 is the free-space Green function for a medium,

$$g_1(\zeta - \mathbf{x}, \omega) \equiv \frac{\exp\left[\dfrac{(2i\omega - \gamma_0)\, r}{v_0}\right]}{4\pi r}, \qquad r \equiv |\zeta - \mathbf{x}|. \tag{4.4}$$

Now we define the spatial Fourier transform for this problem with a factor two in the exponent. The forward spatial transform

$$\widetilde{f}(\kappa) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-2i\kappa.\rho\right] f(\rho)\, d^2\rho, \tag{4.5}$$

and the inverse transform

$$f(\rho) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[2i\kappa.\rho\right] \widetilde{f}(\rho)\, d^2\kappa, \tag{4.6}$$

are defined with the convention $\rho \equiv (x_1, x_2)$. The wave vector κ is defined in terms of two wave numbers k_1 and k_2 by $\kappa \equiv (k_1, k_2)$. Application of the spatial Fourier transform Eq. (4.5) to (4.3), converts the convolution to a multiplication in the κ domain and the result is

$$\int_{0}^{\infty} \epsilon\widetilde{v}_1(\mathbf{x})\, \widetilde{g}_1(\kappa, x_3, \omega)\, dx_3 = -i\pi v_0^4 \frac{\partial}{\partial\omega}\left[\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\right], \tag{4.7}$$

with the transverse Fourier transform of $g_1(\mathbf{x}, \omega)$ being given by

$$\widetilde{g}_1(\kappa, x_3, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[-2i\kappa.\rho + \dfrac{(2i\omega-\gamma_0)\, r}{v_0}\sqrt{\rho^2 + x_3^2}\right]}{4\pi\sqrt{\rho^2+x_3^2}}\, d^2\rho. \tag{4.8}$$

One way to find $\widetilde{g}_1(\kappa, x_3, \omega)$ is by direct performing of the above integration. But the most physically enlightening way of finding the expression is by considering (4.4) to be Green's function for the Helmholtz equation

$$\left[\nabla^2 - \left(\frac{2i\omega-\gamma_0}{v_0}\right)^2\right] g_1(x_1, x_2, x_3, \omega) = -\delta(x_1)\,\delta(x_2)\,\delta(x_3). \tag{4.9}$$

The operator in this equation is different from (2.7), which is due to the factor $2i\omega - \gamma_0$ appearing in the exponent of Eq. (4.4).The Fourier transform of (4.9) yields

$$\left[\frac{d^2}{dx_3^2} - 4\kappa^2 + \frac{4\omega^2}{v_0^2} - \frac{\gamma_0^2}{v_0^2} + \frac{4i\gamma_0\omega}{v_0^2}\right] \widetilde{g}_1(\kappa, x_3, \omega) = -\delta(x_3). \tag{4.10}$$

Note that the factor of 4 multiplying κ^2 is a result of the factor 2 in the exponent of the transform kernel. Since $k^2 = \dfrac{\omega^2}{v_0^2} - \dfrac{\gamma_0^2}{4v_0^2} + \dfrac{i\gamma_0\omega}{v_0^2} = k_1^2 + k_2^2 + k_3^2$, so rewriting the last equation by using the following definition for the vertical wave number k_3:

$$k_3^2 = \frac{\omega^2}{v_0^2} - \frac{\gamma_0^2}{4v_0^2} + \frac{i\gamma_0\omega}{v_0^2} - \kappa^2, \tag{4.11}$$

allows us to write

$$\left[\frac{d^2}{dx_3^2}+(2k_3)^2\right]\widetilde{g}_1(\kappa,x_3,\omega)=-\delta(x_3). \tag{4.12}$$

It is now clear that $\widetilde{g}_1(\kappa,x_3,\omega)$ is just the Green function for the one-dimensional wave equation. Therefore we may write the result

$$\widetilde{g}_1(\kappa,x_3,\omega)=-\frac{\exp\left[2ik_3\left|x_3\right|\right]}{4ik_3}. \tag{4.13}$$

The real and imaginary parts of k_3 are given by

$$\mathrm{Re}(k_3)=\sqrt{\frac{1}{2}\left(\frac{\omega^2}{v_0^2}-\frac{\gamma_0^2}{4v_0^2}-\kappa^2+\sqrt{\left(\frac{\omega^2}{v_0^2}-\frac{\gamma_0^2}{4v_0^2}-\kappa^2\right)^2+\frac{\gamma_0^2\omega^2}{v_0^4}}\right)},$$

$$\mathrm{Im}(k_3)=\frac{\gamma_0\omega}{2v_0^2\mathrm{Re}\,(k_3)}.$$

In the absence of damping i.e. $\gamma_0=0$, the form of k_3 agrees with the form given in the papers [3, 4]. Furthermore, in the absence of damping, k_3 is real-valued if $|\omega|>v_0\kappa$ and purely imaginary if $|\omega|<v_0\kappa$. In [3, 4], there is a restriction on the choice of k_3 to be real-valued because a purely complex-valued k_3 does not give any useful information about the recovered signal. However, in our case, there is no need to put a restriction on the choice of k_3 since it has both real as well as imaginary parts.

On substitution of (4.13) in (4.7), we obtain

$$\int_0^\infty \epsilon\widetilde{v}_1(\mathbf{x})\exp\left[2ik_3\left|x_3\right|\right]dx_3=-4\pi k_3v_0^4\frac{\partial}{\partial\omega}\left[\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\right]. \tag{4.14}$$

The integral equation (4.14) is nearly a forward Fourier transform in x_3. To make this similarity exact, first we observe that $x_3=|x_3|$ in the domain of integration. Furthermore, the lower limit of integration can be extended to $-\infty$, since by assumption $\epsilon v(\mathbf{x})=0$ for $x_3<0$ and hence $\epsilon\widetilde{v}(\mathbf{x})$ is also zero for $x_3<0$. We may thus write (4.14) as

$$\int_{-\infty}^\infty \epsilon\widetilde{v}_1(\mathbf{x})\exp\left[2ik_3x_3\right]dx_3=-4\pi k_3v_0^4\frac{\partial}{\partial\omega}\left[\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\right]. \tag{4.15}$$

Application of the inverse transform to the Eq. (4.15) leads to

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^\infty\left\{\int_{-\infty}^\infty \epsilon\widetilde{v}_1(\mathbf{x})\exp\left[2ik_3x_3\right]dx_3\right\}\exp\left[-2ik_3x_3\right]dk_3\\ =-\int_{-\infty}^\infty 4k_3v_0^4\frac{\partial}{\partial\omega}\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\exp\left[-2ik_3x_3\right]dk_3.\end{aligned} \tag{4.16}$$

This yields the following equation:

$$\epsilon\widetilde{v}_1(\mathbf{x}) = -4v_0^4 \int_{-\infty}^{\infty} k_3 \frac{\partial}{\partial\omega}\left[\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\right] \exp\left[-2ik_3x_3\right] dk_3. \tag{4.17}$$

Now apply the inverse Fourier transform (4.6) to (4.17), to obtain

$$\epsilon v_1(\mathbf{x}) = -\frac{4v_0^4}{\pi^2} \int_{-\infty}^{\infty} k_3 \frac{\partial}{\partial\omega}\left[\frac{\widetilde{u}_S(\kappa,\omega)}{\omega^2}\right] \exp\left[2i\left(\kappa.\rho - k_3x_3\right)\right] d^3k, \tag{4.18}$$

where d^3k corresponds to the triple integration. In the right-hand side of (4.18) the $\zeta-$ dependence is not explicit, and to make this dependence explicit we write $\widetilde{u}_S(\kappa,\omega)$ in its spatial Fourier representation (that is we apply (4.5)). Thus we have

$$\epsilon v_1(\mathbf{x}) = -\frac{4v_0^4}{\pi^2} \int_{-\infty}^{\infty} k_3 d^3k \int_{-\infty}^{\infty} d^2\zeta \frac{\partial}{\partial\omega}\left[\frac{u_S(\zeta,\omega)}{\omega^2}\right] \exp\left[2i\left(\kappa.\left(\rho-\zeta\right) - k_3x_3\right)\right]. \tag{4.19}$$

Since the data are recorded in the time domain, the inversion formula should be written as a function of time. Rewriting $u_S(\zeta,\omega)$ in its temporal Fourier representation, we obtain

$$\begin{aligned}\epsilon v_1(\mathbf{x}) = -\frac{4iv_0^4}{\pi^2} \int_{-\infty}^{\infty} d^2\zeta \int_{-\infty}^{\infty} k_3 d^3k \exp\left[2i\left(\kappa.\left(\rho-\zeta\right) - k_3x_3\right)\right] \\ \frac{1}{\omega_0^2}\int_0^{\infty} u_S(\zeta,t)\left[1+\frac{2i}{\omega_0 t}\right] t\exp\left[i\omega_0 t\right] dt.\end{aligned} \tag{4.20}$$

This is an exact solution to the integral equation (4.2). More precisely, (4.2) is an equation in the space-frequency domain, meaning that (4.19) is a solution of (4.2) and the last expression is the result of re-expressing the observed data in space-time. The results given by Eqs. (4.19)–(4.20) agree with the results given in papers [3, 4] if we take damping equal to zero.

5. Recovery of the Damping and an Iterative Procedure to Improve Velocity and Damping Profiles

We consider the expression (4.1), and write it in the form

$$\begin{aligned}\frac{u_S(\zeta,\omega)}{\omega} = \frac{1}{\left(4\pi v_0\right)^2} \int_{x_3>0} \frac{2\epsilon v_1(x)}{v_0 r^2}\left(\omega + i\gamma_0\right)\exp\left[\frac{2i\omega r}{v_0}\right]\exp\left[\frac{-\gamma_0 r}{v_0}\right] d^3x \\ -\frac{i}{\left(4\pi v_0\right)^2}\int_{x_3>0}\frac{\epsilon\gamma_1(\mathbf{x})}{r^2}\exp\left[\frac{2i\omega r}{v_0}\right]\exp\left[\frac{-\gamma_0 r}{v_0}\right] d^3x = X_2(\zeta,\omega) + Y_2(\zeta,\omega),\end{aligned} \tag{5.1}$$

where

$$X_2(\zeta,\omega) = \frac{1}{(4\pi v_0)^2} \int\limits_{x_3>0} \frac{2\epsilon v_1(\mathbf{x})}{v_0 r^2} (\omega + i\gamma_0) \exp\left[\frac{2i\omega r}{v_0}\right] \exp\left[\frac{-\gamma_0 r}{v_0}\right] d^3x,$$

and

$$Y_2(\zeta,\omega) = -\frac{i}{(4\pi v_0)^2} \int\limits_{x_3>0} \frac{\epsilon\gamma_1(\mathbf{x})}{r^2} \exp\left[\frac{2i\omega r}{v_0}\right] \exp\left[\frac{-\gamma_0 r}{v_0}\right] d^3x.$$

Since $\epsilon v_1(\mathbf{x})$ can be recovered from Eq. (4.20), therefore $X_2(\zeta,\omega)$ can be treated as a known function. Expression (5.1) can be written in the form

$$-\frac{i}{(4\pi v_0)^2} \int\limits_{x_3>0} \frac{\epsilon\gamma_1(\mathbf{x})}{r^2} \exp\left[\frac{(2i\omega - \gamma_0) r}{v_0}\right] d^3x = \left(\frac{u_S(\zeta,\omega)}{\omega} - X_2(\zeta,\omega)\right). \tag{5.2}$$

The integral equation (5.2) is of convolution form, the function $\epsilon\gamma_1(x)$ being convolved with the function $r^{-2}\exp[(2i\omega - \gamma_0) r/v_0]$. As in the case of Eq. (4.2), to get the Fourier transform in the closed form, we differentiate (5.2) with respect to ω. The result is

$$\int\limits_{x_3>0} \epsilon\gamma_1(\mathbf{x})\, g_1(\zeta - \mathbf{x},\omega)\, d^3x = 2\pi v_0^3 \frac{\partial}{\partial\omega}\left(\frac{u_S(\zeta,\omega)}{\omega} - X_2(\zeta,\omega)\right), \tag{5.3}$$

where $g_1(\zeta - \mathbf{x},\omega)$ is given by Eq. (4.4). Now we apply exactly the same procedure as that applied to (4.3), to yield

$$\epsilon\gamma(\mathbf{x}) = -\frac{8iv_0^3}{\pi^2} \int\limits_{-\infty}^{\infty} k_3 d^3k \int\limits_{-\infty}^{\infty} d^2\zeta \frac{\partial}{\partial\omega}\left[\frac{u_S(\zeta,\omega)}{\omega} - X_2(\zeta,\omega)\right] \exp\left[2i\left(\kappa.(\rho - \zeta) - k_3x_3\right)\right]. \tag{5.4}$$

Since the data are recorded in the time domain, therefore writing $u_S(\zeta,\omega)$ and $X_2(\zeta,\omega)$ in their causal temporal Fourier representation, further modifies the inversion formula (5.4) to the form

$$\epsilon\gamma_1(\mathbf{x}) = \frac{8v_0^3}{\pi^2} \int\limits_{-\infty}^{\infty} d^2\zeta \int\limits_{-\infty}^{\infty} d^3k k_3 \exp\left[2i\left(\kappa.(\rho - \zeta) - k_3x_3\right)\right] \int\limits_{0}^{\infty} \left[u_S(\zeta,t)\left(\frac{1}{\omega_0} + \frac{i}{\omega_0^2 t}\right) - X_2(\zeta,\omega_0)\right] t \exp[i\omega_0 t]\, dt. \tag{5.5}$$

Now the velocity and damping profiles can be improved by an iterative procedure. We substitute the damping profile from (5.5) in (5.1) and repeat all the steps of this subsection to get another approximation to the velocity profile. This velocity profile can then be used to obtain another approximation of the damping profile.

6. Conclusions

We have derived approximate solutions to the inverse problem of finding the velocity and damping from the observed wavefield. The approximations made are often used in modeling the inverse problem in seismic exploration. The results derived in this work aimed at determining the variation in wavespeed and damping, are accomplished by simplifying the complicated expressions due to the high frequency assumption. It is established in this work that damping of the medium plays a role in getting a more accurate map of the subsurface. An iterative procedure to improve the velocity and damping profiles is also presented.

We have presented a procedure to determine the wavespeed and damping profiles of a medium with three dimensions of parameter variability, when the source and receiver are located at the same place for three-dimensional problems. We also have assumed the constant-background wavespeed and damping. Nevertheless, the inversion procedure presented in this chapter may provide a launching pad to attack more general problems:

- The derivation of inversion formulas for a variety of source-receiver geometries.
- The derivation of inversion formulas for two-dimensional parameter variability.
- The derivation of inversion formulas for variable-background and a variety of source-receiver geometries.

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